Characteristic polynomial and Matching Polynomial

Lecturer: Ilia Averbouch

e-mail: ailia@cs.technion.ac.il
Outline of Lectures 3-4

- Characteristic polynomial: definition and interpretation of the coefficients
- Acyclic polynomials vs. generating matching polynomials
- Relationship between acyclic and characteristic polynomials
- Roots of the characteristic and acyclic polynomials
Definition 1  *Characteristic polynomial of a graph*

Let $G(V, E)$ be a simple undirected graph with $|V| = n$, and let $A_G$ be the (symmetric) adjacency matrix of $G$ with
\[
(A_G)_{j,i} = (A_G)_{i,j} = 1 \text{ if } (v_i, v_j) \in E \text{ and } (A_G)_{j,i} = (A_G)_{i,j} = 0 \text{ otherwise}
\]

- The **characteristic polynomial** of $G$ is defined as
  \[
P(G, \lambda) = \det(\lambda \cdot 1 - A_G)
  \]
- The roots of $P(G, \lambda)$ are the eigenvalues of $A_G$. We will call them also the eigenvalues of $G$. 
Identities and features

**Proposition 1**
*The characteristic polynomial is multiplicative:*

Let $G \sqcup H$ denote the disjoint union of graphs $G$ and $H$. Then:

$$P(G \sqcup H, \lambda) = P(G, \lambda) \cdot P(H, \lambda)$$

Proof:

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A) \det(B)$$

for any square matrices $A$ and $B$, not necessarily of the same order. The claim follows at once from this.
Coefficients of the characteristic polynomial

Let us suppose that the characteristic polynomial of graph $G$ is:

$$P(G, \lambda) = \sum_{i=0}^{n} c_i(G)\lambda^{n-i}$$

We have seen on the 1-st lecture:

(i) $c_0 = 1$

(ii) $c_1 = 0$

(iii) $-c_2 = |E(G)|$ is the number of edges of $G$.

(iv) $-c_3$ is twice the number of triangles of $G$.

We will find general interpretation of the coefficients of $P(G, \lambda)$
Eigenvalues of graph $G$

The following features of the eigenvalues can be derived from the matrix theory:

(i) Since $A_G$ is a symmetric matrix, all the eigenvalues of $G$ are real

(ii) Since $A_G$ is non-negative matrix, its largest eigenvalue is non-negative and it has the largest absolute value. (corollary of Frobenius' theorem) (Gantmacher F.R. Theory of Matrices I,II (2 vol.) Chelsea, New York 1960 vol.2 p.66)

(iii) Since $A_G$ is non-negative matrix, the largest eigenvalue of every principal minor of $A_G$ doesn’t exceed the largest eigenvalue of $A_G$ (Gantmacher F.R. Theory of Matrices I,II (2 vol.) Chelsea, New York 1960 vol.2 p.69)

We will also use those theorems when analyzing the matching polynomial roots.
Definition 2 Acyclic (matching defect) polynomial of a graph

Let $G(V,E)$ be a simple graph (no multiple edges) with $|V| = n$

We denote by $m_k(G)$ the number of $k$-matchings of a graph $G$, with $m_0(G) = 1$ by convention.

We are concerned with properties of the sequence $\{m_0, m_1, m_2, \ldots\}$

- The matching defect polynomial (or acyclic polynomial)

\[
m(G, \lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k}
\]
**Definition 3** *Matching generating polynomial of a graph*

Another (maybe more natural) polynomial to study is **matching generating polynomial**

\[ g(G, \lambda) = \sum_{k} m_k(G)\lambda^k \]

- For every \( k > \left\lfloor \frac{n}{2} \right\rfloor \) number of matchings \( m_k(G) = 0 \)
- Relationship between two the forms:

\[ m(G, \lambda) = \sum_{k} \frac{n}{2} (-1)^k m_k(G)\lambda^{n-2k} = \lambda^n \sum_{k} \frac{n}{2} (-1)^k m_k(G)\lambda^{-2k} = \]

\[ = \lambda^n \sum_{k} m_k(G)((-1) \cdot \lambda^{-2})^k = \lambda^n \sum_{k} m_k(G)(-\lambda^{-2})^k = \lambda^n g(G, (-\lambda^{-2})) \]
Coefficients of the acyclic polynomial

Let us suppose that the acyclic polynomial of graph $G$ is:

$$m(G, \lambda) = \sum_{i=0}^{n} a_i(G) \lambda^{n-i}$$

According to the definition we see:

(i) $a_0 = 1$

(ii) $a_i = 0$ for every odd $i$

(iii) For every $i$, $a_{2i} = (-1)^i m_i(G')$

(iv) In particular, $(-1)^{\frac{n}{2}} a_n$ is a number of perfect matchings of $G'$
Relationship between *acyclic* and *characteristic* polynomials

We want to explore

- Does characteristic polynomial induce acyclic polynomial (NO)
- Does acyclic polynomial induce characteristic polynomial (NO)
- When nevertheless there is a connection and what is that connection?
- How can we use it?
Counter-example 1

The graphs $G_1$ and $G_2$ have the same characteristic polynomial but different acyclic polynomials.

\[ P(G_1, \lambda) = P(G_2, \lambda) = \lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1 \]

On the other hand, we can see that $m_2(G_1) = 9$ but $m_2(G_2) = 7$

Conclusion: Characteristic polynomial doesn’t induce acyclic polynomial.
Counter-example 2

The graphs $G_3$ and $G_4$ have the same acyclic polynomial but different characteristic polynomials.

\[ G_3 = C_3 \sqcup P_2 \quad \text{and} \quad G_4 = P_5 \]

\[ m(G_1, \lambda) = m(G_2, \lambda) = \lambda^5 - 4\lambda^3 + 3\lambda \]

On the other hand, we can see that $G_1$ has a triangle, and $G_2$ has not.

Thus, they definitely have different characteristic polynomials.

Conclusion: Acyclic polynomial doesn’t induce characteristic polynomial.
Example 4 $G = P_2$

Adjacency matrix:

$$A_{P_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(P_2, \lambda) = \det(\lambda \cdot 1 - A_{P_2}) =$$

$$= \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} =$$

$$= \lambda^2 - 1$$
$G = P_2$

Acyclic polynomial:

\[ m_0(P_2) = 1 \]
\[ m_1(P_2) = 1 \]

\[ m(P_2, \lambda) = \sum_{k=0}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^2 - 1 = P(P_2, \lambda) \]

The acyclic polynomial of $P_2$ is equal to its characteristic polynomial, in contrast for its matching generating polynomial, which is

\[ g(P_2, \lambda) = 1 + \lambda \]
Example 5 $G = P_3$

Adjacency matrix:

$$A_{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(P_3, \lambda) = \det(\lambda \cdot 1 - A_{P_3}) =$$

$$= \det \begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} =$$

$$= \lambda^3 - 2\lambda$$
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\[ G = \text{P}_3 \]

Acyclic polynomial:

\[ m_0(P_3) = 1 \]

\[ m_1(P_3) = 2 \]

\[ m(P_3, \lambda) = \sum_{k}^{n/2} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^3 - 2\lambda = P(P_3, \lambda) \]
Example 6 $G = C_3$

Adjacency matrix:

$$A_{C_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(C_3, \lambda) = \det(\lambda \cdot 1 - A_{C_3}) =$$

$$= \det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} =$$

$$= \lambda^3 - 3\lambda - 2$$
$G = C_3$

Acyclic polynomial:

$m_0(C_3) = 1$

$m_1(C_3) = 3$

$$m(C_3, \lambda) = \sum_{k} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^3 - 3\lambda$$

$$P(C_3, \lambda) = \lambda^3 - 3\lambda - 2 \neq m(C_3, \lambda)$$

Note that 2 is twice the number of triangles in $G$. 
Relationship between \textit{acyclic} and \textit{characteristic} polynomials - continued

Let us generalize:

\begin{itemize}
  \item Can we interpret the coefficients of characteristic polynomial?
  \item Can we interpret the coefficients of acyclic polynomial?
  \item Which recurrence relations do they satisfy?
  \item Theorem (I. Gutman, C. Godsil 1981)
\end{itemize}
Definitions

- An **elementary** graph is a simple graph, each component of which is regular and has degree 1 or 2. In other words, it is disjoint union of single edges ($K_2$) or cycles ($C_k$).

- A **spanning elementary subgraph** of $G$ is an elementary subgraph which contains all the vertices of $G$.

- We will denote spanning elementary subgraph of $G$ as $\gamma$
  - $\text{comp}(\gamma)$ is the number of connected components in $\gamma$
  - $\text{cyc}(\gamma)$ is the number of cycles in $\gamma$

- Note that cycle free spanning elementary subgraph of $G$ is actually a perfect matching of $G$. 
Example: Spanning elementary subgraphs

\[ G \]

\[ 1-2-3-4 \]
\[ 5-8 \]
\[ 6-7-9 \]

\[ G \]

\[ 1-2 \]
\[ 3-5 \]
\[ 4-6-7 \]
\[ 8-9 \]

\[ G \]

\[ 1-2 \]
\[ 3-4 \]
\[ 5-8 \]
\[ 6-7-9 \]
Lemma 1 (Harary, 1962)

Let $A$ be the adjacency matrix of some graph $G(V, E)$ with $|V| = n$. Then

$$\det(A) = (-1)^n \sum_{\gamma} (-1)^{\text{comp}(\gamma)} 2^{\text{cyc}(\gamma)}$$

where summation is over all the spanning elementary subgraphs $\gamma$ of $G$. 

Lemma 1: proof

Let us look at the det(A) and interpret its components. Use the definition of a determinant:

if $A_{n \times n} = (a_{ij})$, then

$$\det(A) = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

where summation is over all permutations $\pi$ of the set $\{1, 2, ..., n\}$

Consider the term

$$\prod_{i=1}^{n} a_{i,\pi(i)}$$

Its value is 0 or 1. This term vanishes if for any $i \in \{1, 2, ..., n\}$, $a_{i,\pi(i)} = 0$; that is, if $(v_i, v_{\pi(i)})$ is not an edge of $G$.

Each non-vanishing term corresponds to a disjoint union of directed cycles.
Lemma 1: proof - continued

Therefore, every such term corresponds to a composition of disjoint cycles of length at least 2, which is actually a spanning elementary subgraph \( \gamma \) of the graph \( G \).

Let \( \Gamma : \pi \rightarrow \gamma \) define uniquely, which \( \gamma \) corresponds to certain \( \pi \).

Let \( \Gamma^{-1}(\gamma) = \{\pi : \Gamma(\pi) = \gamma\} \) define the set of \( \pi \) that correspond to certain \( \gamma \).

If \( \Gamma(\pi) = \Gamma(\pi') \) then \( \pi \) and \( \pi' \) are different only by the direction of their cycles (of length greater than 2).

Hence, \( |\Gamma^{-1}(\gamma)| = 2^{cyc(\gamma)} \)
Lemma 1: proof - continued

We can now split the non-vanishing permutations according to the $\gamma$ they correspond.

$$\det(A) = \sum_{\pi} \text{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)} = \sum_{\gamma} \sum_{\pi \in \Gamma^{-1}(\gamma)} \text{sgn}(\pi) \cdot 1$$

The sign of a permutation $\pi$ is defined as $(-1)^{N_e}$, where $N_e$ is the number of even cycles in $\pi$.
If $\Gamma(\pi) = \Gamma(\pi')$ then $\text{sgn}(\pi) = \text{sgn}(\pi')$, we’ll denote it as $\text{sgn}(\gamma)$

Now we can write:

$$\det(A) = \sum_{\gamma} \text{sgn}(\gamma) \sum_{\pi \in \Gamma^{-1}(\gamma)} 1 = \sum_{\gamma} \text{sgn}(\gamma) 2^{\text{cyc}(\gamma)}$$
Lemma 1: proof - end

The sign of spanning elementary subgraph $\gamma$ is $(-1)^{N_e}$, where $N_e$ is the number of even cycles in $\gamma$.

The number of odd cycles in $\gamma$ is congruent to $n$ modulo 2: $n \equiv N_o (mod 2)$

Having $\text{comp}(\gamma) = N_e + N_o$ we obtain:

$$\text{sgn}(\gamma) = (-1)^N = (-1)^{n+N_o+N_e} = (-1)^{n+\text{comp}(\gamma)}$$

From here, every $\gamma$ contributes $(-1)^{n+\text{comp}(\gamma)} 2^{\text{cyc}(\gamma)}$ to the determinant, and finally

$$\det(A) = (-1)^n \sum_{\gamma} (-1)^{\text{comp}(\gamma)} 2^{\text{cyc}(\gamma)}$$

Q.E.D.
Lemma 2

Let $A$ be the adjacency matrix of graph $G$: $A_{n \times n} = (a_{ij})$ and

$$P(G, \lambda) = \det(\lambda \cdot 1 - A) = \sum_{i=0}^{n} c_i \lambda^{n-i}$$ - its characteristic polynomial.

Then

$$(-1)^i c_i = \sum M_{Di}$$

where $M_{Di}$ are the principal minors of $A$ with order $i$ (Minors, whose diagonal elements belong to the main diagonal of $A$)
Proof:

\[
(\lambda \cdot 1 - A) = \begin{pmatrix}
\lambda & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & \lambda & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \lambda
\end{pmatrix}
\]

Let’s analyze the permutations contributing to \(c_i\):
They have exactly \(n - i\) members of the main diagonal \(a_{kk} = \lambda\)
The permutations in the rest rows and columns (which don’t include the main diagonal) will give exactly the determinant of some principal minor of \(A\).

The sign \((-1)^i\) compensates the fact that all the values in \((\lambda \cdot 1 - A)\) are \(-a_{ij}\)

Hence, \((-1)^i c_i = \sum M_{Di}\)

Q.E.D.
General interpretation of the coefficients of $P(G, \lambda)$

Let $G$ be a graph with adjacency matrix $A_G$, and

$$P(G, \lambda) = \det(\lambda \cdot 1 - A_G) = \sum_{i=0}^{n} c_i \lambda^{n-i}$$

be a characteristic polynomial of graph $G$. Then $c_i$ are given by:

$$c_i = \sum_{\gamma_i} (-1)^{\text{comp}(\gamma_i)} 2^{\text{eye}(\gamma_i)}$$

where the summation is over the elementary subgraphs of $G$ with $i$ vertices. Corollary: we can derive now the identities for $c_0$, $c_1$, $c_2$, $c_3$
Coefficients of $P(G, \lambda)$ - continued

Proof:
According to Lemma 2 we have: $(-1)^i c_i = \sum M_{D_i}$ is the sum of all the principal minors of $A_G$ with order $i$;

Each such minor is the determinant of adjacency matrix $A_{H_i}$ of some graph $H_i$ which is an induced subgraph of $G$ with $i$ vertices;

Let $\gamma_{H_i}$ denote a spanning elementary subgraph of $H_i$

Then, by Lemma 1,
\[
(-1)^i c_i = \sum M_{D_i} = \sum_{H_i} \sum_{\gamma_{H_i}} (-1)^{\text{comp}(\gamma_{H_i})} 2^{\text{cyc}(\gamma_{H_i})}
\]

Every elementary subgraph with $i$ vertices $\gamma_i$ of $G$ is contained in exactly one $H_i$. Thus, summarizing over all the $\gamma_i$ we obtain:

\[
c_i = \sum_{\gamma_i} (-1)^{\text{comp}(\gamma_i)} 2^{\text{cyc}(\gamma_i)}
\]

Q.E.D.
Theorem 1 - (C.Godsil, I.Gutman, 1981)

Let $G$ be a simple graph with $n$ vertices and adjacency matrix $A$,

$$m(G, \lambda) = \sum_{i=0}^{\frac{n}{2}} (-1)^i m_i(G) \lambda^{n-2i}$$

be its acyclic polynomial,

$$P(G, \lambda) = \det(\lambda \cdot 1 - A) = \sum_{i=0}^{n} c_i \lambda^{n-i}$$

be its characteristic polynomial.

Let $C$ denote an elementary subgraph of $G$, which contains only cycles;

Let $\text{comp}(C)$ denote the number of components in $C$;

Let $G - C$ denote the induced subgraph of $G$ obtained from $G$ by removing all the vertices of $C$.

Then the following holds:

$$P(G, \lambda) = m(G, \lambda) + \sum_{C} (-2)^{\text{comp}(C)} m(G - C, \lambda)$$

where the summation is over all non-empty $C$. 
Theorem 1 - continued

In the case of a forest we have:

\[ P(F, \lambda) = m(F, \lambda) \]

Moreover, the coefficients satisfy the following identities:

(i) Even coefficients:

\[ c_{2i} = m_i \]

(ii) Odd coefficients:

\[ c_{2i+1} = 0 \]
Theorem 1 - continued

Proof:

Let us look on the coefficients of \( P(G, \lambda) \):

\[
P(G, \lambda) = \sum_{i=0}^{n} c_i \lambda^{n-i} = \sum_{i=0}^{n} \sum_{\gamma_i} (-1)^{\text{comp}(\gamma_i)} 2^\text{cyc}(\gamma_i) \lambda^{n-i}
\]

Let’s split the internal sum by the \( \gamma \)'s having the same set of cycles (including, in particular, empty set).
Let \( C \) denote such a common set of cycles.
Let \( \delta = \gamma_i - C \) denote the rest of \( \gamma_i \), which is a set of disjoint edges.

Then \( \text{cyc}(\gamma_i) = \text{comp}(C) \) and \( \text{comp}(\gamma_i) = \text{comp}(\delta) + \text{comp}(C) \)

Thus we can write:

\[
P(G, \lambda) = \sum_{i=0}^{n} \sum_{C} \sum_{\delta} (-1)^{\text{comp}(\delta)+\text{comp}(C)} 2^{\text{comp}(C)} \lambda^{n-i}
\]
Theorem 1 - continued

Let $|C|$ denote the number of vertices in $C$.

Then we can express $i$ via $|C|$ and $\text{comp}(\delta)$: $i = 2\text{comp}(\delta) + |C|$

Since $C$ is independent of $i$ and $\delta$, we can write now:

$$P(G, \lambda) = \sum_{C} (-2)^{\text{comp}(C)} \sum_{\text{comp}(\delta)=0}^{n-|C|} (-1)^{\text{comp}(\delta)} \lambda^{n-|C|-2\text{comp}(\delta)} =$$

$$= \sum_{C} (-2)^{\text{comp}(C)} \sum_{j=0}^{n-|C|} m_j(G - C) \lambda^{n-|C|-j} = \sum_{C} (-2)^{\text{comp}(C)} m(G - C, \lambda)$$

Now we should distinguish between the case, when $C = \emptyset$, and the rest of the cases.

$$P(G, \lambda) = m(G, \lambda) + \sum_{C \neq \emptyset} (-2)^{\text{comp}(C)} m(G - C, \lambda)$$

Q.E.D.
Corollary 1.1 (C.Godsil, I.Gutman, 1981)

The acyclic polynomial of a graph coincides with the characteristic polynomial if and only if the graph is a forest.

\[ m(G, \lambda) = P(G, \lambda) \iff \text{Forest}(G) \]

Proof:

"⇒" follows trivially from the theorem 1.

"⇐":
Suppose \( G \) is not a forest, and proof that \( m(G, \lambda) \neq P(G, \lambda) \).
Let \( q \) be the smallest cycle in \( G \) and \( |q| \) is its length.
Without loss of generality we can state that there are exactly \( k \geq 1 \) cycles of such a length, denoted as \( \{q_1, \ldots, q_k\} \) in the graph \( G \).

Let \( a_i \) and \( c_i \) be the the coefficients of \( \lambda^{n-i} \) in respectively acyclic and characteristic polynomials.
Corollary 1.1 - continued

We shall prove that the second part of the equation in Theorem 1

\[ \sum_{C \neq \emptyset} (-2)^{\text{comp}(C)} m(G - C, \lambda) \]

makes the difference between the coefficients \( a_{|q|} \) and \( c_{|q|} \).

First, only the summation over \( C \in \{q_1, \ldots, q_k\} \) contribute to the coefficient of \( \lambda^{n-|q|} \), because all the other cycles or combinations of cycles are bigger, and then the degree of \( m(G - C, \lambda) \) will be less than \( \lambda^{n-|q|} \).

Second, every single cycle contributes exactly \((-2)^{\text{comp}(C)}\), because the graph \( G - C \) has exactly one 0-matching.

Thus, \( a_i - c_i = 2k > 0 \), hence the proposition "\( \Rightarrow \)" holds.

Q.E.D.
Corollary 1.2

We can state now:
For every forest $F$, all the roots of its acyclic polynomial are real. They are equal to the eigenvalues of $F$. 
Identities and Recurrences

**Proposition 2**

The acyclic polynomial is multiplicative:

Let $G \sqcup H$ denote the disjoint union of graphs $G$ and $H$. Then:

$$m(G \sqcup H, \lambda) = m(G, \lambda) \cdot m(H, \lambda)$$
Identities and Recurrences - continued

Proof:
Each $k$-matching of $G \sqcup H$ consists of $l$-matching of $G$ and $(k - l)$-matching of $H$.

$$m_k(G \sqcup H) = \sum_{l=0}^{k} m_l(G)m_{k-l}(H)$$

The coefficient of $\lambda^{n-2k}$ in $m(G, \lambda) \cdot m(H, \lambda)$ is equal to

$$\sum_{l=0}^{k} (-1)^l m_l(G)(-1)^{k-l}m_{k-l}(H) =$$
$$= (-1)^k \sum_{l=0}^{k} m_l(G)m_{k-l}(H) = (-1)^k m_k(G \sqcup H)$$

which is equal to the corresponding coefficient of $m(G \sqcup H, \lambda)$

Q.E.D.
Identities and Recurrences - continued

**Proposition 3**
*Edge recurrence:*

Let $G - e$ denote the graph obtained by removing edge $e = (u, v) \in E$ from the graph $G(V,E)$.

Let $G - u - v$ denote the induced subgraph of $G(V,E)$ obtained from $G$ by removing two vertices $u, v \in V$.

Then:

$$m(G, \lambda) = m(G - e, \lambda) - m(G - u - v, \lambda)$$
Proof:
All the $k$-matchings of $G$ are of 2 disjoint kinds: those that use the edge $e$ and those that do not. Every matching that uses the edge $e$ determines uniquely a $(k-1)$-matching in $G - u - v$. Every matching that don't use $e$ is actually a matching in $G - e$.
Therefore:

$$m_k(G) = m_k(G - e) + m_{k-1}(G - u - v)$$

Hence

$$m(G, \lambda) = \sum_{k \geq 0} (-1)^k m_k(G - e) \lambda^{n-2k} + \sum_{k \geq 1} (-1)^k m_{k-1}(G - u - v) \lambda^{n-2k} =$$

$$= \sum_{k \geq 0} (-1)^k m_k(G - e) \lambda^{n-2k} + (-1) \sum_{k-1 \geq 0} (-1)^{(k-1)} m_{k-1}(G - u - v) \lambda^{n-2-2(k-1)} =$$

$$= m(G - e, \lambda) - m(G - u - v, \lambda)$$

Q.E.D.
Identities and Recurrences - continued

Proposition 4

Vertex recurrence:

Let $u \in V$ be a vertex of degree $d$.

Let $G - u$ denote the induced subgraph of $G(V, E)$ obtained from $G$ by removing vertex $u$.

Let $v_i \in V, 1 \leq i \leq d$ denote all the vertices such that $(u, v_i) \in E$ and

Let $G - u - v_i$ denote the induced subgraph of $G(V, E)$ obtained from $G$ by removing two vertices $u, v_i$.

Then:

$$m(G, \lambda) = \lambda \cdot m(G - u, \lambda) - \sum_{i=1}^{d} m(G - u - v_i, \lambda)$$
Proof:
All the $k$-matchings of $G$ are of 2 disjoint kinds: those that use the vertex $u$ and those that do not. The number of $k$-matchings that do not use the vertex $u$ is equal to $m_k(G - u, \lambda)$. The number which do use $u$ is equal to $m_{k-1}(G - u - v_i)$, summed over the vertices $v_i$ adjacent to $u$. Thus,

$$m_k(G') = m_k(G - u) + \sum_{i=1}^{d} m_{k-1}(G - u - v_i)$$

Hence,

$$m(G, \lambda) = \sum_{k \geq 0} (-1)^k m_k(G - u) \lambda^{n-2k} + \sum_{k \geq 1} (-1)^k \sum_{i=1}^{d} m_{k-1}(G - u - v_i) \lambda^{n-2k} =$$
Identities and Recurrences - continued

Having $G - u$ is a graph of $n - 1$ vertices, and $i$ is independent of $k$, we can write

$$m(G, \lambda) = \lambda \cdot \sum_{k \geq 0} (-1)^k m_k(G - u) \lambda^{(n-1)-2k} +$$

$$+ (-1)^d \sum_{i=1}^{d} \sum_{k-1 \geq 0} (-1)^{(k-1)} m_{k-1}(G - u - v_i) \lambda^{(n-2)-2(k-1)} =$$

$$\lambda \cdot m(G - u, \lambda) - \sum_{i=1}^{d} m(G - u - v_i, \lambda)$$

Q.E.D.
Theorem 2

Let $G$ be a connected graph, $v \in V(G)$ be a vertex of degree $d$, and $H_1$ its induced subgraph without the vertex $v$.

Let $w_i(i = 1, ..., d)$ be the vertices adjacent to $v$.

Let $H_i(i = 2, ..., d)$ be graphs which are all isomorphic to $H_1$.

Let $w_i(H_i)$ denote the vertex of $H_i$ corresponding to the vertex $w_i$ in $H_1$.

Let $F_1 = G \sqcup H_2 \sqcup ... \sqcup H_d$

Let $F_2$ be obtained from $F_1$ by replacing the edges $e_i = \{v, w_i\}$ by $e'_i = \{v, w_i(H_i)\}$

Then $m(F_1) = m(F_2)$
Theorem 2 - continued
Theorem 2 - continued

Proof:

For $m(F_1)$, we will apply vertex recurrence on $G$ and $v$,

$$m(F_1) = m(G)m(H_2)...m(H_d) = m(H_2)...m(H_d)[\lambda m(H_1) - \sum_{i=1}^{d} m(H_1 - w_i)]$$

For $m(F_2)$, we will apply vertex recurrence on $F_2$ and $v$,

$$m(F_2) = \lambda m(H_1)...m(H_d) - m(H_1)...m(H_d) \sum_{i=1}^{d} \frac{m(H_i - w_i(H_i))}{m(H_i)}$$

Having $H_1...H_d$ isomorphic we obtain:

$$m(F_1) = (m(H_1))^{d-1}[\lambda m(H_1) - d \cdot m(H_1 - w_i)] =$$

$$= \lambda (m(H_1))^{d} - d(m(H_1))^{d-1} \cdot m(H_1 - w_i) = m(F_2)$$

Q.E.D.
Theorem 2 - continued

Corollary 2.1
For every simple connected graph $G$ and vertex $v \in V(G)$ there is a tree $T(G, v)$ such that $m(G, \lambda)$ divides $m(T(G, v), \lambda)$ and maximum degree of $T$ is not more than maximum degree of $G$.
Proof: By multiple application of Theorem 2.

Corollary 2.2
For every simple graph $G$ there is a forest $F$ such that $m(G, \lambda)$ divides $m(F, \lambda)$, and maximum degree of $F$ is not more than maximum degree of $G$.
Proof: straightforward from proposition 2 and corollary 2.1.

Corollary 2.3
The zeros (roots) of $m(G, \lambda)$, are real.
Proof: straightforward from (2.2), (1.1) and the fact that the roots of the characteristic polynomial of a simple graph are all real.
Corollary 2.4: The roots of acyclic polynomial are symmetrically placed around zero. In other words,

\[ m(G, \lambda) = 0 \iff m(G, -\lambda) = 0 \]

Proof:
According to the definition,

\[ m(G, \lambda) = \sum_{k} \frac{n}{2} (-1)^{k} m_{k}(G) \lambda^{n-2k} \]

Hence, either all the degrees of \( \lambda \)'s are even or all the degrees of \( \lambda \)'s are odd.

In the first case, \( m(G, -\lambda) = m(G, \lambda) \)

In the second case, \( m(G, -\lambda) = -m(G, \lambda) \)

In both the cases,

\[ m(G, \lambda) = 0 \iff m(G, -\lambda) = 0 \]

Q.E.D.
Roots of the matching generating polynomial

**Corollary 2.5:** All the roots of generating matching polynomial are real and negative.

Proof:
The coefficient of \( \lambda^0 \) in \( g(G, \lambda) \) is always 1 (number of zero-matchings by convention). Thus, \( \lambda = 0 \) cannot be a root of \( g(G, \lambda) \)

On the other hand, we know that \( m(G, \lambda) = \lambda^ng(G, (-\lambda^{-2})) \)

Let \( t \) be a root of \( g(G, \lambda) \). We know that \( t \neq 0 \)

Let \( s = (-t)^{-\frac{1}{2}} \), and then \( t = -s^{-2} \)

Hence, \( m(G, s) = s^ng(G, -s^{-2}) = s^ng(G, t) = 0 \), so \( s \) is a root of \( m(G, \lambda) \)

But we know that all the roots of \( m(G, \lambda) \) are real.

Thus, \( t = -s^{-2} \) is real and negative.

Q.E.D.
Theorem 3 - (Heilman and Lieb, 1972)

(L.Lovasz and M.D.Plummer, Matching Theory - Theorem 8.5.8)

Let $G$ be a simple graph with degree $\Delta(G) > 1$ and let $t$ be any root of $m(G, \lambda)$.

Then

$$t \leq 2\sqrt{\Delta(G) - 1}$$
Theorem 3 - proof

Let’s prove it first for trees:

Let $T$ be a tree of maximum degree $\Delta$.
By theorem 1, the roots of acyclic polynomial are actually the eigenvalues of the tree.

On the other hand, the tree $T$ is an induced subgraph of a full $(\Delta - 1)$-ary tree $T'$.
The adjacency matrix of $T$ is a principal minor of the adjacency matrix of $T'$. But the largest eigenvalue of a principal minor doesn’t exceed the largest eigenvalue of the matrix.

The eigenvalues of a complete $d$-ary tree of depth $k$ are:
$\lambda = 2\sqrt{d}\cos(m\pi/(k + 1))$, $m = 1, ..., k$, hence the largest eigenvalue of $T$ is less than $2\sqrt{\Delta - 1}$ as claimed.

(L.Lovasz Combinatorial problems and Exercises (Exercise 11.5)
2-nd ed. Elsevier S.P., Amsterdam and Akademiai Kiado, Budapest 1993)
The general case now follows using Corollary 2.1:

Let $G$ be a graph, and let $H$ be any of its connected components with the maximum degree $\Delta$.

By the Corollary 2.1, there is a tree $T$ such that $m(H, \lambda) | m(T, \lambda)$, and the maximum degree of $T$ doesn’t exceed $\Delta$.

Since any root of $m(H, \lambda)$ is also a root of $m(T, \lambda)$, it follows that every root of $m(H, \lambda)$ doesn’t exceed $2\sqrt{\Delta} - 1$.

By Proposition 2, $m(G, \lambda) = \prod_H m(H, \lambda)$, so any $t$ root of $m(G, \lambda)$ is also a root of some $m(H, \lambda)$.

Hence the equation $t \leq 2\sqrt{\Delta} - 1$ holds for any graph.

Q.E.D.