Lecture 6
What did we do so far?

• We have learned how to build and use sets to model objects used in computation.

• We have learned how to use inductive definitions and induction proofs.

• We have built the set $\mathbb{HF}$. Most of our future objects will be either elements or relations on $\mathbb{HF}$.

• We have learned that there are many sets of finite size.

• We have seen that there are many versions of infinite sets, countable and uncountable.
Outline of lecture 6

- We shall informally introduce **Propositional Logic**.

- We define the **syntax** of Propositional Logic as the set of words **WFF** which model the formulas of Propositional Logic.

- We formulate the property of **unique readability** of formulas.

- We define the **semantics** or **meaning** of formulas of **WFF** as boolean functions using the **rank** of formulas.

- We introduce the basic semantic concepts **satisfiability**, **logical equivalence** and **logical consequence** in **Lecture 7**.
Propositional connectives

We want to analyse the (universal) words

and, or, not
and
if ... then
and
if and only if or iff.

They are called propositional connectives.

They connect formulas of WFF, also called sentences or propositions.
Atomic (simple) propositions

\(P_1\): There will be elections.

\(P_2\): Dinosaurs drink beer.

\(P_3\): \(a \in X\).

\(P_4\): \(F : A \to B\) is a one-one function.

\(P_5\): There are no sets \(A\) such that \(\mathbb{N} \prec A \prec \wp(\mathbb{N})\).

Atomic propositions are **true** or **false**
but we may not know which of the two.
Propositions (statements)

From atomic propositions we may form

If

\(( ( \text{if } P_1 \text{ then } P_2 ) \text{ and } ( P_1 \text{ or } P_2 ) ) \text{ then } P_2.\)

If

\(( ( \text{if } P_3 \text{ then } P_4 ) \text{ and } ( \text{if } P_4 \text{ then not } P_3 ) ) \text{ then } P_5.\)

Read these aloud with the full text of \(P_1, \ldots, P_5.\)

Can you analyse their meaning?
Why do we model Propositional Logic in the world of sets?

- To clarify the meaning of complex sentences.
- To make the meaning of complex sentences computable.
- To computerize the manipulation of formulas of propositional logic.
- To enable computer aided and automated theorem proving (in mathematics and other exact sciences).
The syntax of propositional logic
(An inductive definition)

Let

\[ Symb = \{\land, \lor, \to, \neg, T, F, (,), \} \]

and

\[ Var = \{p_0, p_1, p_2, \ldots, p_i, \ldots\} \]

where \( i \in \mathbb{N} \).

We treat the elements of \( Var \) also like atoms. This means they are coded as sets, but we shall not use any properties of this coding.

\( Symb \) is called the set of logical symbols and parentheses.

\( \land \) is pronounced \textit{and}, \( \lor \) is pronounced \textit{or},
\( \neg \) is pronounced \textit{not}, \( \to \) is pronounced \textit{arrow},
\( T \) is pronounced \textit{true}, \( F \) is pronounced \textit{false}.

\( Var \) is called the set of propositional variables.
Definition 1 (3.1.1)

The set of well formed formulas \( \text{WFF} \) as words

The set of well formed formulas \( \text{WFF} \) is a subset of \( (\text{Symb} \cup \text{Var})^* \), the finite words over \( \text{Symb} \cup \text{Var} \), defined inductively as follows:

**Basis:** (Atomic formulas of \( \text{WFF} \)).
- (i) For each \( i \in \mathbb{N} \), \( p_i \) is in \( \text{WFF} \).
- (ii) \( \text{F} \in \text{WFF} \).
- (iii) \( \text{T} \in \text{WFF} \).

**Closure:**
- (i) If \( \phi_1, \phi_2 \) are in \( \text{WFF} \), so is \( (\phi_1 \land \phi_2) \);
- (ii) If \( \phi_1, \phi_2 \) are in \( \text{WFF} \), so is \( (\phi_1 \lor \phi_2) \);
- (iii) If \( \phi_1, \phi_2 \) are in \( \text{WFF} \), so is \( (\phi_1 \rightarrow \phi_2) \);
- (iv) If \( \phi \) is in \( \text{WFF} \), so is \( \neg \phi \).
Some Greek letters

We shall always use lower case Greek letters to denote elements of WFF.

- $\phi$ is pronounced (in Hebrew) Fee.
- $\phi_0$ is pronounced Fee zero.
- $\phi_n$ is pronounced Fee enn.
- $\psi$ is pronounced Psee.
- $\psi_0$ is pronounced Psee zero.
- $\psi_n$ is pronounced psee enn.
- $\chi$ is pronounced Khee.
- $\theta$ is pronounced Theta.
- $\phi$ and $\emptyset$ are different symbols.

We shall sometimes write

$$\phi(p_1, p_2, \ldots, p_n)$$

meaning that all the variables occurring in the word $\phi$ are among $p_1, p_2, \ldots, p_n$. 6s-2014
Examples 2 (3.1.2)

Well formed formulas

\[ \phi_1, \phi_2 \text{ and } \phi_3 \text{ are formulas of } \text{WFF:} \]

\[ \phi_1 : ((p_1 \rightarrow T) \land \neg(\neg p_3 \lor p_1)) \]

\[ \phi_2 : (((p_1 \rightarrow p_2) \rightarrow p_1) \rightarrow p_3) \]

\[ \phi_3 : \neg(\neg p_1 \land p_2), \quad \phi_4 : (\neg \neg p_1 \lor p_2) \]

We also write (as abbreviations):

\[ (((\phi_1 \land \phi_1) \rightarrow \phi_3) \]

Here \( \phi_1, \phi_2, \phi_3 \) are subroutine calls in the inductive definition.
Examples 3

Words which are not well formed formulas

Not well formed formulas of $\textbf{WFF}$:

\[
\begin{align*}
(((\neg)), \ (\neg p_1)), \ (\neg p_1), \\
p_1 \land p_2, \ (p_1 \land p_2, \ (p_1 \land p_2)), \ p_1p_2\neg , \ \\
(p_1 \rightarrow T) \land \neg(\neg p_3 \lor p_1) \ \\
((p_1 \rightarrow p_2) \rightarrow p_1p_3) \rightarrow p_3
\end{align*}
\]
Definition 4

Formulas of $\text{WFF}_{\rightarrow, \text{False}}$

The set $\text{WFF}_{\rightarrow, \text{F}} \subseteq \text{WFF}$ is defined similarly but using only (i) and (ii) of the basis and (iii) of the closure condition.

Well formed formulas of $\text{WFF}_{\rightarrow, \text{F}}$:

$$(((p_1 \rightarrow \text{F'}) \rightarrow p_1) \rightarrow \text{F'}),$$

$$((p_1 \rightarrow \text{F'}) \rightarrow (p_1 \rightarrow \text{F})).$$

Not well formed formulas of $\text{WFF}_{\rightarrow, \text{F}}$:

$$p_1 \land p_2, (p_1 \land p_2), (\text{T} \lor \text{F})$$
Reading and writing formulas

Given a word \( w \in (Symb \cup Var)^* \)
we show that \( w \in \text{WFF} \) by exhibiting a

**generating sequence.**

\[ p_1, T, p_3 \]
\[ \neg p_3, (p_1 \rightarrow T), (\neg p_3 \lor p_1), \]
\[ \neg (\neg p_3 \lor p_1) \]

hence

\[
((p_1 \rightarrow T) \land \neg (\neg p_3 \lor p_1)) \in \text{WFF}
\]

We want to define various relations and functions inductively on
well-formed formulas \( \text{WFF} \).
Definition 5 (3.1.3)
Well formed tree formulas WFTF

Basis: Var \subseteq WFTF and T, F ∈ WFTF.
They are trees consisting of a single node.

Closure: (i) If T_1, T_2 ∈ WFTF then the tree with root \( \land \) and \( T_1 \) as its left son and \( T_2 \) as its right son is also in WFTF.

(ii) If T_1, T_2 ∈ WFTF then the tree with root \( \lor \) and \( T_1 \) as its left son and \( T_2 \) as its right son is also in WFTF.

(iii) If T_1, T_2 ∈ WFTF then the tree with root \( \rightarrow \) and \( T_1 \) as its left son and \( T_2 \) as its right son is also in WFTF.

(iv) If T_1 ∈ WFTF then the tree with root \( \neg \) and the only subtree \( T_1 \) as its son is also in WFTF.
Definition 6
Alternative view of WFTF

We treat $\neg$ as a unary function and $\land, \lor$ and $\rightarrow$ as binary functions on $\text{WFTF}$.

**Basis:** $\text{Var} \subseteq \text{WFTF}$ and $T, F \in \text{WFTF}$.

**Closure:** (i) If $T_1, T_2 \in \text{WFTF}$ then $\land(T_1, T_2) \in \text{WFTF}$.
(ii) If $T_1, T_2 \in \text{WFTF}$ then $\lor(T_1, T_2) \in \text{WFTF}$.
(iii) If $T_1, T_2 \in \text{WFTF}$ then $\rightarrow (T_1, T_2) \in \text{WFTF}$.
(iv) If $T_1 \in \text{WFTF}$ then $\neg (T_1) \in \text{WFTF}$.

$\text{WFTF}$ is the smallest set containing $\text{Var} \cup \{T, F\}$ which is closed under the functions $\neg, \land, \lor$ and $\rightarrow$.

So we can use inductive definitions and proofs on $\text{WFTF}$. 

Back to outline of Lecture 6
Writing formulas

Let $T \in \mathbf{WFTF}$.

We can think of writing a well formed formula $\phi$ as the process of obtaining a word $\text{write}(T)$ from a tree $T$.

We start at the leaves and just write them down as words.

If we are at a node labeled by a symbol $\square$ from $\{\land, \lor, \rightarrow\}$ and two subtrees $T_1, T_2$ we add the parentheses and write the word

$$(\text{write}(T_1)\square\text{write}(T_2))$$

If a node is labeled $\neg$ followed by a subtree $T$ we write the word

$$\neg\text{write}(T)$$

We easily observe that $\text{write}(T) \in \mathbf{WFF}$, and that writing is unique:
There is exactly one word $\phi$ which can be obtained from $T$ in the above way.
Reading formulas

We can think of reading a formula $\phi$ as the process of obtaining a tree $T$ from the word $\phi$.

This is a more complicated process and has to be proven by induction.

**Theorem 7 (3.1.4) Unique readability of WFF**

*For every well formed formula $\phi \in \text{WFF}$ there is exactly one labelled tree $T$ such that $\text{write}(T) = \phi$.*

The proof is given in one of the Tirgulim. 

*Skip the outline of the proof*
Proof of unique readability, I

Tirgul

The proof uses the Lemmas 8-14 below.

The details of the proof of the theorem and the lemmas are left as an exercise.

The first four lemmas can be proved independently.

To prove lemma 13 one needs all the previous lemmas.

For lemma 14 one needs only lemma 13.

Finally, to prove the theorem, one needs only lemma 14.
Proof of unique readability, II

Tirgul

Lemma 8
If $\phi \in \text{WFF}$ then $\phi \neq \epsilon$, i.e. is not the empty word, and contains at least one variable $p_i \in \text{Var}$, or one of the constants $T,F$.

Lemma 9
Let $\phi \in \text{WFF}$.
Then the number of left parenthesis in $\phi$ equals the number of right parentheses in $\phi$. 

6s-2014
Proof of unique readability, III

Tirgul

**Definition 10**
A word $\alpha \neq \epsilon$ is a **proper initial segment** of a word $\phi$ if there exists a word $\beta$ such that $\beta \neq \epsilon$ and $\phi = \alpha \circ \beta$.

**Lemma 11**
Let $\phi \in \text{WFF}$ and $\alpha$ be a proper initial segment of the word $\phi$. Then either $\alpha$ contains no parentheses or the number of left parentheses of $\alpha$ is bigger than the number of its right parentheses.
Proof of unique readability, IV

Tirgul

**Lemma 12**

Let $\phi \in \text{WFF}$ and $\alpha$ be an initial segment of the word $\phi$. If $\alpha \neq \phi$ and has no parentheses then $\alpha \in \{\neg\}^*$. 

**Lemma 13**

Let $\phi \in \text{WFF}$ and $\alpha$ be a proper initial segment of the word $\phi$. Then $\alpha \notin \text{WFF}$. 
Proof of unique readability, V

Tirgul

Lemma 14

Let $\phi, \alpha, \alpha_1, \beta, \beta_1 \in \text{WFF}$.

(i) Let $\bullet, \bullet_1 \in \{\land, \lor, \rightarrow\}$.

If $\phi = (\alpha \bullet \beta)$ and $\phi = (\alpha_1 \bullet_1 \beta_1)$ then

$\alpha = \alpha_1, \beta = \beta_1$ and $\bullet = \bullet_1$.

(ii) If $\phi = \neg \alpha$ and $\phi = \neg \beta$ then $\alpha = \beta$.

Furthermore, there are no $\alpha_1, \beta_1$ and

no $\bullet \in \{\land, \lor, \rightarrow\}$ such that $\phi = (\alpha_1 \bullet \beta_1)$.

Combining the preceding lemmas gives

a proof of the above theorem. Q.E.D.
Homework

- Change the definition of \textbf{WFF} such that no parentheses are used and show that the Unique Readability Theorem fails for the resulting definition.

- Polish notation of formulas. We define \textit{POL} inductively:
  \textbf{Basis:} $\text{Var} \cup \{T, F\} \in \text{POL}$.  
  \textbf{Closure:} If $u, v \in \text{POL}$ so are $\neg v, \land uv, \lor uv$ and $\implies uv \in \text{POL}$.
  
  Show that the Unique Readability Theorem is true for this definition.
Rank of a Formula

The inductive definition of $WFF$ suggests that formulas are defined in stages $WFF_n$.

Inductive definitions are bottom up constructions.

The Unique Readibility Theorem allows us to view formulas also top down without ambiguity.

We want to make this precise.

Given $\phi$, what is the first $n \in \mathbb{N}$ such that $\phi \in WFF_n$?
Definition 15 (3.1.13)
Rank of a formula

Let $\text{rank}$ be a function $\text{rank} : \text{WFF} \rightarrow \mathbb{N}$.

**Basis:** $\text{WFF}_0 = \{T, F\} \cup \text{Var}$.
If $\phi \in \text{WFF}_0$ then $\text{rank}(\phi) = 0$.
We call the formulas of $\text{WFF}_0$ also *atomic formulas*.

**Closure:** Let $\phi_1, \phi_2 \in \text{WFF}_n$. Then
$\phi_1, (\phi_1 \land \phi_2), (\phi_1 \lor \phi_2), (\phi_1 \rightarrow \phi_2)$ and $\neg \phi_1$ are in $\text{WFF}_{n+1}$.

$\text{rank}(\phi_1)$ is defined to be the smallest $n \in \mathbb{N}$ such that $\phi_1 \in \text{WFF}_n$. 
Proposition-Exercise 16 (3.1.14)

Properties of the rank

Compute the rank of

\[((p_1 \rightarrow p_2) \rightarrow p_1) \rightarrow p_3\]
\[\neg(\neg p_1 \wedge p_2) \rightarrow (\neg \neg p_1 \vee p_2)\]

Show the following:

(i) \( \text{WFF}_n \subseteq \text{WFF}_{n+1} \);

(ii) \( \text{rank}(\phi_1 \wedge \phi_2) = 1 + \max\{\text{rank}(\phi_1), \text{rank}(\phi_2)\} \);

(iii) \( \text{rank}(\phi_1 \vee \phi_2) = 1 + \max\{\text{rank}(\phi_1), \text{rank}(\phi_2)\} \);

(iv) \( \text{rank}(\phi_1 \rightarrow \phi_2) = 1 + \max\{\text{rank}(\phi_1), \text{rank}(\phi_2)\} \);

(v) \( \text{rank}(\neg \phi_1) = 1 + \text{rank}(\phi) \).
Truth table semantics of propositional logic

- The **meaning** of a well formed formula of **WFF** is an element of the set \{0, 1\}, where we think of 0 as "false" and 1 as "true".

- Note that the set \{0, 1\} is a subset of \(\mathbb{N}\) and the intended interpretation of its elements as "true" and "false" is outside the scope of our mathematical framework.

- We can think of the truth tables as **behavioural descriptions of boolean circuits**.

The truth tables of a \(\lor, \land, \neg\) describe the behaviour of an **or**–gate, an **and**–gate, **not**–gate respectively.
The meaning of a proposition

The definition of a meaning function given in the language of sets is the goal of this subsection.

We shall do this in three stages:

- $k$-ary truth tables, which are functions $TT : \{0, 1\}^k \to \{0, 1\}$

- The set $Ass$ of assignments, which are functions $z : Var \to \{0, 1\}$

- The meaning function, which is a function

  \[ M : \text{WFF} \times Ass \to \{0, 1\} \]
Definition 17 (3.1.19)

Truth tables

Let $n \in \mathbb{N}$. An $n$-ary truth table $TT$ is a function $TT : \{0, 1\}^n \to \{0, 1\}$.

With each symbol $\wedge, \vee, \to$ we shall associate binary truth tables $TT_\wedge, TT_\vee, TT_\to$ respectively in the following way:

$TT_\wedge$ is given by
\[
TT_\wedge(0,0) = 0, \quad TT_\wedge(0,1) = 0, \\
TT_\wedge(1,0) = 0, \quad TT_\wedge(1,1) = 1,
\]

$TT_\vee$ is given by
\[
TT_\vee(0,0) = 0, \quad TT_\vee(0,1) = 1, \\
TT_\vee(1,0) = 1, \quad TT_\vee(1,1) = 1,
\]
**Definition 18 (3.1.19, contd)**

*Truth tables (contd)*

$TT\rightarrow$ is given by

$TT\rightarrow(0,0) = 1$, $TT\rightarrow(0,1) = 1$,  
$TT\rightarrow(1,0) = 0$, $TT\rightarrow(1,1) = 1$,

$TT\neg$ is given by

$TT\neg(0) = 1$ and $TT\neg(1) = 0$.

$TT_T$ is given by

$TT_T = 1$ and

$TT_F$ is given by

$TT_F = 0$. 
Definition 19 (3.1.21) 

Truth assignments

A (propositional) truth assignment is a function

\[ z : \text{Var} \rightarrow \{0, 1\}. \]

We denote by \( \text{Ass} \) the set \( \{0, 1\}^{\text{Var}} \) of all truth assignments.

We can think of the variables in \( \text{Var} \) as registers, and of \( z \) as a function reading the contents of the registers in a current state.
Definition 20 (3.1.23)
The meaning function $M$

A meaning function $M$ is a function $M : \text{WFF} \times \text{Ass} \rightarrow \{0, 1\}$.

We shall denote by $M_{PL}$ the meaning function for Propositional Logic defined inductively as follows:

Basis:

\[ M_{PL}(p_i, z) = z(p_i); \]
\[ M_{PL}(T, z) = TT_T = 1; \]
\[ M_{PL}(F, z) = TT_F = 0. \]
Definition 21 (3.1.23, contd)

The meaning function $M$ (contd)

Closure:

\[
M_{PL}((\phi_1 \land \phi_2), z) = TT_\land(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));
\]

\[
M_{PL}((\phi_1 \lor \phi_2), z) = TT_\lor(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));
\]

\[
M_{PL}((\phi_1 \rightarrow \phi_2), z) = TT_\rightarrow(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z));
\]

\[
M_{PL}(\neg \phi, z) = TT_\neg(M_{PL}(\phi, z)).
\]
Example

We let \( z \) be given by

\[
z(p_0) = 1, z(p_1) = 0, z(p_2) = 1, z(p_3) = 0, z(p_4) = 1, \ldots
\]

and compute

\[
M \left( \left( (p_1 \rightarrow T) \land \neg (\neg p_3 \lor p_1) \right), z \right)
\]

\[
M(p_1, z) = 0, M(p_3, z) = 0,
\]

\[
M(\neg p_3, z) = 1,
\]

\[
M((p_1 \rightarrow T), z) = 1,
\]

\[
M((\neg p_3 \lor p_1), z) = 1, M(\neg (\neg p_3 \lor p_1), z) = 0,
\]

\[
M \left( \left( (p_1 \rightarrow T) \land \neg (\neg p_3 \lor p_1) \right), z \right) = 0
\]

Similarly, compute:

\[
M \left( \left( (p_1 \rightarrow p_2) \rightarrow p_1 \right) \rightarrow p_3 \right), z
\]

\[
M(p_1, z) = 0, M(p_2, z) = 1, M(p_3, z) = 0,
\]

\[
M((p_1 \rightarrow p_2), z) = 1,
\]

\[
M(((p_1 \rightarrow p_2) \rightarrow p_1), z) = 0,
\]

\[
M((((p_1 \rightarrow p_2) \rightarrow p_1) \rightarrow p_3), z) = 1
\]
Remarks

- $M_{PL}$ defines also a meaning function $M_{\{\to, F\}}$ for the formulas of $WFF_{\{\to, F\}}$.

As $WFF_{\{\to, F\}} \subseteq WFF$ we set

$$M_{\{\to, F\}} = M_{PL}|_{WFF_{\{\to, F\}} \times Ass}$$

the restriction of $M_{PL}$ to $WFF_{\{\to, F\}} \times Ass$.

- The definition of $M_{PL}$ rests on the Unique Readibility Theorem for the formulas of $WFF$.

  It relies not on the word $\phi$ but on the unique $T_\phi$ such that $\text{write}(T_\phi) = \phi$. 
Examples 22 (3.1.25)

Exercises

(i) Choose several $\phi \in \textbf{WFF}$ and propositional assignments $z$ and compute $M_{PL}(\phi, z)$.

(ii) Let $\phi, \psi \in \textbf{WFF}$ and $z \in \textbf{Ass}$. Show:

(ii.a) If $\phi = (\phi_1 \rightarrow \phi_1)$ then $M_{PL}(\phi, z) = 1$.

(ii.b) if $\phi = (\phi_1 \land \neg \phi_1)$ then $M_{PL}(\phi, z) = 0$.

(ii.c) If $\phi = (\neg \phi_1 \lor \phi_2)$ and $\psi = (\phi_1 \rightarrow \phi_2)$ then $M_{PL}(\phi, z) = M_{PL}(\psi, z)$. 
Homework

Finite dependency of the meaning function

It is obvious from the definitions, that the function $M_{PL}$ only depends on finitely many values of $z$. The next proposition makes this precise.

Prove the following (by induction);

**Proposition 23**

Let $\phi \in \text{WFF}$ be a formula with all its propositional variables in the set $\{p_1, p_2, ..., p_n\}$.

Let $z_1$ and $z_2$ be two propositional assignments such that for every $i \leq n$ $z_1(p_i) = z_2(p_i)$.

Then $M_{PL}(\phi, z_1) = M_{PL}(\phi, z_2)$. 
Proof of Proposition 23:

We prove by induction:

**Basis:**
If $\phi \in WFF_0$ then $\phi = p_i$ for some $i \leq n$
or $\phi = T$ or $\phi = F$.
In all these cases $M_{PL}(\phi, z)$ depends only on the value of $z(p_i)$ or is constant.

**Closure:**
If $\phi \in WFF_{n+1}$ and the proposition is true for all $\phi_1, \phi_2 \in WFF_i$ $i \leq n,$ we have four cases.

Let $\phi = (\phi_1 \land \phi_2)$.
As $M_{PL}((\phi_1 \land \phi_2), z) = TT_\land(M_{PL}(\phi_1, z), M_{PL}(\phi_2, z))$
and $TT_\land$ does not depend on $z$,
the proposition is also true for $\phi$.

The cases for $\lor, \to,$ and $\neg$ are left to the reader.
Definition 24 (3.1.27)

Truth table associated with $\phi$

We associate with each well formed formula $\phi \in \text{WFF}$ a truth table $TT_\phi$, in the following way:

Let $\phi \in \text{WFF}$ and let $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$ be all the variables occurring in $\phi$.

Let $TT_\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ the truth table defined by

$$TT_\phi(x_1, x_2, \ldots, x_n) = M_{PL}(\phi, z)$$

with $z(p_{i_j}) = x_j$ for $j = 1, 2, \ldots, n$.

The finite dependency of the meaning function assures us that this is indeed a function in $n$ variables.
Examples of truth tables, I

Let

\[ T_{\text{Scheffer}}(x_1, x_2) = 1 \text{ iff } x_1 = x_2 = 0 \]

and

\[ \phi_{\text{Scheffer}} = (\neg p_1 \land \neg p_2). \]

Then \( T_{\text{Scheffer}} = T_{\phi_{\text{Scheffer}}}. \)
Examples of truth tables, II

Let

\[ T_1(x_1, x_2) = \begin{cases} 
1 & \text{if } x_1 = x_2 \\
0 & \text{else} 
\end{cases} \]

and

\[ \phi_1(p_1, p_2) = ((p_1 \land p_2) \lor (\neg p_1 \land \neg p_2)) \]

Then \( T_1 = T_{\phi_1} \).
Examples of truth tables, III

Let

\[
T_2(x_1, x_2, x_3) = \begin{cases} 
1 & \text{if } x_1 = x_2 \text{ and } x_3 = 0 \\
1 & \text{if } x_3 = 1 \\
0 & \text{else}
\end{cases}
\]

and

\[
\phi_2(p_1, p_2, p_3) = ((\neg p_3 \land ((p_1 \land p_2) \lor (\neg p_1 \land \neg p_2))) \lor p_3)
\]

Then \( T_2 = T_{\phi_2} \).
Is every truth table $TT$ obtainable from a formula $\phi$ as $TT_\phi$?

Let $TT$ be an $n$-ary truth table.

Does there exists a formula $\phi \in WFF$ such that $TT = TT_\phi$?

A positive answer to this question, would give us a justification for the choice of the basic truth tables

$TT_\land, TT_\lor, TT_\neg$

underlying the semantics of $WFF$. 
Theorem 25 (3.1.28)

Functional completeness

Let $TT$ be an $n$-ary truth table. Then there exists a formula $\phi \in \text{WFF}$ such that $TT = TT_\phi$.

The formula can actually be found in $\text{WFF}_{\rightarrow,F}$ or in $\text{WFF}_{\lor,\neg}$.
Proof of Theorem 25

For $n = 0$ there are two constant truth tables.

For $n > 0$, let

$$x = \langle x_1, x_2, \ldots, x_n \rangle \in \{0, 1\}^n$$

such that $TT(x) = 1$.

Let

$$\ell_i = \begin{cases} p_i & \text{if } x_i = 1 \\ \neg p_i & \text{if } x_i = 0. \end{cases}$$

Let $C_x$ be the conjunction of all the $\ell_i$,

$$C_x = ((\ldots (l_1 \land l_2) \land \ldots \land l_n))$$

Now let $\phi$ be the disjunction of all the $C_x$ such that $TT(x) = 1$.

It is now easy to verify that $TT_\phi = TT$. Q.E.D.
We now generalize our notion of well formed formulas. This will not be used in the sequel, but it is useful for checking whether one understands the material presented so far.

**Definition 26 (The set of well formed formulas $WFF_S$):**

Let $S = \{s_1, s_2, \ldots, s_m\}$ be a set of symbols and $n(s_i) \in \mathbb{N}$ be a natural number called the arity of $s_i$.

Let $S_1 = S \cup \{(,;),\}$.

Let $\text{Var}$ be the set of propositional variables.

$WFF_S$ is inductively defined as a subset of $(S_1 \cup \text{Var})^*$.

**Basis:**

$\text{Var} \subseteq WFF_S$.

**Closure:**

If $s_i \in S$ and $n(s_i) = k$ and $\phi_1, \phi_2, \ldots, \phi_k \in WFF_S$ then $s_i(\phi_1; \phi_2; \ldots; \phi_k) \in WFF_S$. 
Sets and Logic (234293) WS 2014/5

Tirgul: \( \text{WFF}_S \) (continued)

Example 27 (Exercise:)
Let \( S = \{ \text{F, } \neg, \land, kuku \} \)
with \( n(\text{F}) = 0, n(\neg) = 1, n(\land) = 2, \)
and \( n(kuku) = 3. \)

Write some formulas of \( \text{WFF}_S \).

Theorem 28 (Exercise:)
Formulate and prove the Unique Readibility Theorem for \( \text{WFF}_S \).

The following will be useful later:

Theorem 29
Let \( S \) be countable. Then \( \text{WFF}_S \) is countable.
Tirgul: $\textbf{WFF}_S$ (continued)

Definition 30 (Semantics for $\textbf{WFF}_S$)
Let $S = \{s_1, s_2, ..., s_n\}$ be a set of symbols with arities $n(s_i) \in \mathbb{N}$.

For each $i \leq n$ let $TT_i$ be an $n(i)$-ary truth table.

Let $z$ be a propositional assignment.
We define a meaning function $M_S$ inductively as follows:

**Basis:**
$M_S(T, z) = 1$,
$M_S(F, z) = 0$ and
$M_S(p_i, z) = z(p_i)$.

**Closure:**
For every $i \leq n$ with $n(s_i) = k$ and $\phi_1, \phi_2, ..., \phi_k \in \textbf{WFF}_S$

$$M_S(s_i(\phi_1; \phi_2; ...; \phi_k), z) = TT_i(M_S(\phi_1), M_S(\phi_2), ..., M_S(\phi_k))$$
Tirgul: \( \text{WFF}_S \) (continued)

Let \( S = \{ \text{F}, \neg, \land, kuku \} \)
with \( n(\text{F}) = 0, n(\neg) = 1, n(\land) = 2 \),
and \( n(kuku) = 3 \).

Let \( TT_{kuku} \) be an 3-ary truth table.

Show the following by induction.

**Theorem 31**
For every formula \( \phi \in \text{WFF}_S \)
there is a formula \( \psi \in \text{WFF} \) such that for every assignment \( z \) we have
\[
M_S(\phi, z) = M(\psi, z)
\]