Characteristic polynomial and Matching Polynomial

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Outline of Lectures 3-4

- Characteristic polynomial: definition and interpretation of the coefficients
- Acyclic polynomials vs. generating matching polynomials
- Relationship between acyclic and characteristic polynomials
- Roots of the characteristic and acyclic polynomials

Definition 1 *Characteristic polynomial of a graph*

Let $G(V, E)$ be a simple undirected graph with $|V| = n$, and
Let A_G be the (symmetric) adjacency matrix of G

with
$(A_G)_{j,i} = (A_G)_{i,j} = 1$ if $(v_i v_j) \in E$ and
$(A_G)_{j,i} = (A_G)_{i,j} = 0$ otherwise

- The **characteristic polynomial** of G is defined as

$$P(G, \lambda) = \det(\lambda \cdot 1 - A_G)$$

- The roots of $P(G, \lambda)$ are the eigenvalues of A_G. We will call them also the eigenvalues of G.

Identities and features

Proposition 1
The characteristic polynomial is multiplicative:

Let $G \sqcup H$ denote the disjoint union of graphs G and H. Then:

$$P(G \sqcup H, \lambda) = P(G, \lambda) \cdot P(H, \lambda)$$

Proof:

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A) \det(B)$$

for any square matrices A and B, not necessarily of the same order.
The claim follows at once from this.

Coefficients of the characteristic polynomial

Let us suppose that the characteristic polynomial of graph G is:

$$P(G,\lambda) = \sum_{i=0}^{n} c_i(G)\lambda^{n-i}$$

We have seen on the 1-st lecture:

(i) $c_0 = 1$

(ii) $c_1 = 0$

(iii) $-c_2 = \mid E(G) \mid$ is the number of edges of G.

(iv) $-c_3$ is twice the number of triangles of G.

We will find general interpretation of the coefficients of $P(G,\lambda)$

Eigenvalues of graph G

The following features of the eigenvalues can be derived from the matrix theory:

(i) Since A_G is a symmetric matrix, all the eigenvalues of G are real

(ii) Since A_G is non-negative matrix, its largest eigenvalue is non-negative and it has the largest absolute value. (corollary of Frobenius' theorem) (Gantmacher F.R. Theory of Matrices I,II (2 vol.) Chelsea, New York 1960 vol.2 p.66)

(iii) Since A_G is non-negative matrix, the largest eigenvalue of every principal minor of A_G doesn't exceed the largest eigenvalue of A_G (Gantmacher F.R. Theory of Matrices I,II (2 vol.) Chelsea, New York 1960 vol.2 p.69)

We will also use those theorems when analyzing the matching polynomial roots.

Definition 2 *Acyclic (matching defect) polynomial of a graph*

Let $G(V, E)$ be a simple graph (no multiple edges) with $|V| = n$

We denote by $m_k(G)$ the number of k-matchings of a graph G, with $m_0(G) = 1$ by convention.

We are concerned with properties of the sequence $\{m_0, m_1, m_2 ...\}$

- The **matching defect polynomial** (or **acyclic polynomial**)

$$m(G, \lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k}$$

Definition 3 *Matching generating polynomial of a graph*

Another (maybe more natural) polynomial to study is
matching generating polynomial

$$g(G,\lambda) = \sum_{k}^{n} m_k(G)\lambda^k$$

- For every $k > \lfloor \frac{n}{2} \rfloor$ number of matchings $m_k(G) = 0$
- Relationship between two the forms:

$$m(G,\lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G)\lambda^{n-2k} = \lambda^n \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G)\lambda^{-2k} =$$

$$= \lambda^n \sum_{k}^{\frac{n}{2}} m_k(G)((-1)\cdot\lambda^{-2})^k = \lambda^n \sum_{k}^{\frac{n}{2}} m_k(G)(-\lambda^{-2})^k = \lambda^n g(G,(-\lambda^{-2}))$$

Coefficients of the acyclic polynomial

Let us suppose that the acyclic polynomial of graph G is:

$$m(G,\lambda) = \sum_{i=0}^{n} a_i(G)\lambda^{n-i}$$

According to the definition we see:

(i) $a_0 = 1$

(ii) $a_i = 0$ for every odd i

(iii) For every i, $a_{2i} = (-1)^i m_i(G)$

(iv) In particular, $(-1)^{\frac{n}{2}} a_n$ is a number of perfect matchings of G

Relationship between **acyclic** and **characteristic** polynomials

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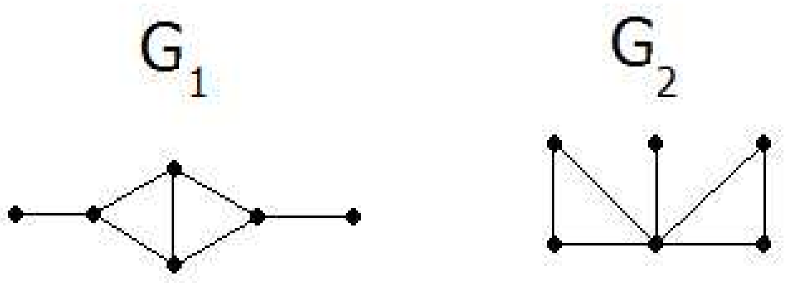
We want to explore

- Does characteristic polynomial induce acyclic polynomial (NO)
- Does acyclic polynomial induce characteristic polynomial (NO)
- When nevertheless there is a connection and what is that connection?
- How can we use it?

Counter-example 1

The graphs G_1 and G_2 have the same characteristic polynomial but different acyclic polynomials.



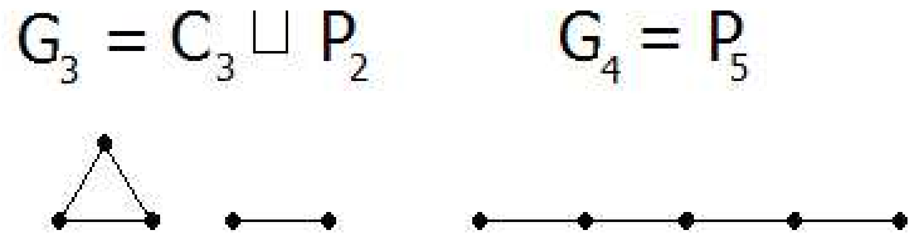
$$P(G_1, \lambda) = P(G_2, \lambda) = \lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$$

On the other hand, we can see that $m_2(G_1) = 9$ but $m_2(G_2) = 7$

Conclusion: Characteristic polynomial doesn't induce acyclic polynomial.

Counter-example 2

The graphs G_3 and G_4 have the same acyclic polynomial but different characteristic polynomials.



$m(G_1, \lambda) = m(G_2, \lambda) = \lambda^5 - 4\lambda^3 + 3\lambda$

On the other hand, we can see that G_1 has a triangle, and G_2 has not.

Thus, they definitely have different characteristic polynomials.

Conclusion: Acyclic polynomial doesn't induce characteristic polynomial.

Example 4 $G = P_2$

Adjacency matrix:

$$A_{P_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(P_2, \lambda) = \det(\lambda \cdot 1 - A_{P_2}) =$$

$$= \det \begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} =$$

$$= \lambda^2 - 1$$

$G = P_2$

Acyclic polynomial:

$$m_0(P_2) = 1$$

$$m_1(P_2) = 1$$

$$m(P_2, \lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^2 - 1 = P(P_2, \lambda)$$

The acyclic polynomial of P_2 is equal to its characteristic polynomial,

in contrast for its matching generating polynomial, which is

$$g(P_2, \lambda) = 1 + \lambda$$

Example 5 $G = P_3$

Adjacency matrix:

$$A_{P_3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(P_3, \lambda) = \det(\lambda \cdot 1 - A_{P_3}) =$$

$$= \det \begin{pmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} =$$

$$= \lambda^3 - 2\lambda$$

$$G = P_3$$

Acyclic polynomial:

$$m_0(P_3) = 1$$

$$m_1(P_3) = 2$$

$$m(P_3, \lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^3 - 2\lambda = P(P_3, \lambda)$$

Example 6 $G = C_3$

Adjacency matrix:

$$A_{C_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$P(C_3, \lambda) = \det(\lambda \cdot 1 - A_{C_3}) =$$

$$= \det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{pmatrix} =$$

$$= \lambda^3 - 3\lambda - 2$$

$G = C_3$

Acyclic polynomial:

$$m_0(C_3) = 1$$

$$m_1(C_3) = 3$$

$$m(C_3, \lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k} = \lambda^3 - 3\lambda$$

$$P(C_3, \lambda) = \lambda^3 - 3\lambda - 2 \neq m(C_3, \lambda)$$

Note that 2 is twice the number of triangles in G.

Relationship between **acyclic** and **characteristic** polynomials - continued

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Let us generalize:

- Can we interpret the coefficients of characteristic polynomial?
- Can we interpret the coefficients of acyclic polynomial?
- Which recurrence relations do they satisfy?
- Theorem (I.Gutman, C.Godsil 1981)

Definitions

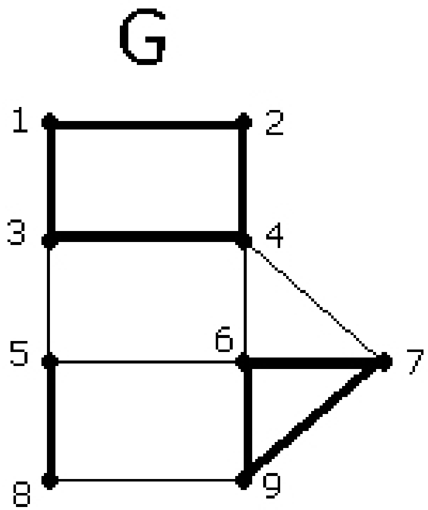
- An **elementary** graph is a simple graph, each component of which is regular and has degree 1 or 2.
  In other words, it is disjoint union of single edges (K_2) or cycles (C_k)

- A **spanning elementary subgraph** of G is an elementary subgraph which contains all the vertices of G.

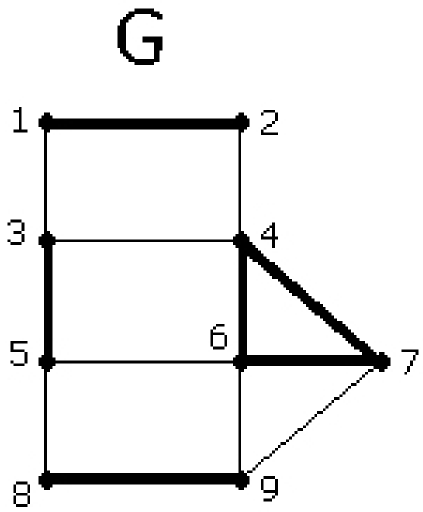
- We will denote spanning elementary subgraph of G as γ
  $comp(\gamma)$ is the number of connected components in γ
  $cyc(\gamma)$ is the number of cycles in γ

- Note that cycle free spanning elementary subgraph of G is actually a perfect matching of G

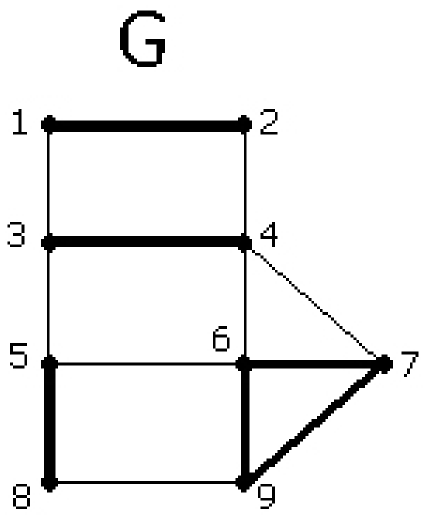
Example: Spanning elementary subgraphs



1-2-3-4
5-8
6-7-9



1-2
3-5
4-6-7
8-9



1-2
3-4
5-8
6-7-9

Lemma 1 (Harary,1962)

Let A be the adjacency matrix of some graph $G(V, E)$ with $|V| = n$.
Then

$$\text{det}(A) = (-1)^n \sum_{\gamma} (-1)^{comp(\gamma)} 2^{cyc(\gamma)}$$

where summation is over all the spanning elementary subgraphs γ of G

Lemma 1: proof

Let us look at the det(A) and interpret its components. Use the definition of a determinant:
if $A_{n\times n} = (a_{ij})$, then

$$\det(A) = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}$$

where summation is over all permutations π of the set $\{1, 2, ..., n\}$

Consider the term

$$\prod_{i=1}^{n} a_{i,\pi(i)}$$

Its value is 0 or 1. This term vanishes if for any $i \in \{1, 2, ...n\}$, $a_{i,\pi(i)} = 0$; that is, if $(v_i, v_{\pi(i)})$ is not an edge of G.

Each non-vanishing term corresponds to a **disjoint union of directed cycles**.

Lemma 1: proof - continued

Therefore, every such term corresponds to a
composition of disjoint cycles of length at least 2,
which is actually a **spanning elementary subgraph** γ of the graph G

Let $\Gamma : \pi \to \gamma$ define uniquely, which γ corresponds to certain π.

Let $\Gamma^{-1}(\gamma) = \{\pi : \Gamma(\pi) = \gamma\}$ define the set of π that correspond to certain γ

If $\Gamma(\pi) = \Gamma(\pi')$ then π and π' are different only by the direction of their cycles (of length greater than 2).

Hence, $|\Gamma^{-1}(\gamma)| = 2^{cyc(\gamma)}$

Lemma 1: proof - continued

We can now split the non-vanishing permutations according to the γ they correspond.

$$\det(A) = \sum_{\pi} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi(i)} = \sum_{\gamma} \sum_{\pi \in \Gamma^{-1}(\gamma)} sgn(\pi) \cdot 1$$

The sign of a permutation π is defined as $(-1)^{N_e}$,
where N_e is the number of even cycles in π.
If $\Gamma(\pi) = \Gamma(\pi')$ then $sgn(\pi) = sgn(\pi')$, we'll denote it as $sgn(\gamma)$

Now we can write:

$$\det(A) = \sum_{\gamma} sgn(\gamma) \sum_{\pi \in \Gamma^{-1}(\gamma)} 1 = \sum_{\gamma} sgn(\gamma) 2^{cyc(\gamma)}$$

Lemma 1: proof - end

The sign of spanning elementary subgraph γ is $(-1)^{N_e}$,
where N_e is the number of even cycles in γ.

The number of odd cycles in γ is congruent to n modulo 2: $n \equiv N_o (mod 2)$

Having $comp(\gamma) = N_e + N_o$ we obtain:

$$sgn(\gamma) = (-1)_e^N = (-1)^{n+N_o+N_e} = (-1)^{n+comp(\gamma)}$$

From here, every γ contributes $(-1)^{n+comp(\gamma)}2^{cyc(\gamma)}$ to the determinant,
and finally

$$\det(A) = (-1)^n \sum_{\gamma} (-1)^{comp(\gamma)} 2^{cyc(\gamma)}$$

Q.E.D.

Lemma 2

Let A be the adjacency matrix of graph G: $A_{n\times n} = (a_{ij})$ and

$P(G, \lambda) = \det(\lambda \cdot 1 - A) = \sum_{i=0}^{n} c_i \lambda^{n-i}$ - its characteristic polynomial.

Then

$$(-1)^i c_i = \sum M_{Di}$$

where M_{Di} are the principal minors of A with order i (Minors, whose diagonal elements belong to the main diagonal of A)

Lemma 2 - end

Proof:

$$(\lambda \cdot 1 - A) = \begin{pmatrix} \lambda & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda \end{pmatrix}$$

Let's analyze the permutations contributing to c_i:
They have exactly $n - i$ members of the main diagonal $a_{kk} = \lambda$
The permutations in the rest rows and columns (which don't include the main diagonal) will give exactly the determinant of some principal minor of A.

The sign $(-1)^i$ compensates the fact that all the values in $(\lambda \cdot 1 - A)$ are $-a_{ij}$

Hence, $(-1)^i c_i = \sum M_{Di}$

Q.E.D.

General interpretation of the coefficients of $P(G,\lambda)$

Let G be a graph with adjacency matrix A_G, and

$$P(G,\lambda) = \det(\lambda \cdot 1 - A_G) = \sum_{i=0}^{n} c_i \lambda^{n-i}$$

be a characteristic polynomial of graph G.
Then c_i are given by:

$$c_i = \sum_{\gamma_i} (-1)^{comp(\gamma_i)} 2^{cyc(\gamma_i)}$$

where the summation is over the elementary subgraphs of G with i vertices.
Corollary: we can derive now the identities for c_0, c_1, c_2, c_3

Coefficients of $P(G, \lambda)$ - continued

Proof:
According to Lemma 2 we have: $(-1)^i c_i = \sum M_{Di}$ is the sum of all the principal minors of A_G with order i;

Each such minor is the determinant of adjacency matrix A_{H_i} of some graph H_i which is an induced subgraph of G with i vertices;

Let γ_{H_i} denote a spanning elementary subgraph of H_i

Then, by Lemma 1,

$$(-1)^i c_i = \sum M_{Di} = \sum_{H_i} \sum_{\gamma_{H_i}} (-1)^{comp(\gamma_{H_i})} 2^{cyc(\gamma_{H_i})}$$

Every elementary subgraph with i vertices γ_i of G is contained in exactly one H_i. Thus, summarizing over all the γ_i we obtain:

$$c_i = \sum_{\gamma_i} (-1)^{comp(\gamma_i)} 2^{cyc(\gamma_i)}$$

Q.E.D.

Theorem 1 - (C.Godsil, I.Gutman, 1981)

Let G be a simple graph with n vertices and adjacency matrix A,

$m(G, \lambda) = \sum_{i=0}^{\frac{n}{2}} (-1)^i m_i(G) \lambda^{n-2i}$ be its acyclic polynomial,

$P(G, \lambda) = \det(\lambda \cdot 1 - A) = \sum_{i=0}^{n} c_i \lambda^{n-i}$ be its characteristic polynomial.

Let C denote an elementary subgraph of G, which contains only cycles;

Let $comp(C)$ denote the number of components in C;

Let $G - C$ denote the induced subgraph of G obtained from G by removing all the vertices of C.

Then the following holds:

$$P(G, \lambda) = m(G, \lambda) + \sum_{C} (-2)^{comp(C)} m(G - C, \lambda)$$

where the summation is over all non-empty C.

Theorem 1 - continued

In the case of a forest we have:

$$P(F, \lambda) = m(F, \lambda)$$

Moreover, the coefficients satisfy the following identities:

(i) Even coefficients:

$$c_{2i} = m_i$$

(ii) Odd coefficients:

$$c_{2i+1} = 0$$

Theorem 1 - continued

Proof:

Let us look on the coefficients of $P(G, \lambda)$:

$$P(G,\lambda) = \sum_{i=0}^{n} c_i \lambda^{n-i} = \sum_{i=0}^{n} \sum_{\gamma_i} (-1)^{comp(\gamma_i)} 2^{cyc(\gamma_i)} \lambda^{n-i}$$

Let's split the internal sum by the γ's having the same set of cycles (including, in particular, empty set).
Let C denote such a common set of cycles.
Let $\delta = \gamma_i - C$ denote the rest of γ_i, which is a set of disjoint edges.

Then $cyc(\gamma_i) = comp(C)$ and $comp(\gamma_i) = comp(\delta) + comp(C)$

Thus we can write:

$$P(G,\lambda) = \sum_{i=0}^{n} \sum_{C} \sum_{\delta} (-1)^{comp(\delta)+comp(C)} 2^{comp(C)} \lambda^{n-i}$$

Theorem 1 - continued

Let $|C|$ denote the number of vertices in C.

Then we can express i via $|C|$ and $comp(\delta)$: $i = 2comp(\delta) + |C|$

Since C is independent of i and δ, we can write now:

$$P(G,\lambda) = \sum_{C}(-2)^{comp(C)} \sum_{comp(\delta)=0}^{n-|C|} \sum_{\delta}(-1)^{comp(\delta)}\lambda^{n-|C|-2comp(\delta)} =$$

$$= \sum_{C}(-2)^{comp(C)} \sum_{j=0}^{n-|C|} m_j(G-C)\lambda^{n-|C|-j} = \sum_{C}(-2)^{comp(C)} m(G-C,\lambda)$$

Now we should distinguish between the case, when $C = \emptyset$, and the rest of the cases.

$$P(G,\lambda) = m(G,\lambda) + \sum_{C\neq\emptyset}(-2)^{comp(C)} m(G-C,\lambda)$$

Q.E.D.

Corollary 1.1 (C.Godsil, I.Gutman, 1981)

The acyclic polynomial of a graph coincides with the characteristic polynomial if and only if the graph is a forest.

$$m(G,\lambda) = P(G,\lambda) \Leftrightarrow Forest(G)$$

Proof:

"$\Leftarrow$" follows trivially from the theorem 1.

"$\Rightarrow$":
Suppose G is not a forest, and proof that $m(G,\lambda) \neq P(G,\lambda)$.
Let q be the smallest cycle in G and $|q|$ is its length.
Without loss of generality we can state that there are exactly $k \geq 1$ cycles of such a length, denoted as $\{q_1, ..., q_k\}$ in the graph G.

Let a_i and c_i be the the coefficients of λ^{n-i} in respectively acyclic and characteristic polynomials.

Corollary 1.1 - continued

We shall prove that the second part of the equation in Theorem 1

$\sum_{C \neq \emptyset} (-2)^{comp(C)} m(G - C, \lambda)$

makes the difference between the coefficients $a_{|q|}$ and $c_{|q|}$.

First, only the summation over $C \in \{q_1, ..., q_k\}$ contribute to the coefficient of $\lambda^{n-|q|}$, because all the other cycles or combinations of cycles are bigger, and then the degree of $m(G - C, \lambda)$ will be less than $\lambda^{n-|q|}$.

Second, every single cycle contributes exactly (-2), because the graph $G - C$ has exactly one 0-matching.

Thus, $a_i - c_i = 2k > 0$, hence the proposition "$\Rightarrow$" holds.
Q.E.D.

Corollary 1.2

We can state now:
For every forest F, all the roots of its acyclic polynomial are real. They are equal to the eigenvalues of F.

Identities and Recurrences

Proposition 2
The acyclic polynomial is multiplicative:

Let $G \sqcup H$ denote the disjoint union of graphs G and H. Then:

$$m(G \sqcup H, \lambda) = m(G, \lambda) \cdot m(H, \lambda)$$

Identities and Recurrences - continued

Proof:
Each k-matching of $G \sqcup H$ consists of l-matching of G and $(k-l)$-matching of H.

$m_k(G \sqcup H) = \sum_{l=0}^{k} m_l(G) m_{k-l}(H)$

The coefficient of λ^{n-2k} in $m(G,\lambda) \cdot m(H,\lambda)$ is equal to

$\sum_{l=0}^{k} (-1)^l m_l(G)(-1)^{k-l} m_{k-l}(H) =$
$= (-1)^k \sum_{l=0}^{k} m_l(G) m_{k-l}(H) = (-1)^k m_k(G \sqcup H)$

which is equal to the corresponding coefficient of $m(G \sqcup H, \lambda)$

Q.E.D.

Identities and Recurrences - continued

Proposition 3
Edge recurrence:

Let $G - e$ denote the graph obtained by removing edge $e = (u, v) \in E$ from the graph $G(V, E)$

Let $G - u - v$ denote the induced subgraph of $G(V, E)$ obtained from G by removing two vertices $u, v \in V$

Then:

$$m(G, \lambda) = m(G - e, \lambda) - m(G - u - v, \lambda)$$

Identities and Recurrences - continued

Proof:
All the k-matchings of G are of 2 disjoint kinds: those that use the edge e and those that do not. Every matching that uses the edge e determines uniquely a $(k-1)$-matching in $G-u-v$. Every matching that don't use e is actually a matching in $G-e$.
Therefore:

$$m_k(G) = m_k(G-e) + m_{k-1}(G-u-v)$$

Hence

$$m(G,\lambda) = \sum_{k\geq 0}(-1)^k m_k(G-e)\lambda^{n-2k} + \sum_{k\geq 1}(-1)^k m_{k-1}(G-u-v)\lambda^{n-2k} =$$

$$= \sum_{k\geq 0}(-1)^k m_k(G-e)\lambda^{n-2k} + (-1)\sum_{k-1\geq 0}(-1)^{(k-1)} m_{k-1}(G-u-v)\lambda^{n-2-2(k-1)} =$$

$$= m(G-e,\lambda) - m(G-u-v,\lambda)$$

Q.E.D.

Identities and Recurrences - continued

Proposition 4
Vertex recurrence:

Let $u \in V$ be a vertex of degree d.

Let $G - u$ denote the induced subgraph of $G(V, E)$ obtained from G by removing vertex u

Let $v_i \in V, 1 \leq i \leq d$ denote all the vertices such that $(u, v_i) \in E$ and

Let $G - u - v_i$ denote the induced subgraph of $G(V, E)$ obtained from G by removing two vertices u, v_i

Then:

$$m(G, \lambda) = \lambda \cdot m(G - u, \lambda) - \sum_{i=1}^{d} m(G - u - v_i, \lambda)$$

Identities and Recurrences - continued

Proof:
All the k-matchings of G are of 2 disjoint kinds: those that use the vertex u and those that do not. The number of k-matchings that do not use the vertex u is equal to $m_k(G-u,\lambda)$. The number which do use u is equal to $m_{k-1}(G-u-v_i)$, summed over the vertices v_i adjacent to u. Thus,

$$m_k(G) = m_k(G-u) + \sum_{i=1}^{d} m_{k-1}(G-u-v_i)$$

Hence,

$$m(G,\lambda) = \sum_{k\geq 0}(-1)^k m_k(G-u)\lambda^{n-2k} + \sum_{k\geq 1}(-1)^k \sum_{i=1}^{d} m_{k-1}(G-u-v_i)\lambda^{n-2k} =$$

Identities and Recurrences - continued

Having $G-u$ is a graph of $n-1$ vertices, and i is independent of k, we can write

$$m(G,\lambda) = \lambda \cdot \sum_{k\geq 0}(-1)^k m_k(G-u)\lambda^{(n-1)-2k} +$$

$$+(-1)\sum_{i=1}^{d}\sum_{k-1\geq 0}(-1)^{(k-1)} m_{k-1}(G-u-v_i)\lambda^{(n-2)-2(k-1)} =$$

$$\lambda \cdot m(G-u,\lambda) - \sum_{i=1}^{d} m(G-u-v_i,\lambda)$$

Q.E.D.

Theorem 2

Let G be a connected graph, $v \in V(G)$ be a vertex of degree d, and H_1 its induced subgraph without the vertex v.

Let $w_i (i = 1, ..., d)$ be the vertices adjacent to v.

Let $H_i (i = 2, ..., d)$ be graphs which are all isomorphic to H_1.

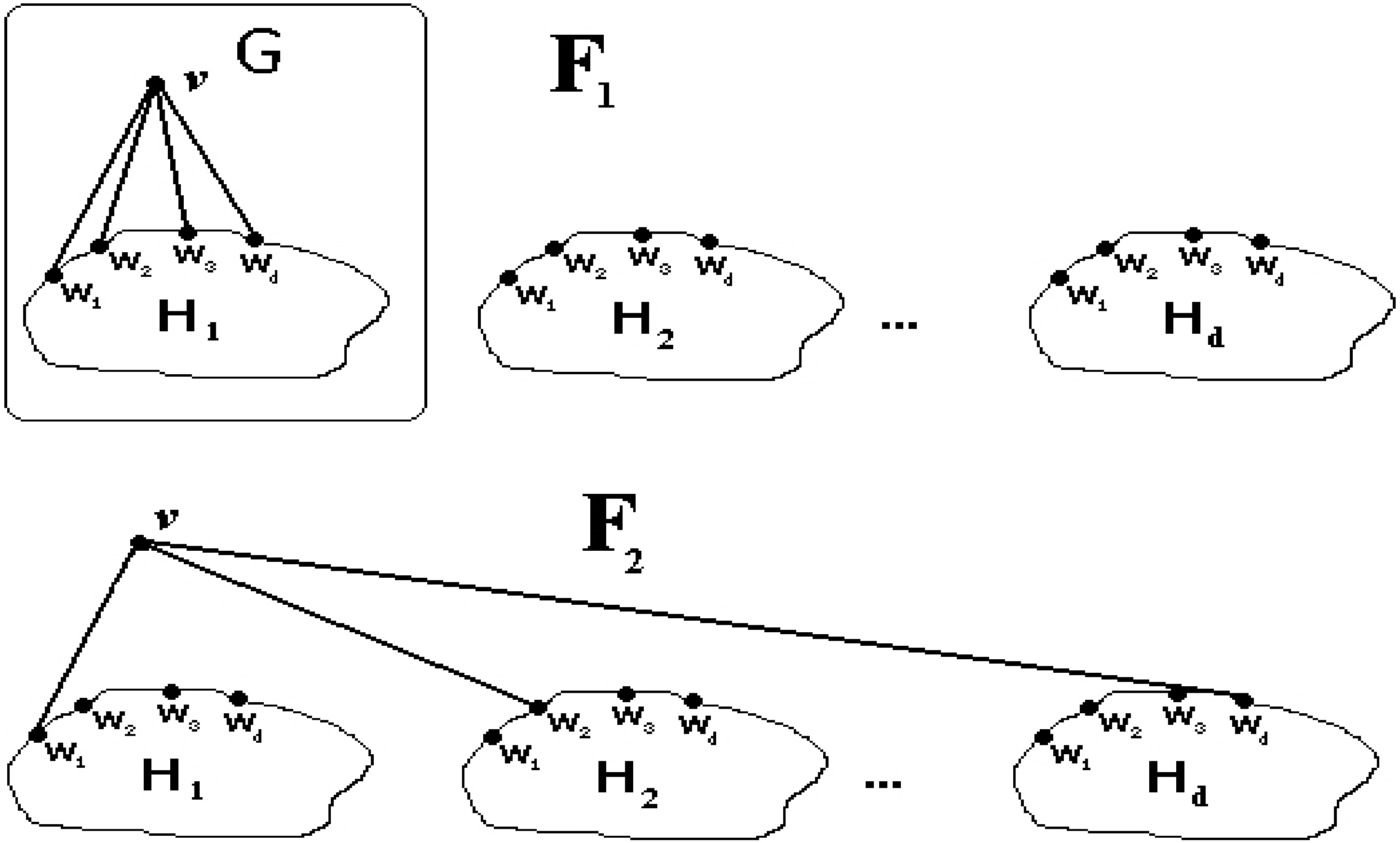
Let $w_i(H_i)$ denote the vertex of H_i corresponding to the vertex w_i in H_1.

Let $F_1 = G \sqcup H_2 \sqcup ... \sqcup H_d$

Let F_2 be obtained from F_1 by replacing the edges
$e_i = \{v, w_i\}$ by $e'_i = \{v, w_i(H_i)\}$

Then $m(F_1) = m(F_2)$

Theorem 2 - continued



Theorem 2 - continued

Proof:

For $m(F_1)$, we will apply vertex recurrence on G and v,

$$m(F_1) = m(G)m(H_2)...m(H_d) = m(H_2)...m(H_d)[\lambda m(H_1) - \sum_{i=1}^{d} m(H_1 - w_i)]$$

For $m(F_2)$, we will apply vertex recurrence on F_2 and v,

$$m(F_2) = \lambda m(H_1)...m(H_d) - m(H_1)...m(H_d) \sum_{i=1}^{d} \frac{m(H_i - w_i(H_i))}{m(H_i)}$$

Having $H_1 ... H_d$ isomorphic we obtain:

$$m(F_1) = (m(H_1))^{d-1}[\lambda m(H_1) - d \cdot m(H_1 - w_i)] =$$

$$= \lambda(m(H_1))^d - d(m(H_1))^{d-1} \cdot m(H_1 - w_i) = m(F_2)$$

Q.E.D.

Theorem 2 - continued

Corollary 2.1
For every simple connected graph G and vertex $v \in V(G)$ there is a tree $T(G,v)$ such that $m(G,\lambda)$ divides $m(T(G,v),\lambda)$ and maximum degree of T is not more than maximum degree of G.
Proof: By multiple application of Theorem 2.

Corollary 2.2
For every simple graph G there is a forest F such that $m(G,\lambda)$ divides $m(F,\lambda)$, and maximum degree of F is not more than maximum degree of G.
Proof: straightforward from proposition 2 and corollary 2.1.

Corollary 2.3
The zeros (roots) of $m(G,\lambda)$, are real.
Proof: straightforward from (2.2), (1.1) and the fact that the roots of the characteristic polynomial of a simple graph are all real.

Roots of the acyclic polynomial

Corollary 2.4: *The roots of acyclic polynomial are symmetrically placed around zero.* In other words,

$$m(G,\lambda) = 0 \Leftrightarrow m(G,-\lambda) = 0$$

Proof:
According to the definition,

$$m(G,\lambda) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) \lambda^{n-2k}$$

Hence, either all the degrees of λ's are even or all the degrees of λ's are odd.

In the first case, $m(G,-\lambda) = m(G,\lambda)$

In the second case, $m(G,-\lambda) = -m(G,\lambda)$

In both the cases,

$$m(G,\lambda) = 0 \Leftrightarrow m(G,-\lambda) = 0$$

Q.E.D.

Roots of the matching generating polynomial

Corollary 2.5: *All the roots of generating matching polynomial are real and negative.*

Proof:
The coefficient of λ^0 in $g(G, \lambda)$ is always 1 (number of zero-matchings by convention). Thus, $\lambda = 0$ cannot be a root of $g(G, \lambda)$

On the other hand, we know that $m(G, \lambda) = \lambda^n g(G, (-\lambda^{-2}))$

Let t be a root of $g(G, \lambda)$. We know that $t \neq 0$

Let $s = (-t)^{-\frac{1}{2}}$, and then $t = -s^{-2}$

Hence, $m(G, s) = s^n g(G, -s^{-2}) = s^n g(G, t) = 0$, so s is a root of $m(G, \lambda)$

But we know that all the roots of $m(G, \lambda)$ are real.

Thus, $t = -s^{-2}$ is real and negative.

Q.E.D.

Theorem 3 - (Heilman and Lieb, 1972)

(L.Lovasz and M.D.Plummer, Matching Theory - Theorem 8.5.8)

Let G be a simple graph with degree $\Delta(G) > 1$ and
let t be any root of $m(G, \lambda)$.

Then

$$t \leq 2\sqrt{\Delta(G) - 1}$$

Theorem 3 - proof

Let's prove it first for trees:

Let T be a tree of maximum degree Δ.
By theorem 1, the roots of acyclic polynomial are actually the eigenvalues of the tree.

On the other hand, the tree T is an induced subgraph of a full $(\Delta - 1)$-ary tree T'.
The adjacency matrix of T is a principal minor of the adjacency matrix of T'. But the largest eigenvalue of a principal minor doesn't exceed the largest eigenvalue of the matrix.

The eigenvalues of a complete d-ary tree of depth k are:
$\lambda = 2\sqrt{d}\cos(m\pi/(k+1)), m = 1, ..., k$, hence the largest eigenvalue of T is less than $2\sqrt{\Delta - 1}$ as claimed.
(L.Lovasz Combinatorial problems and Exercises (Exercise 11.5)
2-nd ed. Elsevier S.P., Amsterdam and Akademiai Kiado, Budapest 1993)

Theorem 3 - continued

The general case now follows using Corollary 2.1:

Let G be a graph, and let H be any of its connected components with the maximum degree Δ.

By the Corollary 2.1, there is a tree T such that $m(H,\lambda)|m(T,\lambda)$, and the maximum degree of T doesn't exceed Δ.

Since any root of $m(H,\lambda)$ is also a root of $m(T,\lambda)$, it follows that every root of $m(H,\lambda)$ doesn't exceed $2\sqrt{\Delta - 1}$.

By Proposition 2, $m(G,\lambda) = \prod_H m(H,\lambda)$, so any t root of $m(G,\lambda)$ is also a root of some $m(H,\lambda)$.
Hence the equation $t \leq 2\sqrt{\Delta - 1}$ holds for any graph.

Q.E.D.