The speed of hereditary properties of graphs

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P – property of graphs (e.g. 3-col)

P^n – the set of graphs with vertex set $[n]$ satisfying P

$f_P(n) = |P^n|$ - the speed of P

Example 1:

P: G in P iff G is a clique

for all n: $|P^n| = 1$

Example 2:

P: G in P iff G is a disjoint union of at most 2 cliques

for all n: $|P^n| = ?$

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P: G in P iff G is a disjoint union of at most 2 cliques

for all n: $|P^n| = 2^{n-1}$

P is monotone if it is closed under taking subgraphs.

P is hereditary if it is closed under taking induced subgraphs.

Being acyclic or planar are monotone properties (hence hereditary).

Being a clique is hereditary (but not monotone).

Every hereditary property can be defined by a family (possibly infinite) of forbidden induced subgraphs.

For example, being an empty graph is defined by a forbidden family:

???

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So the size of P^n is at least as the number of non isomorphic labelings of G.

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$$\{K_2\}$$

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Amazingly, for hereditary properties,

if $P^n < n^n$, then P is well determined by its speed $|P^n|$.

Theorem [Scheinerman and Zito 1994]:

let P be hereditary. Then one of the following holds:

(i) For all large enough n, $|P^n|$ is identically 0,1 or 2.

(ii) $|P^n|=\Theta(n^k)$ for some positive integer k.

(iii) For some positive c_1, c_2, $c_1^n < |Pn| < c_2^n$.

(iv) For some $c > 0$, $n^{cn}<|P^n|$.

Main result:

There is a much finer hierarchy of speeds.

In fact, for properties with $|P^n| < n^{(1+o(1))n}$, we almost precisely determine the functions allowed to appear.

Surprisingly, we can describe the types of properties which occur at each level.

A slight abuse of growth terms:

Exponential growth:

we write $f(n) = O(k^n)$ even if

$f(n) = O(n^t k^n)$ for some constant t.

Factorial growth:

f(n) is factorial whenever

$f(n) > n^{cn}$ for some constant c.

Canonical property 1:

let G be an infinite graph, define P(G) := {H: H $\leq$ G}

clearly P(G) is hereditary.

for example, if G = $K_{1,\infty}$, P(G) = {H: H is a star or an empty graph}, and $|P^n(G)|$ = ???.

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Canonical property 2 (“template” prop.):

Let H be a simple graph with loops allowed, over $V(H) := \{1,\ldots,k\}$.

Let b,c be functions:

$b: V(H) \rightarrow \{1,2,\ldots\} \cup \infty$

$c: \binom{V(H)}{2} \rightarrow \{1,2,\ldots\} \cup \infty$

then P(H,b,c) contains all graphs G,

s.t. V(G) can be partitioned into $\{V_1,\ldots,V_k\}$ and the following holds:

(1) $|V_i| \leq b(i)$

(2) $G[V_i]$ is clique if {i,i} is a loop in H and it is an empty graph otherwise.

(3) If {i,j} in E(H), then the order of $G[V_i,V_j] \leq c(\{i,j\})$. Otherwise, the order of the bipartite complement of $G[V_i,V_j] \leq c(\{i,j\})$.

For example, $P(K_2,\infty,\infty)$ is the collection of bipartite graphs.

What is $P(H,1,1)$???

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What is $P(H,1,1)$: subgraphs of H

In general, $P(H,b,c)$ is hereditary.

Lemma:

Let H be a graph on k+1 vertices and x be one of them. If $c \equiv 1$, $b(v)=1$ for all vertices except x, and $b(x)=\infty$, then

$$|P^n(H,b,c)|=O(n^k)$$

Proof:

Let G be a graph in $P^n(H,b,c)$. G is completely defined by the labels that appear on vertices other than $v=b^{-1}(\infty)$. There are at most $\binom{n}{k}k!$ ways to label those vertices.

Why canonical properties are important?

Minimal properties with $|P^n|<n^n$, can be expressed as canonical properties with appropriate H,b, and c.

(Minimal property: every its sub-property has a smaller order of growth).

Polynomial growth:

If $|P^n|$ is bounded then $|P^n| \leq 2$ and P is one of: {}, {K_n}, {E_n}, {K_n,E_n} [Scheinerman and Zito].

If $|P^n| \geq 3$, then there is G in P^n which is neither empty nor complete.

There is v with d(v),đ(v)≥1. Hence there are ≥n-1 ways to choose v's neighborhood.

Lemma: if $|P^n| \geq 3$ then $|P^n| \geq n-1$

We write $x \sim y$ when $\Gamma(x) \setminus \{y\} = \Gamma(y) \setminus \{x\}$.

We call the equivalence classes of ~ the homogeneous sets.

Note: if $x \sim y$ then x and y are in the same orbit of the automorphism group. In addition, the homogeneous sets span either a clique or an empty graph.

Note: if x~y in G, then G-x and G-y are isomorphic.

How about the converse?

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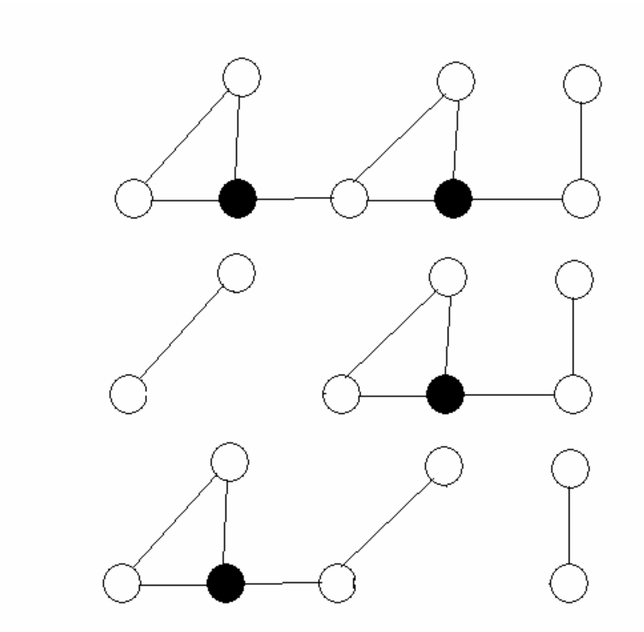
How about the converse?

G

G-x

G-y



Suppose $V(G)=A \cup B$, and B is a homogeneous set (note, every v in A…).

If A is of minimal order, then A is called the head and B the body of G (if $|A|=0$…)

Lemma 5: If G has a head with $<V(G)/2$ vertices, then G has a unique head.

Proof: assume two heads $H_1,H_2<V(G)/2$. $(V(G)\setminus H_1)\cap(V(G)\setminus H_2)=V(G)\setminus(H_1 \cup H_2)\neq\Phi$.

Let $B=(V(G)\setminus H_1)\cup(V(G)\setminus H_2)=V(G)\setminus(H_1\cap H_2)$.

For all x_1 in $V(G)\setminus H_1$, x_2 in $V(G)\setminus H_2$ and x in $V(G)\setminus(H_1 \cup H_2)$, we have $x_1\sim x\sim x_2$ and by transitivity, B is a head (contradicting the minimality of H_1).

Our purpose:

prove that if $|P^n|=\Theta(n^k)$
then every G (large enough) that satisfies P
has an unique head of size k.

Lemma 6:

If $|P^n|$ has $O(n^k)$ growth, then for large enough n, if G is in P^n then every vertex of G has a degree or a codegree at most k.

Proof: $\binom{n-1}{d(x)} > cn^k$

Lemma: Given a graph G, either G or its complement has $\leq 2k$ vertices with degree $\leq k$.

Proof:Suppose that the graph G is a counterexample. Clearly then $|V(G)| \geq 4k+2$.

Let A be a subset of [x in V(G) : $d(x) \geq k$]

and B subset of [x in V(G) : $d^c(x) \geq k$]

such that $|A|=|B|=2k+1$.

By the definition of A, between A and B there are at most $|A|k$ edges.

By the definition of B, between A and B there are at most $|B|k$ non-edges.

Between A and B there are $|A||B|$ places for edges, so $|A||B| \leq (|A|+|B|)k$ and $2k+1=|A| \leq 2k$.

Without loss of generality, suppose G
contains at most 2k vertices with codegree at most k.

Let T denote that vertex set, and let G* denote the graph that we get from G by deletion of T.

By Lemma 6 we could choose N sufficiently large that each vertex in G* has degree at most k.

Let $M_t = tK_2$, the graph consisting of t disjoint edges.

The next two lemmas constrain the edges of G*.

Lemma 8: If G* has at least $f(k, t)=(2(k-1)^2+1)(t-1)+1$ edges, then it contains an induced M_t .

Proof: Suppose G* contains more than f(k, t) edges. Pick an arbitrary edge (u_1,v_1) from E(G*). Then choose another edge (u_2,v_2) from E(G*) such that both u_2,v_2 are not in $\Gamma(u_1) \cup \Gamma(v_1)$. Continue choosing edges; Because for each vertex x we have $|\Gamma(x)|\leq k$, we remove at each step at most $|\Gamma(u_j) \cup \Gamma(v_j)|(k-1)+1\leq(2k-2)(k-1)+1$ edges from the set of edges we can choose, and after t-1 steps there is still at least one edge left.

So we have independent edges $(u_1,v_1)\dots(u_t, v_t)$ which span an M_t.

Lemma 9: There exists an m, independent from n and G, such that the number of edges in G* is at most m.

Proof: Assume that there is no such m. There is an N and a constant c such that for each n>N, we have $|P^n|<cn^k<(n/2)!$.
Since the number of edges in G* is unbounded, there is a G in P such that G* has at least f(k,N) edges.
By the previous lemma G contains an induced M_N. Since P is hereditary, M_N is in P^{2N}, but there are >N! ways to label a graph M_N, so $|P^{2N}|>c(2N)^k$.
Hence m=f(k, N) will work as a bound on number of edges.

Theorem 10: If $|P^n|$ has $O(n^k)$ order growth, then there is an integer N such that, if n>N and G is in P^n, then G has a unique head A. Furthermore, $|V(A)| \leq k$.

Proof. Let G be in P^n and define G* and T as before. We partition the vertices of G into two sets.

Let A=[x in V(G) : x in T, x is not an isolated vertex in G*, or $T \setminus \Gamma(x) \neq \Phi$].

Let B=V(G)\A. Then B is a homogeneous set, because it is an independent set and each x in B is adjacent to every element of T and to no element of G\T.

So if n is big enough, then, by Lemma 5, A is the unique head of G. Hence $|A| \leq k$ since there are at least $O(n^{|A|})$ ways to label G.

Note: If we initially assumed that G contains at most 2k vertices with degree (rather then codegree) k, then B would be a complete subgraph of G, but the rest of the argument remains the same.

We shall need the following lemma to establish a lower bound on the polynomials.

Lemma 11: If $|P^n|=0(n^k)$, then for each sufficiently large n, there is G in P^n with a head of order k.

Proof: If for every n there is an N>n such that P^N has a graph with a head of order k, then since P is hereditary P^n contains such a graph as well. So assume not. Then for n large enough each G in P^n has head of size at most k-1.
To maximize $|P^n|$, we allow every graph with a head of order at most k-1. There are at most $(n^{k-1})2^{k-1}2^{k(k-1)}$ different graphs for a head of order k-1, two choices for the homogeneous set (independent or clique), and each vertex of the head is either adjacent or not adjacent to every vertex of the homogeneous set.

So, summing for all head sizes $0<i<k$, $|P^n|=O(n^{k-1})$.
Corollary:

if $|P^n|$ has a smaller speed than $O(n^k)$, then it has speed $O(n^{k-1})$.

Theorem 13. For $k>1$, let L_k and U_k be properties defined as follows:

L_k={G : G contains a clique with order at most k and the remaining vertices are isolated}

and U_k={G : in G all but at most k vertices are G-equivalent}

If P is a property with $|P^n|=\Theta(n^k)$, then, for sufficiently large values of n, $|L_n^k| \leq |P^n| \leq |U_n^k|$,

where $|L_n^k|=\binom{n}{k}+\ldots+\binom{n}{2}+1$

and $|U_n^k| \leq 1/k!\ (2^{k(k+1)+1})\ n^k$.

We can now describe, using the canonical properties P(H,b,c), exactly what properties with polynomial growth look like.

Let A be a simple graph. Any graph H which has one more vertex than A and has a vertex identified so that removing that vertex leaves a graph isomorphic to A will be said to be of the form A*{x}.

We allow a loop at x but at no other vertex.

Consider a graph G in a property P with growth $|P^n|=\Theta(n^k)$.

We are going to build a canonical property P(H, b, c) containing G.

By Theorem 10, G has a unique head of order at most k. We define the type graph of G as a graph A*{x}, where A is the head, and we let xy be an edge in E(A*{x}) iff y is adjacent to the homogeneous set of G.

Hence the type graph A*{x} of G has a loop at x if and only if the homogeneous set of G is a clique in G.

Clearly the type graph is well-defined and captures the structure of G. Further,with b(a)=1 for all vertices a of the head A and b(x)=∞, and with c≡1, thecanonical property P(A*{x}, b, c) contains G.

Thus we call (A*{x}, b, c) the type of G.

Recall that $|P^n(A^*\{x\}, b, c)|$ has polynomial order growth.

These properties, in fact, form the basis for all properties with polynomial order growth.

Theorem 14: If $|P^n|=O(n^k)$, then there exist graphs $A_1*\{x_1\}, ..., A_r*\{x_r\}$ such that, for n sufficiently large,

$$\bigcup_{i=1}^{r} P^n(A_i*\{x_i\}, b_i, 1)=P^n$$

where $b_i(A_i)\equiv 1$ and $b_i(x_i)=\infty$ for all i.

Proof: By Theorem 10, for sufficiently large n, every G in P^n has a unique head of order at most k. Since there are a finite number of graphs of order at most k, 2^k choices for how the body is connected to the head, and two choices for the loop at x, there are a finite number of types in P.

Each of these types are of the form $(A*\{x\}, b, 1)$, where $b(A)\equiv 1$ and $b(x)=\infty$.

Two properties P and Q are equivalent if their symmetric difference is finite.

That implies that there exists an N such that $P^n=Q^n$ for all $n>N$.

Corollary: For each $k > 0$, there are only a finite number of non-equivalent hereditary properties with polynomial order growth $\Theta(n^k)$.

The minimal properties with polynomial growth turn out to be exactly the canonical properties with proper order growth.
For the polynomial order, since we have shown that all properties have polynomial growth, a minimal property shall be one in which all proper subproperties have a lower order polynomial growth.

Lemma 17. The minimal properties for speed $\Theta(n^k)$ are those which consist of exactly one type, i.e., $P(A^*\{x\}, b, 1)$ where $(A^*\{x\}, b, 1)$ is a polynomial order type.

Lemma 17. The minimal properties for speed $\Theta(n^k)$ are those which consist of exactly one type, i.e., $P(A^*\{x\}, b, 1)$ where $(A^*\{x\}, b, 1)$ is a polynomial order type.

Proof. It is clear that a property which contains more than one type (of order n^k) can not be minimal. So we only need to show that given a type $(A^*\{x\}, b, 1)$, the property $P=P(A^*\{x\}, b, 1)$ is minimal.
Suppose $|P^n|=O(n^k)$ and let $P'<P$.
We need to prove $|P'^n|=O(n^{k-1})$.
By Theorem 14, for large n, there are types $\{(A_i^*\{x_i\}, b_i, 1)\}$ such that define P'^n.

Since $P'<P$, $A_i \neq A$ for all i.
Hence $A_i<A$ for each i. That is, $|A_i|<k$, so $|P'^n|=O(n^{k-1})$.