Overview

We want to combine

- Second Order Logic and
- BSS-computations

to define a general class of graph polynomials.

SOL-formulas, I: Variables and Interpretations

We have variables

- v_i for each $i \in \mathbb{N}$.
  These are **individual variables**.
- $U_{r,i}$ for ech $r, i \in \mathbb{N}, r \geq 1$.
  These are **relation variables**. r is the arity of $U_{r,i}$.

We denote the set of variables by $\mathbf{VAR}$.

Given a **non-empty** but **finite** set A, an A-**interpretation** is a map

$$I_A : \mathbf{VAR} \to A \cup \bigcup_r P(A^r)$$

such that $I_A(v_i) \in A$ and $I_A(U_{r,i}) \subseteq A^r$.

SOL-formulas, II: Atomic Formulas and their meaning

Atomic formulas are of the form

- $(v_i \simeq v_j)$ with **set of free variables** $\{v_i, v_j\}$.
- $U_{r,j}(v_{i_1}, v_{i_2}, \ldots, v_{i_r})$ with **set free variables** $\{U_{r,j}\} \cup \{v_{i_m} : 1 \leq m \leq r\}$.

The **meaning function** M is a function which assigns to a formula and an A-interpretation its truth value, i.e.

$$M : \text{Formulas} \times \text{Interpretations} \rightarrow \{0, 1\}$$

M is defined inductively. For atomic formulas we put

- $M((v_i \simeq v_j), I_A) = 1$ iff $I_A(v_i) = I_A(v_j)$.
- $M(U_{r,j}(v_{i_1}, v_{i_2}, \ldots, v_{i_r}), I_A) = 1$ iff $(I_A(v_{i_1}), I_A(v_{i_2}), \ldots, I_A(v_{i_r})) \in I_A(U_{r,j})$.

$\mathbf{SOL}$-formulas, III: Formulas

We define now inductively the set of SOL-formulas $\mathbf{SOL}$.
If $\phi \in \mathbf{SOL}$ we denote by $free(\phi)$ its set of free variables.

- Atomic formulas ϕ are in $\mathbf{SOL}$ with $free(\phi)$ as defined before.
- If ϕ_1 and ϕ_2 are in $\mathbf{SOL}$ then ϕ of the form
  $(\phi_1 \vee \phi_2)$,
  $(\phi_1 \wedge \phi_2)$ or
  $(\phi_1 \rightarrow \phi_2)$
  is in $\mathbf{SOL}$ with $free(\phi) = free(\phi_1) \cup free(\phi_2)$.
- If ϕ_1 is in $\mathbf{SOL}$ then $\phi = \neg\phi_1$ is in $\mathbf{SOL}$ with $free(\phi) = free(\phi_1)$.
- If ϕ_1 is in $\mathbf{SOL}$ then ϕ of the form
  $\exists v_j \phi$, $\forall v_j \phi$,
  is in $\mathbf{SOL}$ with $free(\phi) = free(\phi_1) - \{v_j\}$.
- If ϕ_1 is in $\mathbf{SOL}$ then ϕ of the form
  $\exists U_{r,j} \phi$ or $\forall U_{r,j} \phi$
  is in $\mathbf{SOL}$ with $free(\phi) = free(\phi_1) - \{U_{r,j}\}$.

SOL-formulas, IV: Formulas and their meaning

The meaning function for formulas is defined as usual.

- For $\phi = \neg \phi_1$ we put $M(\neg \phi_1, I_A) = 1 - M(\phi_1, I_A)$.

- For each of the binary boolean connectives $\bullet \in \{\wedge, \vee, \rightarrow, \}$ we have the corresponding truth table $T_\bullet$.
  We put $M((\phi_1 \bullet \phi_2), I_A) = T_\bullet(M(\phi_1, I_A), M(\phi_2, I_A))$.

- Let $V \in \mathbf{VAR}$. For $\phi = \exists V \phi_1$ we put $M(\phi, I_A) = 1$ iff there is an A-Interpretation J_A such that for all variables different from V the interpretations I_A and J_A coincide, and such that $M(\phi_1, J_A) = 1$.

- Let $V \in \mathbf{VAR}$. For $\phi = \forall V \phi_1$ we put $M(\phi, I_A) = 1$ iff for every A-Interpretation J_A such that for all variables different from V the interpretations I_A and J_A coincide, we have $M(\phi_1, J_A) = 1$.

Satisfiability and $\mathbf{SOL}$-consequence

The standard notions of logic apply here as well.
Note we have restricted our interpretations of $\mathbf{SOL}$ to **finite sets**.

- A set of formulas Σ is f-**satisfiable** if there is an A-interpretation I_A with A finite such that for all $\phi \in \Sigma$ we have $M(\phi, I_A) = 1$.

- A formula ψ f-follows from a set of formulas Σ, denoted by $\Sigma \models_f \psi$, if for every A-interpretation I_A with A finite such that for all $\phi \in \Sigma$ we have $M(\phi, I_A) = 1$ we have also $M(\psi, I_A) = 1$.

- If $\tau \subset \mathbf{VAR}$, we call A-interpretations restricted to τ also τ-structures or τ-models.

$\mathbf{SOL}$-polynomials $\mathbf{SOL}(\mathcal{R})$, I: Indeterminates

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Let $\mathcal{R}$ be a commutative ring
with addition, multiplication,
and neutral elements 0 for addition and 1 for multiplication.

We first define the set of **indeterminates** $\mathbb{X}$.
Our indeterminates depend on variables in $\mathbf{VAR}$.

- For each $i \in \mathbb{N}$
  X_i is an indeterminate in $\mathbb{X}$.

- For each finite sequence of variables $\bar{V} = (V_1, V_2, \ldots, V_k) \in \mathbf{VAR}^k$
  and $i \in \mathbb{N}$ $X_{\bar{V},i}$ is an indeterminate in $\mathbb{X}$.

- $free(X_i) = \emptyset$.
  $free(X_{\bar{V},i}) = \{V_1, V_2, \ldots, V_k\}$

$\mathbf{SOL}$-polynomials $\mathbf{SOL}(\mathcal{R})$, II: Terms $\mathbb{T}(\mathcal{R})$

- Each $a \in \mathcal{R}$ is a term of $\mathbb{T}(\mathcal{R})$ with $free(a) = \emptyset$.
- Each indeterminate X_i and $X_{\bar{V},i} \in \mathbb{X}$ is a term of $\mathbb{T}(\mathcal{R})$ with $free(X_i) = \emptyset$ and $free(X_{\bar{V},i}) = \{V_1, V_2, \ldots, V_k\}$
- If t_1 and t_2 are in $\mathbb{T}(\mathcal{R})$ so are $t_1 + t_2$ and $t_1 \cdot t_2$.
  $free(t_1 + t_2) = free(t_1 \cdot t_2) = free(t_1) \cup free(t_2)$.
- If ϕ is a $\mathbf{SOL}$-formula, $tv(\phi)$ is a term in $\mathbb{T}(\mathcal{R})$ and $free(tv(\phi)) = free(\phi)$.

$\mathbf{SOL}$-polynomials $\mathbf{SOL}(\mathcal{R})$, III: Terms $\mathbb{T}(\mathcal{R})$

Let ϕ, ψ be $\mathbf{SOL}$-formulas and s, t, be terms of $\mathbb{T}(\mathcal{R})$.

- If $\bar{v} = (v_{i_1}, v_{i_2}, \dots v_{i_m})$ is a vector of individual variables, then

$$\sum_{\bar{v}:\phi} s(\bar{v}) \qquad \text{and} \qquad \prod_{\bar{v}:\phi} s(\bar{v})$$

  are terms with set of free variables $free(s) \cup free(\phi) - \{v_{i_1}, v_{i_2}, \dots v_{i_m}\}$

- If $\bar{V} = (V_1, V_2, \dots V_m)$ is a vector of individual and relation variables, then

$$\sum_{\bar{V}:\psi} t(\bar{V})$$

  is a term with set of free variables $free(t) \cup free(\phi) - \{V_1, V_2, \dots V_m\}$.

Here summation and multiplication act like quantifiers.

We do not allow products over the range of relation variables.

$\mathbf{SOL}$-polynomials $\mathbf{SOL}(\mathcal{R})$, IV: Substitution of free variables.

Let $\theta(v_1, \ldots, v_r)$ be a formula with $v_k \in free(\theta)$ for $1 \leq k \leq r$.
Let $U_{r,j}$ be an r-ary relation variable, and v_i be an individual variable.
Furthermore, let t be a term in $\mathbb{T}(\mathcal{R})$.

- Let t_1 be the term

$$\sum_{v_i : v_i = v_k} t$$

  is a term with $free(t_1) = (free(t) - \{v_i\}) \cup \{v_k\}$.

- If $U_{r,j} \notin free(\theta)$, then the term t_1 given by

$$t_1 = \sum_{U_{r,j} : \forall \bar{v} (U_{r,j}(\bar{v}) \leftrightarrow \theta(\bar{v}))} (t)$$

  is a term with $free(t_1) = (free(t) \cup free(\theta)) - \{U_{r,j}, v_1, \ldots, v_r\}$.

The sum has only one term in each case, so the sum has only the effect of changing the free variables in t.

Evaluation of $\mathbf{SOL}$-polynomials $\mathbf{SOL}(\mathcal{R})$

If an A-interpretation I_A is given, each term $t \in \mathbb{T}(\mathcal{R})$ has a natural interpretation as a **polynomial** in $\mathcal{R}[\mathbb{X}]$.

- If ϕ has no free variables and ϕ is a tautology, i.e. $\neg\phi$ is not satisfiable, then $\sum_{v:\phi} t = \prod_{v:\phi} t = t$.

- If ϕ has no free variables and ϕ is a contradiction, i.e. ϕ is not satisfiable, then $\sum_{v:\phi} t = 0$ and $\prod_{v:\phi} t = 1$.

- If for $t \in \mathbb{T}(\mathcal{R})$ we have $\tau = free(t)$ then we speak of a τ-**polynomial**, which is an invariant of τ-structures.

- If for $t \in \mathbb{T}(\mathcal{R})$, τ just contains one binary relation symbol, t is a **graph polynomial**.

We denote this interpretation by $\mathfrak{P}(t, I_A)$.

Equivalence of terms of $\mathbb{T}(\mathcal{R})$

Let $t_1, t_2 \in \mathbb{T}(\mathcal{R})$ be two terms.

- t_1 and t_2 are **equivalent over** $\mathcal{R}$ if for every A-interpretation I_A we have $\mathfrak{P}(t_1, I_A) = \mathfrak{P}(t_2, I_A)$.

  We write $t_1 \sim_{\mathcal{R}} t_2$.

- ϕ and ψ are logically equivalent iff $tv(\phi)$ and $tv(\psi)$ are equivalent over $\mathcal{R}$ for every ring $\mathcal{R}$.

- t_1 **induces** t_2 **over** $\mathcal{R}$ if for any two interpretations I_A and I_B such that $\mathfrak{P}(t_1, I_A) = \mathfrak{P}(t_1, I_B)$ we also have $\mathfrak{P}(t_2, I_A) = \mathfrak{P}(t_2, I_B)$.

  We write $t_1 \models_{\mathcal{R}} t_2$.

Undecidability of equivalence

We have seen in the second lecture:

Theorem:

- Equivalence between terms $t_1 \sim_{\mathcal{R}} t_2$ is undecidable over every ring $\mathcal{R}$.
- The induction relation $t_1 \models_{\mathcal{R}} t_2$ is undecidable over every ring $\mathcal{R}$.
- The same holds even when terms are restricted to those with one free binary relation variable.

$\mathbf{SOL}$-monomials

We define inductively the **$\mathbf{SOL}$-monomials**.

- Ring elements a and indeterminates $X_{\bar{V}}$ are $\mathbf{SOL}$-monomials.
- For every $\mathbf{SOL}$-formula ϕ the term $tv(\phi)$ is an $\mathbf{SOL}$-monomial.
- The product of two $\mathbf{SOL}$-monomials t_1 and t_2 is an $\mathbf{SOL}$-monomial.
- If t_1 is a $\mathbf{SOL}$-monomial, ϕ is an $\mathbf{SOL}$-formula and $\bar{v}$ is a sequence of individual variables then

$$\prod_{\bar{v}:\phi} t$$

  is an $\mathbf{SOL}$-monomial.

Summation Normal Form (**SNF**)

We define inductively the the terms in Summation Normal Form **SNF**.

- $\mathbf{SOL}$-monomials are in **SNF**.
- If y_1 and t_2 are in **SNF**, so is $t_1 + t_2$.
- If t_1 is in **SNF**, ϕ is an $\mathbf{SOL}$-formula and $\bar{v}$ is a sequence of individual variables, then so is
$$\sum_{\bar{v}:\phi} t$$
- If t_1 is in **SNF**, ψ is an $\mathbf{SOL}$-formula and $\bar{V}$ is a sequence of individual or relation variables, then so is
$$\sum_{\bar{V}:\psi} t$$

SNF-Theorem for terms in $\mathbb{T}(\mathcal{R})$

Theorem:

For every term t in $\mathbb{T}(\mathcal{R})$ there is a term t_1 in $\mathbb{T}(\mathcal{R})$ in **SNF** such that

- $t \sim_{\mathcal{R}} t_1$ for every ring $\mathcal{R}$.
- $free(t) = free(t_1)$

Proof of the **SNF**-Theorem, I

Recall that J^I denotes the set of function $f : I \rightarrow J$.

We first observe the following:

Lemma 1: Let $t_{i,j}$ be terms with $i \in I$ and $j \in J$. Then

$$\prod_{i \in I} \sum_{j \in J} t_{i,j} = \sum_{F \in J^I} \prod_{i \in I} t_{i,F(i)}$$

Q.E.D.

We want to analize this for $J = P(I)$.

Lemma 2: There is a one-one correspondence between $P(I)^I$ and $P(I \times I)$.

Proof:

If $R \subseteq I \times I$ we define a function $F_R : I \rightarrow P(I)$ by

$$F_R(i) = \{j \in I : (i,j) \in R\}$$

In the other direction, if $F : I \rightarrow P(I)$, we define $R_F \subseteq I \times I$ by

$$R_F = \{(i,j) \in I^2 : i \in I, j \in F(i)\}$$

Q.E.D.

Proof of the **SNF**-Theorem, II

It is enough to prove it for products of terms in **SNF**.

We spell out the details for $\bar{v} = (v)$ and $\bar{U} = (U_{1,1}) = (U)$.

Lemma 3: Let

$$t_v = \sum_{U:\psi(v,U)} s(v,U).$$

and $W_{2,1} = W \notin free(t_v)$. Then

$$\prod_{v:\phi(v)} t_v = \prod_{v:\phi(v)} \left(\sum_{U:\psi(v,U)} s(v,U) \right) = \sum_{W:} \left(\sum_{U:\forall u(U(u) \leftrightarrow \exists w W(u,w))} \left(\prod_{v:\phi(v) \wedge \psi(v,U)} s(v,U) \right) \right)$$

Proof: Use Lemma 2 with $J = P(I)$ and the fact that the innermost sum ist just there to change the arity of the variable of summation.

Q.E.D.

Proof of the **SNF**-Theorem, III

The general case looks as follows.

Lemma 4: Let

$$\bar{U} = (U_{r_1,1}, \ldots, U_{r_m,m}), \qquad \bar{v} = (v_1, \ldots, v_k), \qquad \bar{W} = (W_{r_1+k,1}, \ldots, W_{r_m+k,m})$$

and

$$t_{\bar{v}} = \sum_{\bar{U}:\psi(\bar{v},\bar{U})} s(\bar{v}, \bar{U}).$$

Then

$$\prod_{\bar{v}:\phi(\bar{v})} t(\bar{v}) = \prod_{\bar{v}:\phi(\bar{v})} \left(\sum_{\bar{U}:\psi(\bar{v},\bar{U})} s(\bar{v}, \bar{U}) \right) =$$

$$\sum_{\bar{W}:} \left(\sum_{\bar{U}:\bigwedge_{i=1}^{m} (\forall \bar{u}(U_{r_i,i}(\bar{u}) \leftrightarrow \exists \bar{w} W_{r_i+k,i}(\bar{u},\bar{w})))} \left(\prod_{\bar{v}:\phi(\bar{v}) \wedge \psi(\bar{v},\bar{U})} s(\bar{v}, \bar{U}) \right) \right)$$

Q.E.D.

Coding finite structures in the ring, I: ring assignments.

Let $\mathfrak{z} : \mathbb{X} \to \mathcal{R}$ be a function which gives each indeterminate in $\mathbb{X}$ a value in the ring $\mathcal{R}$.

If $\mathfrak{z}$ additionally satisfies

- For all but finitely many $X_{v_i} \in \mathbb{X}$ we have $\mathfrak{z}(X) = 0$.
- For all $i \in \mathbb{N}$, if $\mathfrak{z}(X_{v_i}) = 0$ then also $\mathfrak{z}(X_{v_{i+1}}) = 0$.
- The set $\{(i,j) \in \mathbb{N}^2 : \mathfrak{z}(X_{v_i,v_j}) \neq 0\}$ is the graph of a unary function with domain $\{v_i : i \in \mathbb{N}\}$.
- For all individual variables $\bar{v} = (v_1, v_2, \ldots, v_r)$ and all $U_{r,j}$ we have that if $\mathfrak{z}(X_{\bar{v}}, U_{r,j}) \neq 0$ then for all $1 \leq i \leq r$, $\mathfrak{z}(X_{v_i}) \neq 0$.

then $\mathfrak{z}$ is called a **ring assignment** wra.

If $\mathfrak{z} : \mathbb{X} \to \{0,1\} \subset \mathcal{R}$ then it is called **discrete ring assignment** dra.

Coding finite structures in the ring, II: The structure $\mathfrak{A}(\mathfrak{z})$

We define an A-interpretation $I_A(\mathfrak{z}) : \mathbf{VAR} \to A \cup \bigcup_r P(A^r)$ as follows:

- The universe is the set

$$A(\mathfrak{z}) = A = \{i \in \mathbb{N} : \mathfrak{z}(X_{v_i}) \neq 0\}$$

- Each individual variable v_i is interpreted by

$$I_A(\mathfrak{z})(v_i) = j \text{ iff } \mathfrak{z}(X_{v_i,v_j}) \neq 0$$

- Each relation variable $U_{r,j}$ is interpreted by

$$I_A(U_{r,j}) = \{(i_1, i_2, \ldots, i_r) \in \mathbb{N}^r : \mathfrak{z}(X_{v_{i_1},v_{i_2},\ldots,v_{i_r},U_{r,j}}) \neq 0\}$$

SOL-polynomials and BSS-programs.

Given a ring assignment $\mathfrak{z}$ and its corresponding $I_A(\mathfrak{z})$-interpretation, and an **SOL**-polynomial $t \in \mathbb{T}(\mathcal{R})$,

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Complexity classes for **SOL**-polynomials.

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