The Tutte Polynomial

Graph Polynomials

238900 winter 05/06

The Rank Generation Polynomial

Reminder

$$S(G;x,y)=\sum_{F\subseteq E(G)} x^{k\langle F\rangle - k\langle E\rangle} y^{n\langle F\rangle}$$

$k\langle F\rangle =$ Number of connected components in G(V,F)

$$n\langle F\rangle = |F| - |V| + k\langle F\rangle$$

$$r\langle F\rangle = |V| - k\langle F\rangle$$

The Rank Generation Polynomial

Theorem 2

$$S(G;x,y)=\begin{cases}(x+1)S(G-e;x,y) & e \text{ is a bridge} \\ (y+1)S(G-e;x,y) & e \text{ is a loop} \\ S(G-e;x,y)+S(G/e;x,y) & \text{else}\end{cases}$$

In addition, $S(E_n;x,y) = 1$ for $G(V,E)$ where $|V|=n$ and $|E|=0$

$G-e$ Is the graph obtained by omitting edge *e*

G/e Is the graph obtained by contracting edge *e*

The Rank Generation Polynomial

Theorem 2 – Proof

Let us define :

G’	=	G – e
G’’	=	G / e
r’<G>	=	r<G’>
n’<G>	=	n<G’>
r’’<G>	=	r<G’’>
n’’<G>	=	n<G’’>

The Rank Generation Polynomial

Theorem 2 – Proof

Let us denote :

$$\left(F \cup e\right) \Rightarrow \left(F \cup \{e\}\right)$$

$$\left(E - e\right) \Rightarrow \left(E - \{e\}\right)$$

The Rank Generation Polynomial

Observations

$$\forall e \in E, \quad \left(F \subset E - e\right) \rightarrow$$

$$\begin{cases} r\langle F \rangle = r'\langle F \rangle \\ n\langle F \rangle = n'\langle F \rangle \\ r\langle E \rangle - r\langle F \cup e \rangle = r''\langle E - e \rangle - r''\langle F \rangle = \\ \qquad = r\langle G'' \rangle - r''\langle F \rangle \end{cases}$$

The Rank Generation Polynomial

Observations

$$\forall e \in E, \quad (F \subset E - e) \rightarrow$$

$$\begin{cases} r\langle F \rangle = r'\langle F \rangle \\ n\langle F \rangle = n'\langle F \rangle \\ r\langle E \rangle - r\langle F \cup e \rangle = r''\langle E - e \rangle - r''\langle F \rangle = \\ \qquad = r\langle G'' \rangle - r''\langle F \rangle \end{cases}$$

$e \notin F$ (applies to the first two equations)

The Rank Generation Polynomial

Observations

$$\forall e \in E, \quad (F \subset E - e) \rightarrow$$

$$\begin{cases} r\langle F \rangle = r'\langle F \rangle \\ n\langle F \rangle = n'\langle F \rangle \\ r\langle E \rangle - r\langle F \cup e \rangle = r''\langle E - e \rangle - r''\langle F \rangle = \\ \qquad = r\langle G'' \rangle - r''\langle F \rangle \end{cases}$$

$$k\langle E \rangle = k\langle E - e \rangle$$

$$k\langle F \cup e \rangle = k\langle F \rangle$$

$$\begin{cases} |V| \Rightarrow |V| & loop \\ |V| \Rightarrow |V| - 1 & else \end{cases}$$

The Rank Generation Polynomial

Observations

$$\forall e \in E, \quad (F \subset E - e) \rightarrow$$

$$\begin{cases} r\langle F \rangle = r'\langle F \rangle \\ n\langle F \rangle = n'\langle F \rangle \\ r\langle E \rangle - r\langle F \cup e \rangle = r''\langle E - e \rangle - r''\langle F \rangle = \\ G'' = (G')'' \quad = r\langle G'' \rangle - r''\langle F \rangle \end{cases}$$

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The Rank Generation Polynomial

Observations

1

$$r(E) = \begin{cases} r'(E-e)+1 & e \text{ is a bridge} \\ r'(E-e) & \text{else} \end{cases}$$

2

$$n(F \cup e) = \begin{cases} n''\langle F \rangle + 1 & e \text{ is a loop} \\ n''\langle F \rangle & \text{else} \end{cases}$$

$|E| \Rightarrow |E|-1$

$|V| \Rightarrow |V|-1$

$k \Rightarrow k$

$|E| \Rightarrow |E|-1$

$|V| \Rightarrow |V|$

$k \Rightarrow k$

The Rank Generation Polynomial

Reminder

$$S(G;x,y) = \sum_{F \subseteq E(G)} x^{k\langle F\rangle - k\langle E\rangle} y^{n\langle F\rangle}$$

$$= \sum_{F \subseteq E(G)} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle}$$

The Rank Generation Polynomial

Let us define

$$S(G;x,y) = S_0(G;x,y) + S_1(G;x,y)$$

$$S_0(G;x,y) = \sum_{F \subseteq E(G), e \notin F} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle}$$

$$S_1(G;x,y) = \sum_{F \subseteq E(G), e \in F} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle}$$

The Rank Generation Polynomial

$$S_0(G;x,y) = \sum_{F \subseteq E(G), e \notin F} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle} =$$

$$\sum_{F \subseteq E-e} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle} =$$

1

$$\left\{ \begin{array}{ll} \sum_{F \subseteq E(G')} x^{r'\langle E-e\rangle + 1 - r'\langle F\rangle} y^{n'\langle F\rangle} & e \text{ is a bridge} \\ \sum_{F \subseteq E(G')} x^{r'\langle E-e\rangle - r'\langle F\rangle} y^{n'\langle F\rangle} & \text{otherwise} \end{array} \right\}$$

The Rank Generation Polynomial

$$S_0(G;x,y) =$$

$$\left\{\begin{array}{ll} \sum_{F\subseteq E(G')} x^{r'\langle E-e\rangle+1-r'\langle F\rangle} y^{n'\langle F\rangle} & e \text{ is a bridge} \\ \sum_{F\subseteq E(G')} x^{r'\langle E-e\rangle-r'\langle F\rangle} y^{n'\langle F\rangle} & \text{otherwise} \end{array}\right\} =$$

$$\left\{\begin{array}{ll} xS(G-e;x,y) & e \text{ is a bridge} \\ S(G-e;x,y) & \text{otherwise} \end{array}\right\}$$

The Rank Generation Polynomial

$$S_1(G;x,y) = \sum_{F \subseteq E(G), e \in F} x^{r\langle E\rangle - r\langle F\rangle} y^{n\langle F\rangle} =$$

$$\sum_{F \subseteq E-e} x^{r\langle E\rangle - r\langle F\cup e\rangle} y^{n\langle F \cup e\rangle} =$$

3

2

$$\begin{cases} \sum_{F \subseteq E(G'')} x^{r\langle G''\rangle - r''\langle F\rangle} y^{n''\langle F\rangle + 1} & e \text{ is a loop} \\ \sum_{F \subseteq E(G'')} x^{r\langle G''\rangle - r''\langle F\rangle} y^{n''\langle F\rangle} & \text{otherwise} \end{cases}$$

The Rank Generation Polynomial

$$S_1(G;x,y)=$$

$$\left\{\begin{matrix} \sum_{F\subseteq E(G'')} x^{r\langle G''\rangle - r''\langle F\rangle} y^{n''\langle F\rangle+1} & e \text{ is a loop} \\ \sum_{F\subseteq E(G'')} x^{r\langle G''\rangle - r''\langle F\rangle} y^{n''\langle F\rangle} & \text{otherwise} \end{matrix}\right\} =$$

$$\left\{\begin{matrix} yS(G/e;x,y) & e \text{ is a loop} \\ S(G/e;x,y) & \text{otherwise} \end{matrix}\right\}$$

The Rank Generation Polynomial

$$S(G;x,y) = S_0(G;x,y) + S_1(G;x,y) =$$

$$\begin{Bmatrix} xS(G-e;x,y) & e \text{ is a bridge} \\ S(G-e;x,y) & \text{otherwise} \end{Bmatrix}$$

+

$$\begin{Bmatrix} yS(G/e;x,y) & e \text{ is a loop} \\ S(G/e;x,y) & \text{otherwise} \end{Bmatrix}$$

The Rank Generation Polynomial

$$S(G;x,y)=S_0(G;x,y)+S_1(G;x,y)=$$

$$\begin{cases} xS(G-e;x,y)+S(G/e;x,y) & e \text{ is a bridge} \\ S(G-e;x,y)+yS(G/e;x,y) & e \text{ is a loop} \\ S(G-e;x,y)+S(G/e;x,y) & \text{otherwise} \end{cases}$$

The Rank Generation Polynomial

$$S(G;x,y)=S_0(G;x,y)+S_1(G;x,y)=$$

$$\begin{cases} xS(G-e;x,y)+S(G/e;x,y) & e \text{ is a bridge} \\ S(G-e;x,y)+yS(G-e;x,y) & e \text{ is a loop} \\ S(G-e;x,y)+S(G/e;x,y) & \text{otherwise} \end{cases}$$

Obvious

The Rank Generation Polynomial

$$S(G;x,y) = S_0(G;x,y) + S_1(G;x,y) =$$

$$\begin{cases} xS(G-e;x,y) + S(G/e;x,y) & e \text{ is a bridge} \\ (y+1)S(G-e;x,y) & e \text{ is a loop} \\ S(G-e;x,y) + S(G/e;x,y) & \text{otherwise} \end{cases}$$

Obvious

The Rank Generation Polynomial

$$S(G;x,y) = S_0(G;x,y) + S_1(G;x,y) =$$

$$\begin{cases} xS(G-e;x,y) + S(G-e;x,y) & e \text{ is a bridge} \\ (y+1)S(G-e;x,y) & e \text{ is a loop} \\ S(G-e;x,y) + S(G/e;x,y) & \text{otherwise} \end{cases}$$

$$\forall F \subseteq E-e \quad n''\langle F\rangle = n'\langle F\rangle,$$

$$r''\langle E-e\rangle - r''\langle F\rangle = r'\langle E-e\rangle - r'\langle F\rangle$$

$$G' \Rightarrow G'' \quad |V| \Rightarrow |V|-1 \quad k+1 \Rightarrow k$$

The Rank Generation Polynomial

$$S(G;x,y) = S_0(G;x,y) + S_1(G;x,y) =$$

$$\begin{cases} (x+1)S(G-e;x,y) & e \text{ is a bridge} \\ (y+1)S(G-e;x,y) & e \text{ is a loop} \\ S(G-e;x,y) + S(G/e;x,y) & \text{otherwise} \end{cases}$$

Obvious

Q.E.D

The Tutte Polynomial

$$T_G(x,y) = T(G;x,y) = S(G;x-1,y-1) =$$

$$\sum_{F \subseteq E} (x-1)^{r\langle E\rangle - r\langle F\rangle} (y-1)^{n\langle F\rangle} =$$

$$\frac{1}{(x-1)^{k\langle E\rangle}(y-1)^{|V(G)|}} \sum_{F \subseteq E(G)} ((x-1)(y-1))^{k\langle F\rangle} (y-1)^{|F|}$$

The Tutte Polynomial

Reminder

$$T(G;x,y) = \begin{cases} x \cdot T(G-e;x,y) & \text{if } e \text{ is a bridge} \\ y \cdot T(G-e;x,y) & \text{if } e \text{ is a loop} \\ T(G-e;x,y) + T(G/e;x,y) & \text{else} \end{cases}$$

4

$$T(E_n;x,y) = 1$$

5

The Universal Tutte Polynomial

$$U(G; x, y, \alpha, \sigma, \tau) = \alpha^{k(G)} \sigma^{n(G)} \tau^{r(G)} T\left(G; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right)$$

The Universal Tutte Polynomial

Theorem 6

$$U(G;x,y,\alpha,\sigma,\tau)=$$

$$\begin{cases} x\cdot U(G-e;x,y) & e \text{ is a bridge} \\ y\cdot U(G-e;x,y) & e \text{ is a loop} \\ \sigma\cdot U(G-e;x,y)+\tau\cdot U(G/e;x,y) & \text{otherwise} \end{cases}$$

$$U(E_n;x,y,\alpha,\sigma,\tau)=\alpha^n$$

The Universal Tutte Polynomial

Theorem 6 - Proof

$$U\left(E_n; x, y, \alpha, \sigma, \tau\right) =$$

$$\alpha^{|V|}\sigma^{0-|V|+|V|}\tau^{|V|-|V|}T\left(E_n; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right) =$$

$$\alpha^{n} \cdot 1 \cdot 1 \cdot 1 = \alpha^{n}$$

5

The Universal Tutte Polynomial

If e is a bridge

$$(e \text{ is a bridge}) \rightarrow \begin{cases} k(G-e) = k(G)+1 \\ n(G-e) = n(G) \\ r(G-e) = r(G)-1 \end{cases}$$

$$U(G;x,y,\alpha,\sigma,\tau) =$$

$$\alpha^{k(G-e)-1}\sigma^{n(G-e)}\tau^{r(G-e)+1}\frac{\alpha x}{\tau}T\left(G-e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)$$

4

The Universal Tutte Polynomial

If e is a bridge

$$U(G; x, y, \alpha, \sigma, \tau) =$$

$$\alpha^{k(G-e)-1}\sigma^{n(G-e)}\tau^{r(G-e)+1}\frac{\alpha x}{\tau}T\left(G-e; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right)$$

$$= \alpha^{k(G-e)}\sigma^{n(G-e)}\tau^{r(G-e)}xT\left(G-e; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right) =$$

$$xU(G-e; x, y, \alpha, \sigma, \tau)$$

The Universal Tutte Polynomial

If e is a loop

$$(e \text{ is a loop}) \rightarrow \begin{cases} k(G-e) = k(G) \\ n(G-e) = n(G) - 1 \\ r(G-e) = r(G) \end{cases}$$

$$U(G; x, y, \alpha, \sigma, \tau) =$$

$$\alpha^{k(G-e)} \sigma^{n(G-e)+1} \tau^{r(G-e)} \frac{y}{\sigma} T\left(G-e; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right)$$

4

The Universal Tutte Polynomial

If e is a bridge

$$U(G;x,y,\alpha,\sigma,\tau)=$$

$$\alpha^{k(G-e)}\sigma^{n(G-e)+1}\tau^{r(G-e)}\frac{y}{\sigma}T\left(G-e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)$$

$$=\alpha^{k(G-e)}\sigma^{n(G-e)}\tau^{r(G-e)}yT\left(G-e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)=$$

$$yU(G-e;x,y,\alpha,\sigma,\tau)$$

The Universal Tutte Polynomial

If *e* is neither a loop nor a bridge

$$\neg\left(\begin{array}{l}(e \text{ is a bridge})\\ \vee \\ (e \text{ is a loop})\end{array}\right) \rightarrow \begin{cases} k(G-e)=k(G/e)=k(G) \\ n(G-e)=n(G)-1 \\ n(G/e)=n(G) \\ r(G-e)=r(G) \\ r(G/e)=r(G)-1 \end{cases}$$

The Universal Tutte Polynomial

If *e* is neither a loop nor a bridge

$$U(G;x,y,\alpha,\sigma,\tau)=$$

$$\alpha^{k(G)}\sigma^{n(G)}\tau^{r(G)}\cdot$$

$$\left[T\left(G-e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)+T\left(G/e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)\right]$$

4

The Universal Tutte Polynomial

If e is neither a loop nor a bridge

$$U(G;x,y,\alpha,\sigma,\tau)=$$

$$\alpha^{k(G-e)}\sigma^{n(G-e)+1}\tau^{r(G-e)}T\left(G-e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)+$$

$$\alpha^{k(G/e)}\sigma^{n(G/e)}\tau^{r(G/e)+1}T\left(G/e;\frac{\alpha x}{\tau},\frac{y}{\sigma}\right)$$

The Universal Tutte Polynomial

If *e* is neither a loop nor a bridge

$$U(G; x, y, \alpha, \sigma, \tau) =$$

$$\sigma U(G - e; x, y, \alpha, \sigma, \tau) +$$

$$\tau U(G / e; x, y, \alpha, \sigma, \tau)$$

The Universal Tutte Polynomial

Theorem 6 - Proof

$$U(G; x, y, \alpha, \sigma, \tau) =$$

$$\begin{cases} x \cdot U(G-e; x, y) & e \text{ is a bridge} \\ y \cdot U(G-e; x, y) & e \text{ is a loop} \\ \sigma \cdot U(G-e; x, y) + \tau \cdot U(G/e; x, y) & \text{otherwise} \end{cases}$$

Q.E.D

The Universal Tutte Polynomial

Theorem 7

$$\forall V(G;x,y,\alpha,\sigma,\tau) \quad s.t.\, V(G;x,y,\alpha,\sigma,\tau) =$$

$$\begin{Bmatrix} x \cdot V(G-e;x,y) & e \text{ is a bridge} \\ y \cdot V(G-e;x,y) & e \text{ is a loop} \\ \sigma \cdot V(G-e;x,y) + \tau \cdot V(G/e;x,y) & \text{otherwise} \end{Bmatrix}$$

$$V(E_n;x,y,\alpha,\sigma,\tau) = \alpha^n$$

then $V(G;x,y,\alpha,\sigma,\tau) = U(G;x,y,\alpha,\sigma,\tau)$

The Universal Tutte Polynomial

Theorem 7 – proof

$$V(G;x,y,\alpha,\sigma,\tau)=$$

$$\begin{cases} x\cdot V(G-e;x,y) & e \text{ is a bridge} \\ y\cdot V(G-e;x,y) & e \text{ is a loop} \\ \sigma\cdot V(G-e;x,y)+\tau\cdot V(G/e;x,y) & \text{otherwise} \end{cases}$$

V(G) is dependant entirely on *V(G-e)* and *V(G/e).*

In addition – $V(E_n;x,y,\alpha,\sigma,\tau)=U(E_n;x,y,\alpha,\sigma,\tau)$

Q.E.D

Evaluations of the Tutte Polynomial

Proposition 8

Let G be a connected graph.

$T(G,1,1)$ is the number of spanning trees of G

Shown in the previous lecture

Evaluations of the Tutte Polynomial

Proposition 8

Let G be a connected graph.

$T(G,1,2)$ is the number of connected spanning sub-graphs of G

$$T(G;1,2) = S(G;0,1) = \sum_{F \subseteq E(G)} 0^{k\langle F\rangle - k\langle E\rangle} 1^{n\langle F\rangle}$$

$$= \sum_{(F \subseteq E(G)) \wedge (k(F) = k(E))} 1^{n\langle F\rangle} = \sum_{(F \subseteq E(G)) \wedge (k(F) = k(E))} 1$$

Evaluations of the Tutte Polynomial

Proposition 8

Let G be a connected graph.

$T(G,1,2)$ is the number of connected spanning sub-graphs of G

$$T(G;1,2) = \sum_{(F \subseteq E(G)) \wedge (k(F) = k(E))} 1$$

sub-graphs of G

connected

Evaluations of the Tutte Polynomial

Proposition 8

Let G be a connected graph.

$T(G,2,1)$ is the number of (edge sets forming) spanning forests of G

$$T(G;2,1) = S(G;1,0) = \sum_{F \subseteq E(G)} 1^{k\langle F\rangle - k\langle E\rangle} 0^{n\langle F\rangle}$$

$$= \sum_{(F \subseteq E(G)) \wedge (n\langle F\rangle = 0)} 1$$

Spanning forests

Evaluations of the Tutte Polynomial

Proposition 8

Let G be a connected graph.

$T(G,2,2)$ is the number of spanning sub-graphs of G

$$T(G;2,2) = S(G;1,1) = \sum_{F \subseteq E(G)} 1^{k\langle F\rangle - k\langle E\rangle} 1^{n\langle F\rangle}$$

$$= \sum_{F \subseteq E(G)} 1$$

The Universal Tutte Polynomial

Theorem 9

The chromatic polynomial and the Tutte polynomial are related by the equation :

$$\chi(G;x) = (-1)^{r(G)} x^{k(G)} T(G;1-x,0)$$

The Universal Tutte Polynomial

Theorem 9 – proof

Claim :

$$\chi(G;x) = U\left(G, \frac{x-1}{x}, 0, x, 1, -1\right)$$

The Universal Tutte Polynomial

Theorem 9 – proof

We must show that :

$$\chi(E_n; x) = U\left(E_n, \frac{x-1}{x}, 0, x, 1, -1\right)$$

The Universal Tutte Polynomial

Theorem 9 – proof

and :

$$\chi\left(G;\frac{x-1}{x}\right)=$$

$$\left\{\begin{array}{ll}\frac{x-1}{x}\cdot\chi\left(G-e;\frac{x-1}{x}\right) & e \text{ is a bridge} \\ 0 & e \text{ is a loop} \\ \chi\left(G-e;\frac{x-1}{x}\right)-\chi\left(G/e;\frac{x-1}{x}\right) & \text{otherwise}\end{array}\right\}$$

The Universal Tutte Polynomial

Theorem 9 – proof

$$\chi(E_n; x) = x^n$$

Obvious. (empty graphs)

$$U\left(E_n, \frac{x-1}{x}, 0, x, 1, -1\right) = \alpha^n = x^n$$

$$\chi(E_n; x) = U\left(E_n, \frac{x-1}{x}, 0, x, 1, -1\right)$$

The Universal Tutte Polynomial

Theorem 9 – proof

$$\forall(e \text{ loop}) \quad \chi(G;x)=0$$

Obvious

$$\forall(e \neg(\text{bridge} \vee \text{loop})) \quad \chi(G;x)=\chi(G-e;x)-\chi(G/e;x)$$

Chromatic polynomial property

$$\forall(e \text{ bridge}) \quad \chi(G;x)=\frac{x-1}{x}\chi(G-e;x)$$

Chromatic polynomial property

The Universal Tutte Polynomial

Theorem 9 – proof

Thus :

$$\chi\left(G;\frac{x-1}{x}\right)=$$

$$\left\{\begin{array}{ll}\dfrac{x-1}{x}\cdot\chi\left(G-e;\dfrac{x-1}{x}\right) & e\text{ is a bridge}\\ 0 & e\text{ is a loop}\\ \chi\left(G-e;\dfrac{x-1}{x}\right)-\chi\left(G/e;\dfrac{x-1}{x}\right) & \text{otherwise}\end{array}\right\}$$

The Universal Tutte Polynomial

Theorem 9 – proof

And so, due to the uniqueness we shown in Theorem 7 :

$$\chi(G;x) = U\left(G, \frac{x-1}{x}, 0, x, 1, -1\right)$$

The Universal Tutte Polynomial

Theorem 9 – proof

Remembering that :

$$U(G; x, y, \alpha, \sigma, \tau) =$$

$$\alpha^{k(G)} \sigma^{n(G)} \tau^{r(G)} T\left(G; \frac{\alpha x}{\tau}, \frac{y}{\sigma}\right)$$

The Universal Tutte Polynomial

Theorem 9 – proof

Remembering that :

$$\chi(G;x) = U\left(G, \frac{x-1}{x}, 0, x, 1, -1\right) =$$

$$x^{k(G)} 1^{n(G)} (-1)^{r(G)} T(G, -(x-1), 0) =$$

$$x^{k(G)} (-1)^{r(G)} T(G, 1-x, 0)$$

Q.E.D

Evaluations of the Tutte Polynomial

Let G be a connected graph.

$T(G,2,0)$ is the number of acyclic orientations of G.

We know that (Theorem 9) –

$$\chi(G;x) = (-1)^{r(G)} x^{k(G)} T(G;1-x,0)$$

Evaluations of the Tutte Polynomial

So –

$$T(G,2,0) = T(G,1-(-1),0)$$

And thus –

$$T(G,2,0) = \frac{\chi(G;x)}{(-1)^{r\langle G\rangle} x^{k\langle G\rangle}} = \frac{\chi(G;-1)}{(-1)^{r\langle G\rangle + k\langle G\rangle = |V|}} =$$

$$(-1)^{-|V|}\chi(G;-1) = (-1)^{|V|}\chi(G;-1)$$

Acyclic Orientations of Graphs

Let G be a connected graph without loops or multiple edges.

An *orientation* of a graph is received after assigning a direction to each edge.

An orientation of a graph is *acyclic* if it does not contain any directed cycles.

Acyclic Orientations of Graphs

Proposition 1.1

$\chi(G,x)$ is the number of pairs (σ,υ) where σ is a map $\sigma : V \to \{1,2,3,\ldots,x\}$ and υ is an orientation of G, subject to the following :

- The orientation is acyclic
- $((u \to v) \in \upsilon) \Rightarrow \sigma(u) > \sigma(v)$

Acyclic Orientations of Graphs

Proof

The second condition forces the map to be a proper coloring.

The second condition is immediately implied from the first one.

Acyclic Orientations of Graphs

Proof

Conversely, if the map is proper, than the second condition defines a unique acyclic orientation of G.

Hence, the number of allowed mappings is simply the number of proper coloring with x colors, which is by definition $\chi(G,x)$

Acyclic Orientations of Graphs

$\tilde{\chi}(G,x)$ be the number of pairs (σ,υ) where σ is a map $\sigma : V \rightarrow \{1,2,3,\ldots,x\}$ and υ is an orientation of G, subject to the following :

- The orientation is acyclic
- $((u \rightarrow v) \in \upsilon) \Rightarrow \sigma(u) \geq \sigma(v)$

Acyclic Orientations of Graphs

Theorem 1.2

$$\forall x \in N \quad \tilde{\chi}(G, x) = (-1)^{|V|} \chi(G, -x)$$

$$= T(G, 2, 0)$$

Acyclic Orientations of Graphs

Proof

The chromatic polynomial is uniquely determined by the following :

$$\chi(G_0, x) = x$$ G_0 is the one vertex graph

$$\chi(G+H, x) = \chi(G, x)\chi(H, x)$$ Disjoint union

$$\chi(G, x) = \chi(G-e, x) - \chi(G/e, x)$$

Acyclic Orientations of Graphs

Proof

We now have to show for the new polynomial :

Obvious

$$\tilde{\chi}(G_0, x) = x$$ G_0 is the one vertex graph

$$\tilde{\chi}(G+H, x) = \tilde{\chi}(G, x)\tilde{\chi}(H, x)$$ Disjoint union

$$\tilde{\chi}(G, x) = \tilde{\chi}(G-e, x) - \tilde{\chi}(G/e, x)$$

Obvious

Acyclic Orientations of Graphs

Proof

We need to show that :

$$\tilde{\chi}(G,x) = \tilde{\chi}(G-e,x) - \tilde{\chi}(G/e,x)$$

Let :

$$\sigma : V(G-e) \to \{1,2,3,\ldots,x\}$$

Acyclic Orientations of Graphs

Proof

Let :

$$\sigma : V(G-e) \rightarrow \{1,2,3,\ldots,x\}$$

Let υ be an acyclic orientation of G-e compatible with σ

Let : $e = \{u,v\}$

Acyclic Orientations of Graphs

Proof

Let υ_1 be an orientation of G after adding *{u → v}* to υ

Let υ_2 be an orientation of G after adding *{v → u}* to υ

Acyclic Orientations of Graphs

Proof

We will show that for each pair $\left(\sigma,\upsilon\right)$ exactly one of the orientations $\left(\upsilon_1,\upsilon_2\right)$ is acyclic and compatible with σ ,expect for $\tilde{\chi}\left(G/e,x\right)$ of them, in which case both $\left(\upsilon_1,\upsilon_2\right)$ are acyclic orientations compatible with σ

Acyclic Orientations of Graphs

Proof

Once this is done, we will know that –

$$\tilde{\chi}(G,x)=\tilde{\chi}(G-e,x)-\tilde{\chi}(G/e,x)$$

due to the definition of $\tilde{\chi}(G,x)$

Acyclic Orientations of Graphs

Proof

For each pair (σ, υ) where -

$$\sigma : V(G-e) \rightarrow \{1,2,3,\ldots,x\}$$

and υ is an acyclic orientation compatible with σ one of these three scenarios must hold :

Acyclic Orientations of Graphs

Proof

Case 1 –

$$\sigma(u) > \sigma(v)$$

Clearly υ_2 is not compatible with σ while υ_1 is compatible. Moreover, υ_1 is acyclic :

$$u \rightarrow v \rightarrow w_1 \rightarrow w_2 \rightarrow \ldots \rightarrow u$$

$$\sigma(u) > \sigma(v) \geq \sigma(w_1) \geq \sigma(w_2) \geq \ldots \geq \sigma(u)$$

Impossible cicle

Acyclic Orientations of Graphs

Proof

Case 2 –

$$\sigma(u) < \sigma(v)$$

Clearly υ_1 is not compatible with σ while υ_2 is compatible. Moreover, υ_2 is acyclic :

$$v \rightarrow u \rightarrow w_1 \rightarrow w_2 \rightarrow \ldots \rightarrow v$$

$$\sigma(v) > \sigma(u) \geq \sigma(w_1) \geq \sigma(w_2) \geq \ldots \geq \sigma(v)$$

Impossible cicle

Acyclic Orientations of Graphs

Proof

Case 3 –

$$\sigma(u) = \sigma(v)$$

Both are compatible with σ

At least one is also acyclic. Suppose not, then:

υ_1 contains $u \to v \to w_1 \to w_2 \to \ldots \to u$

υ_2 contains $v \to u \to w_{1'} \to w_{2'} \to \ldots \to v$

Acyclic Orientations of Graphs

Proof

υ_1 contains $u \to v \to w_1 \to w_2 \to \ldots \to u$

υ_2 contains $v \to u \to w_{1'} \to w_{2'} \to \ldots \to v$

$\Downarrow$

υ contains

$u \to w_{1'} \to \ldots \to v \to w_1 \to \ldots \to u$

Impossible cicle

Acyclic Orientations of Graphs

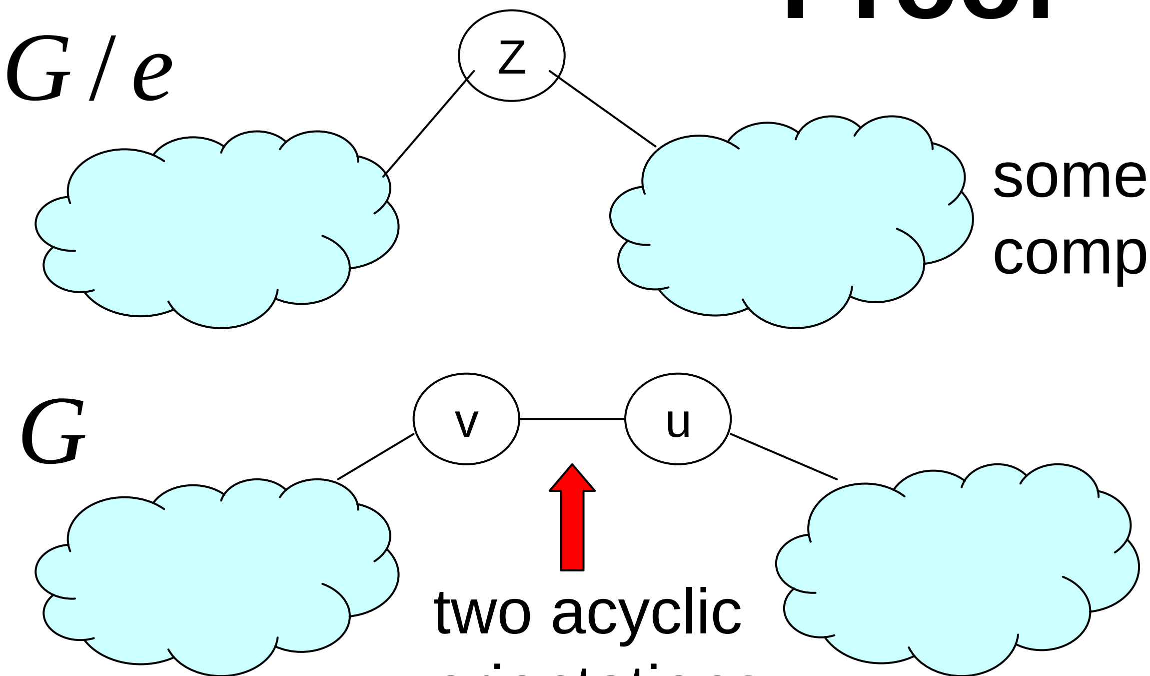
Proof

We now have to show that both υ_1 and υ_2 are acyclic for exactly $\tilde{\chi}(G/e,x)$ pairs of (σ,υ) with $\sigma(u)=\sigma(v)$

Let z denote the vertex identifying {u,v} in G/e

Acyclic Orientations of Graphs

Proof



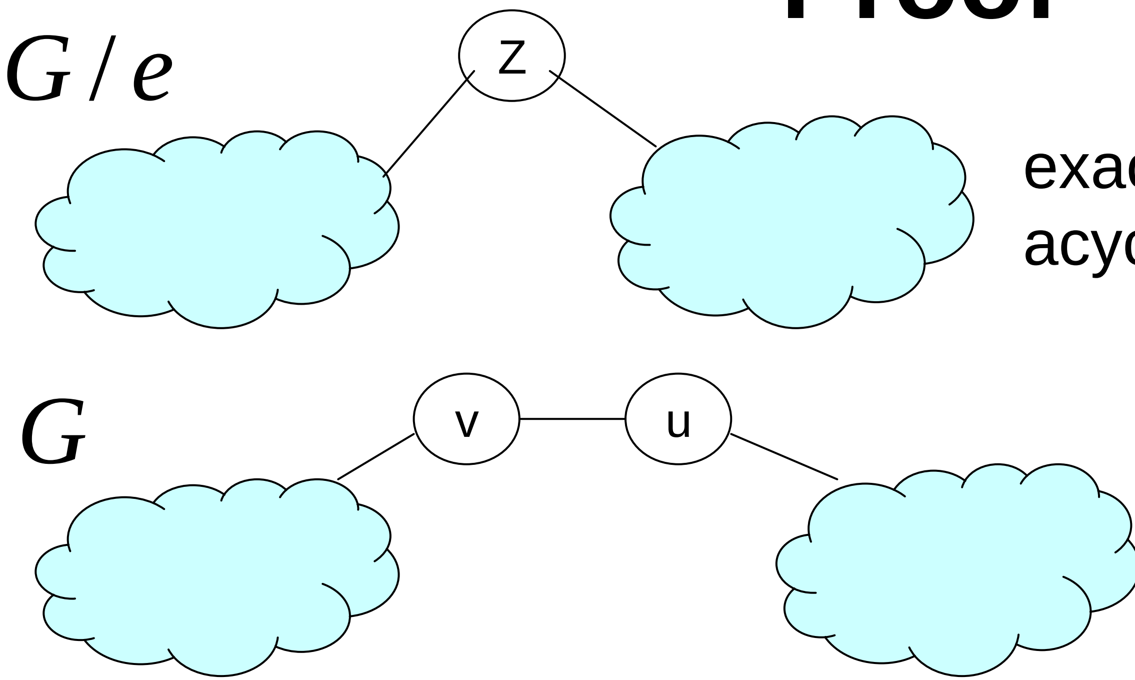
some acyclic orientation compatible with σ

impossible to add a circle by the new edge {u,v}

two acyclic orientations, compatible with σ

Acyclic Orientations of Graphs

Proof



exactly one, necessarily acyclic, compatible with σ

Some two acyclic orientations, compatible with σ

All other vertices of G remains the same

Acyclic Orientations of Graphs

Proof

And so both υ_1 and υ_2 are acyclic for exactly $\tilde{\chi}(G/e,x)$ pairs of (σ,υ) with $\sigma(u)=\sigma(v)$

And so –

$$\tilde{\chi}(G,x)=\tilde{\chi}(G-e,x)-\tilde{\chi}(G/e,x)$$

Acyclic Orientations of Graphs

Proof

It is obvious that for *x = 1* every orientation is compatible with $\sigma : V \to \{1\}$

And so the expression count the number of acyclic orientations in G

Q.E.D