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Recursively enumerable subsets of R^q in two computing models Blum–Shub–Smale machine and Turing machine

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Abstract

In this paper we compare recursively enumerable subsets of R^q in two computing models over real numbers: the Blum–Shub–Smale machine and the oracle Turing machine. We prove that any Turing RE open subset of R^q is a BSS RE set, while a Turing RE closed set may not be a BSS RE set. As an application we show that the Julia set of any computable hyperbolic polynomial is decidable in the Turing computing model. © 1998—Elsevier Science B.V. All rights reserved

1. Introduction

In this paper we compare recursively enumerable subsets of R^q, or RE subsets for abbreviation, defined in two computing models over real numbers. One model is the classical oracle Turing machine, the other is the model introduced by Blum et al. 1989 [2].

The work of Turing, Gödel, Church and others in the 30s forms the core of classical computation theory. Although much of the classical theory of computation deals with computing over the natural numbers, certain approaches have considered other underlying domains. One such approach is recursive analysis, which studies the computability of reals and continuous functions of real variables. The subject is a natural development of computability theory for functions from natural numbers to natural numbers, and has been well studied, e.g. [6, 10]. Let R be the set of real numbers. In recursive analysis the standard definition for "computable open sets" of R^q, now commonly known as recursively enumerable open sets, or RE open sets, goes as follows: an open set U of R^q is RE open if it is the union of a sequence of q-balls, $\{x : |x - a_i| < r_i\}$, where $\{r_i\}$

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and $\{a_i\}$ are computable sequences of real numbers and q-vectors of R^q, respectively [9]. This definition is in accordance with the notion of classical RE sets of the natural numbers, i.e. when the set N of the natural numbers is considered as a subspace of R^q, RE open sets of N are the same as the ordinary RE sets in the classical recursion theory. The computing model adopted here is oracle Turing machines where, roughly, an oracle Turing machine is a classical Turing machine equipped with an oracle which can supply rational approximations of real numbers on demand.

On the other hand, Blum et al. [2] developed a model for computation over ordered rings in 1989. This model provides an interplay between algebra, analysis, scientific computation, and topology. In this computational model, a set $Y \subset R^q$ is RE if it is the halting set of some machine over R. Y is said to be decidable (recursive) if Y and its complement are both RE sets. This theory, as the recursive analysis approach, also reflects the classical computation theory over Z (e.g. in the case where R is the ring of integers Z, the computable functions are ordinary recursive functions, RE sets are ordinary RE sets of Z). By contrast with oracle Turing machines, a real number in this model is viewed as a mathematical entity, not as its decimal or binary expansion. Thus, for example, rather than feed rational approximations of a real number into a machine, one can input real numbers directly into the machine.

Both the Blum–Shub–Smale model (BSS model) and the Turing model bring the theory of computation into the domain of topology and make it possible to investigate the effectiveness of constructions in topology. To begin the study, it is natural to ask: Do these two computational models give the same objects? The answer to this question is No, for while the Cantor middle-third set is a Turing RE closed set (Proposition 3.1 below), it is not a BSS RE set as shown in [2]. Furthermore, it is shown in [2] that most Julia sets are not BSS RE over the reals and therefore are not decidable in the BSS computing model. In contrast, we will show that this is not the case in the Turing computing model. More precisely, we show in Section 4 that the Julia set of any computable hyperbolic polynomial is decidable in the Turing model.

The Julia set example demonstrates how different BSS and Turing RE closed sets can be. On the other hand, there are certain connections between the two models since both of them reflect the classical theory of computation over integers. It is shown that, for example, every Turing RE open set is a BSS RE set (Theorem 3.1 below). Several other connections between BSS RE sets and Turing RE sets are also given in the paper.

The paper is organized as follows. Section 2 contains the definitions for Turing RE open sets, Turing RE closed sets, and BSS RE sets. All sets here are subsets of R^q. In Section 3, we compare BSS RE sets with Turing RE sets. Proposition 3.1, coupled with an example in [2], shows that a Turing RE closed set may not be a BSS RE closed set, while Theorem 3.1 asserts that every Turing RE open set is a BSS RE set. Theorem 3.2 gives a partial answer to the question when a BSS RE open set is Turing RE open. In Section 4, we investigate which of the Julia sets of polynomials are Turing RE closed sets. We first show that the Julia set of any computable hyperbolic polynomial is a Turing RE closed set. Combining this result with the proposition [2]: The basin of attraction of a polynomial is a BSS RE set over R, we further prove that the Julia set of any computable hyperbolic polynomial is in fact decidable in the Turing computing model.

2. Preliminary

We first review the definitions of computable reals, computable functions of real variable, RE (recursively enumerable) open sets, and RE closed sets of R^q in the Turing computing model.

Let N, Z, and R be the sets of natural numbers, integers, and real numbers respectively. We take as known the idea of a recursive function from N^p to N^q and a Turing machine over N. A sequence of rationals $\{r_n\}$ is *computable* if there exist three recursive functions s, b, and c from N to N such that

$$r_n = (-1)^{s(n)}\frac{b(n)}{c(n)},$$

where $c(n) \neq 0$. A computable double or triple sequence of rationals is defined similarly.

Definition 2.1. A real number x is said to be *computable* if there exist both a computable sequence $\{r_n\}$ of rationals and a recursive function e from N to N such that

$$k \geqslant e(n) \quad \text{implies } |x - r_k| \leqslant \frac{1}{2^n}.$$

A point $(x_1, x_2, \ldots, x_q) \in R^q$ is computable if $x_1, x_2, \ldots, x_q$ are computable real numbers.

Definition 2.2. A sequence $\{x_n\}$ of real numbers is *computable* (as a sequence) if there is a computable double sequence $\{r_{nk}\}$ of rationals and a recursive function e from N^2 to N such that

$$|x_n - r_{nk}| \leqslant \frac{1}{2^N} \quad \text{whenever } k \geqslant e(n, N).$$

According to the definition, there are only countably many computable reals and computable sequences of reals. Therefore there exists a real number such that no computable sequence converges to it. Later we will use the following variant of a computable sequence of reals. A sequence $\{x_n\}$ of real numbers is computable if there is a computable double sequence $\{r_{nk}\}$ of rational numbers such that for all $k, n \in N$,

$$|r_{nk} - x_n| \leqslant 2^{-k}.$$

We now introduce the definition of RE and recursive open/closed subsets of R^q.

Definition 2.3. (1) An open set U in R^q is called *recursively enumerable* (or RE in short) [9] if there exist computable sequences $\{x_n\}$ and $\{r_n\}$, $x_n \in R^q$ and $r_n \in (0, \infty)$, such that

$$U = \bigcup_{n=0}^{\infty} \{x \in R^q : |x - x_n| < r_n\}.$$

A closed set K is called *recursively enumerable* [11] if there exists a computable sequence $\{x_n\}$ in K such that the closure of $\{x_n\}$ is K.

(2) An open (respectively closed) set U in R^q is called recursive (or computable, or decidable) if both U and its complement are RE sets [11].

The empty set is accepted as a RE open set as well as a RE closed set. The RE (resp. recursive) sets defined above will be called Turing RE (resp. Turing recursive) sets from now on. Observe that if α is a noncomputable real, then the singleton set $\{\alpha\}$ is not a Turing RE closed set.

In classical recursion theory a recursive set is defined as a set with a computable characteristic function. An alternative description for a recursive set there is that a set is recursive if and only if both the set and its complement are RE. Since computable functions of real variables must be continuous, only trivial characteristic functions are computable. Thus, the alternative approach is used in Definition 2.3(2) to define non-trivial recursive subsets of R^q which is also in accordance with the classical definition. Then in the case where the set N of natural numbers is considered as a subspace of R, we have the same RE sets and recursive sets in N as the classical ones.

We turn now to the definition of computable functions, which is due to Grzegorczyk [7].

Definition 2.4. A function f from R^p to R^q is said to be computable if:
- f *is sequentially computable*, that is, f carries computable sequences $\{x_n\}$ to computable sequences $\{f(x_n)\}$; and
- f *is effectively uniformly continuous*, that is, there is a recursive function e from $N \times N$ to N such that for all points $x, y \in R^p$,

$$|x-y| \leqslant \frac{1}{2^{e(k,n)}} \quad \text{and} \quad x,y \in [-k,k] \quad \text{implies} \quad |f(x)-f(y)| \leqslant \frac{1}{2^n}.$$

As is well known a function f of real variables is determined if we know the values of f on a dense set of points and that f is continuous. The above definition simply effectivizes these notions. For there are only countably many computable sequences of real vectors, the sequential computability implies that there are only countably many computable functions of real variables.

A computable sequence of functions is defined in a similar way just as that was done for computable sequences of reals. According to the definition, it is easy to see that most of the elementary continuous functions, e.g. $e^x, \sin x, \cos x$, etc., are computable. In particular, a polynomial is computable if and only if all of its coefficients are computable numbers.

We now pass from the Turing model to the Blum et al. computing model, which will be called BSS model or BSS machine throughout the paper. We restrict our attention to machines over real numbers with finite dimensions. The reader is referred to [2] for ∞-dimensional machines over an arbitrary ordered ring.

Definition 2.5 (*Blum et al.* [2]). A machine M over R consists of an *input space* $\bar{I}$, *output space* $\bar{O}$ and *state space* $\bar{S}$, together with a connected directed graph whose nodes labeled $1,\ldots,N$ are of certain types and with associated functions. Here $\bar{I},\bar{O}$, and $\bar{S}$ are each R^l, R^m and R^n, respectively, with $l,m,n<\infty$. The directed graph of the machine M has 4 types of nodes as follows:

(a) Exactly one *input node*, characterized as having no incoming edge, and one outgoing edge. Associated with this input node is a linear injective map $I:\bar{I}\to\bar{S}$ (which just takes the input and puts it into the machine), and $\beta(1)$ the *next node*.

(b) *Output nodes*, characterized by having no outgoing edges. To each such node, n, is associated a linear map $O_n:\bar{S}\to\bar{O}$.

(c) *Computing nodes*. Each such node has a single outgoing edge, so that a next node $\beta(n)$ is defined. To n is associated a polynomial map $g_n:\bar{S}\to\bar{S}$.

(d) A *branch node n* has two outgoing edges, giving us next nodes $\beta^-(n)$ and $\beta^+(n)$. To n is associated a polynomial $h_n:\bar{S}\to R$ with $\beta^-(n)$ associated to the condition $h_n(x)<0$, $\beta^+(n)$ to $h_n(x)\geqslant 0$:

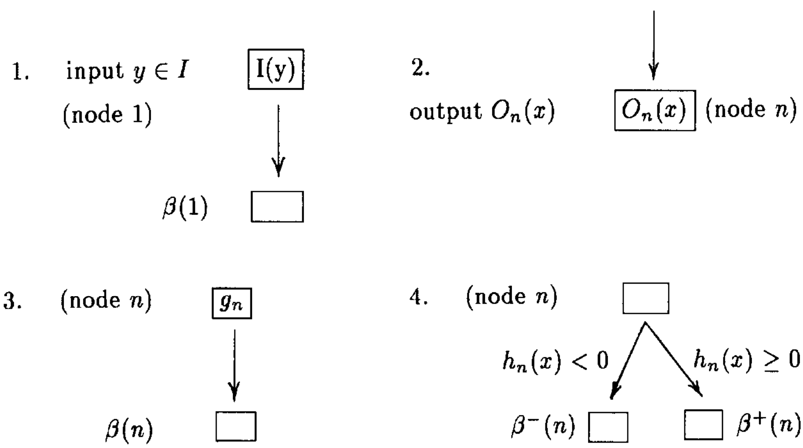

Fig. 1.

Definition 2.6. If we denote $Q_M\subseteq\bar{I}$ as the set of y where the computation halts. Then M defines an input–output map $\varphi_M:Q_M\to\bar{O}$. A function $f:R^l\to R^m$ is called computable over R if there is a machine M over R such that the domain of f is Q_M and $f(x)=\varphi_M(x)$ for all $x\in Q_M$. A set $Y\subset R^l$ is called BSS RE over R if $Y=Q_M$ for some BSS machine M over R. It is said to be *decidable* if both Y and its complement are BSS RE over R.

Definition 2.7. If a BSS RE set in R^l is an open (respectively closed) set, then it is called a BSS RE open (respectively closed) set.

According to the definition, every interval is a BSS RE set. It has been proved in [2] that computable functions over Z in the BSS model are the classical Turing computable functions over Z. This implies that the BSS machine is equivalent to the Turing machine over Z. Therefore, both models define the same RE sets over Z. At this point it is natural to ask whether these two computing models over R give rise the same objects in R^q? We study this problem in the following sections.

3. BSS RE sets and Turing RE sets in R^q

We concentrate on working with subsets of R for simplicity. All results in this section can be easily generalized to subsets of R^q.

Let α be a noncomputable real. Then the singleton set $\{\alpha\}$ is not a Turing RE closed set. It is however a RE closed set in the BSS model, as we have mentioned in the last section. On the other hand, it has been proved in [2] that any BSS RE set over R is a countable union of basic semialgebraic sets. Here a basic semialgebraic set is a subset of R^n defined by a set of polynomial inequalities of the form

$$h_i(x) < 0, \quad i = 1, \ldots, l,$$

$$h_j(x) \leqslant 0, \quad j = l + 1, \ldots, m.$$

Since a basic semialgebraic set only has a finite number of connected components, it follows that the Cantor middle-third set is not a BSS RE closed set as shown in [2]. But the Cantor middle-third set is a Turing RE closed set.

Proposition 3.1. *The Cantor middle-third set C is a Turing RE closed set.*

Proof. Let

$$S_0 = \{1, 2\},$$

$$S_1 = S_0 \cup \{3^2 - a : a \in S_0\},$$

$$S_n = S_{n-1} \cup \{3^{n+1} - a : a \in S_{n-1}\},$$

and for $n = 2, 3, \ldots$, we arrange $S_n = \{a_1^n, \ldots, a_{j(n)}^n\}$ with $j(n) = 2^{n+1}$. Define $e : N \to N$ as

$$e(0) = 2, \quad e(n) = e(n-1) + j(n).$$

Thus, e is a recursive function on N. For each $i = 1, \ldots, j(n)$, let $x_{e(n-1)+i} = a_i^n / 3^{n+1}$. Then $\{x_n\}$ is a computable sequence of rationals because this procedure is effective. Since $\{x_n\}$ is the set of end points of removed middle third intervals, it is dense in C. We therefore conclude that C is a Turing RE closed set. □

The above example demonstrates that Turing RE closed sets and BSS RE closed sets behave quite differently. One reason for such difference is that a Turing RE closed set is defined as the closure of a computable sequence. The closure operator often involves "too much" limit process which is difficult to deal with effectively.

A Turing RE open set $U = \bigcup_{n=0}^{\infty} \{x : |x - x_n| < r_n\}$ behaves more "machine-like". Here, roughly, one might imagine a real number x fed to the machine which computes $|x - x_1|, |x - x_2|, \ldots$; if for some i, $|x - x_i| < r_i$, then the computation halts and the machine outputs x. Otherwise the machine does not halt. The following result thus comes with no surprise.

Theorem 3.1. *Every Turing RE open set is a BSS RE open set.*

Before the proof we recall that an open subset U of R^q is Turing RE open if there exist computable sequences $\{x_n\}$ and $\{a_n\}$, $x_n \in R^q$ and $a_n \in (0, \infty)$, such that

$$U = \bigcup_{n=0}^{\infty} \{x \in R^q : |x - x_n| < a_n\}.$$

Without loss of generality, x_n (resp. a_n) can be assumed to be rational vector (resp. rational). We state and prove this fact in the Lemma 3.1 below.

Lemma 3.1. *An open subset U of R^q is Turing RE open if and only if there exist computable sequences $\{y_n\}$ of rational vectors and $\{r_n\}$ of rationals, $y_n \in R^q$ and $r_n \in (0, \infty)$, such that*

$$U = \bigcup_{n=0}^{\infty} \{x \in R^q : |x - y_n| < r_n\}.$$

Proof. Only necessity need to be proved. Let

$$U = \bigcup_{n=0}^{\infty} \{x : |x - x_n| < a_n\}$$

be a Turing RE open set, where $\{x_n\}$ and $\{a_n\}$ are computable sequences of real vectors and positive real numbers, respectively. Thus, there are two computable double sequences $\{x_{mn}\}$ of rational vectors and $\{a_{mn}\}$ of rationals such that for all $n \in N$,

$$|x_{mn} - x_n| < \frac{1}{2^m}, \qquad |a_{mn} - a_n| < \frac{1}{2^m}.$$

We now write each disk $\{x : |x - x_n| < a_n\}$ as the countable union of the disks $\{x : |x - x_{mn}| < a_{mn} - 1/2^{m-1}\}$ for $m \in N$. (For each $n \in N$, since a_n is a positive real number, there is an $n_0 \in N$ such that for all $m \geqslant n_0$, $a_{mn} - 1/2^{m-1} > 0$. Moreover, since $\{a_n\}$ is a computable sequence, the procedure of determining n_0 on input n is effective. Thus, without loss of generality, we can assume that $a_{mn} - 1/2^{m-1} > 0$ for all $m \in N$.) This can be done because for each x, if $|x - x_n| < a_n$, then there exists an m such that $|x - x_n| < a_n - 1/2^{m-2}$. It follows that $|x - x_{mn}| < |x - x_n| + |x_n - x_{mn}| < a_n - 1/2^{m-2} + (1/2^m) < a_{mn} + (1/2^m) - (1/2^{m-2}) + 1/2^m = a_{mn} - (1/2^{m-1})$. On the other hand, if $|x - x_{mn}| < a_{mn} - 1/2^{m-1}$, we have $|x - x_n| < |x - x_{mn}| + |x_{mn} - x_n| < a_{mn} - (1/2^{m-1}) + 1/2^m = a_{mn} - 1/2^m < a_n$.

Let $\phi : N \to N \times N$ be a recursive ordering on $N \times N$. Then

$$\{y_n\} = \{x_{\phi(n)}\} \quad \text{and} \quad \{r_n\} = \left\{ a_{\phi(n)} - \frac{1}{2^{Proj_1 \phi(n) - 1}} \right\}$$

are computable sequences of rational vectors and rationals, where $Proj_1 \phi$ is the projection of ϕ on its first coordinate, which is a recursive function, and as we have just shown that

$$U = \bigcup_{n=0}^{\infty} \{x \in R^q : |x - y_n| < r_n\}.$$

The proof is complete. □

We now come to the proof of Theorem 3.1. Let

$$U = \bigcup_{n=0}^{\infty} \{x \in R : |x - y_n| < r_n\}$$

be a Turing RE open set, where $\{y_n\}$ and $\{r_n\}$ are computable sequences of rationals. Let f and g be two recursive functions defined on N such that $f(n) = y_n$ and $g(n) = r_n$. In [2] it is proved that the classical recursive functions are BSS computable. Thus both f and g are BSS computable functions.

The following machine halts on an input x if and only if $x \in U$, which proves that U is a BSS RE set. The computation proceeds as follows. Input $x \in R$ as the fourth coordinate of a point in R^4 with the other coordinates being 1. Then compute $f(k)$ and $g(k)$. Replace $y_{k-1} = (k, f(k-1), g(k-1), x)$ by $y_k = (k+1, f(k), g(k), x)$. Output x if $|y_k^4 - y_k^2| < y_k^3$, where $y_k^i = i$th coordinate of y_k. □

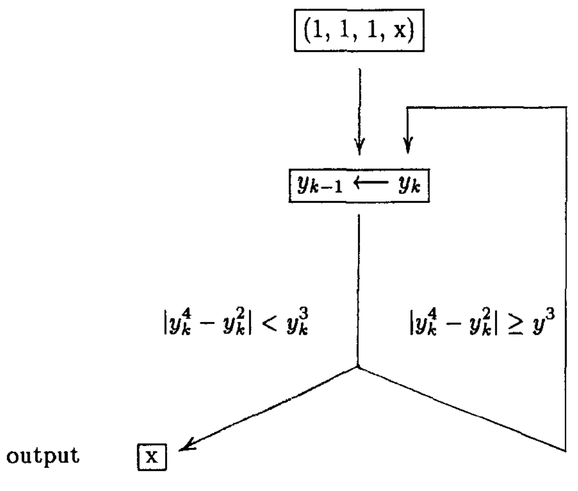

Fig. 2.

While every Turing RE open set is also a BSS RE open set, the converse is not true. For example, for any two real numbers α and β with $\alpha < \beta$, one can construct a BSS machine M such that its halting set Q_M is the open interval (α, β). However, if either

α or β is not Turing computable, the open interval may not necessarily be a Turing RE open set. For instance, let α be a positive real number such that no computable sequence converges to it. Then $(0,\alpha)$ is not a Turing RE open set. For if it is, then $(0,\alpha)=\bigcup_{n=0}^{\infty}\{x:|x-x_n|<r_n\}$, where $\{x_n\}$ and $\{r_n\}$ are two computable sequences of real numbers. Let $\tilde{c}_n=x_n+r_n$, and $c_n=\max(\tilde{c}_0,\dots,\tilde{c}_n)$. Then $\{c_n\}$ is a computable sequence which converges to α. This is a contradiction. The further question is under what conditions a BSS RE open set is Turing RE open? The following theorem provides a partial answer.

Theorem 3.2. *For a BSS RE open set Q_M, if computing nodes $g_1,\dots,g_m$ and branch nodes $h_1,\dots,h_n$ are Turing computable polynomials, and Q_M is defined by inequalities of the type*

$$h_j(g_{k_i}(\cdots g_{k_2}(g_{k_1}(x))))>0 \quad or \quad h_l(g_{k_m}(\cdots g_{k_2}(g_{k_1}(x))))<0,$$

then Q_M is a Turing RE open set.

Proof. We prove the case where M can be described by the following flow chart. The other cases can be proved similarly:

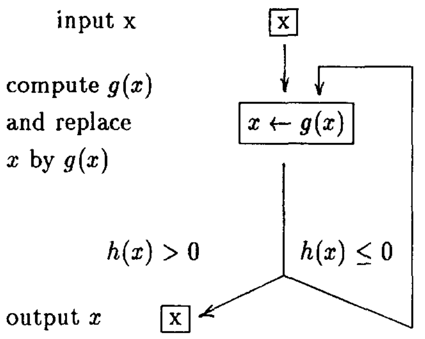

Fig. 3.

The halting set of M is

$$Q_M=\bigcup_{n=0}^{\infty}(h\circ g^n)^{-1}(0,\infty).$$

By the hypothesis both g and h are Turing computable polynomials. Therefore, the sequence $\{h\circ g^n\}$ is a Turing computable sequence of polynomials. Thus, there exists a recursive function $e:N\times N\times N\to N$ such that for all $k,m,n\in N$,

$$|x-y|<\frac{1}{2^{e(k,m,n)}} \quad \text{and} \quad x,y\in[-k,k]$$

imply

$$|h\circ g^n(x)-h\circ g^n(y)|<\frac{1}{2^m}.$$

Let $\{r_l\}$ be the computable sequence of rationals of R. We construct a computable double sequence of rationals in the following way:

$$r_{kl} = \begin{cases} r_l & \text{if } r_l \in (-k,k), \\ 0 & \text{otherwise.} \end{cases}$$

For each $(m,n) \in N \times N$, let $\{r_{kl,mn}\}$ be a subsequence of $\{r_{lk}\}$ such that

$$h(g^n(r_{kl,mn})) > \frac{1}{2^m}.$$

Then $\{r_{kl,mn}\}$ is a quadruple computable sequence, since $\{h(g^n(r_{kl}))\}$ is a computable sequence of rationals and $h(g^n(r_{kl})) > 1/2^m$ can be determined effectively. Let

$$s_{kl,mn} = \frac{1}{2^{e(k,m,n)}} \quad \text{and} \quad x_{kl,mn} = r_{kl,mn}.$$

Then

$$V = \bigcup \{x \in (-k,k) : |x - x_{kl,mn}| < s_{kl,mn},\ k,l,m, \text{ and } n \in N\}$$

is a Turing RE open set. We next show that $V = Q_M$, which will complete the proof. Suppose $x \in Q_M$. Then there exist positive integers m and n such that $h(g^n(x)) > 1/2^{m-1}$. Assume $x \in (-k,k)$. Select a r_{kl} from the dense sequence $\{r_{kl}\}$ satisfying $|r_{kl} - x| < 1/2^{e(k,m,n)}$. As a result we have $|h(g^n(r_{kl})) - h(g^n(x))| < 1/2^m$. This last inequality implies that $h(g^n(r_{kl})) > h(g^n(x)) - 1/2^m > (1/2^{m-1}) - (1/2^m) = 1/2^m$. Therefore $r_{kl} = x_{kl,mn}$ and $x \in (-k,k) \cap \{y : |y - x_{kl,mn}| < s_{kl,mn}\} \subset V$. On the other hand, if $x \in V$, then $x \in (-k,k) \cap \{y : |y - x_{kl,mn}| < s_{kl,mn}\}$ for certain k,l,m, and n. Consequently $|h(g^n(x)) - h(g^n(x_{kl,mn}))| < 1/2^m$, which in turn implies that $h(g^n(x)) > h(g^n(x_{kl,mn})) - 1/2^m > (1/2^m) - (1/2^m) = 0$. Therefore $x \in h \circ g^n(0,\infty) \subset Q_M$. □

Problem 3.1. Now if a BSS RE open set is generated by the flow chart below:

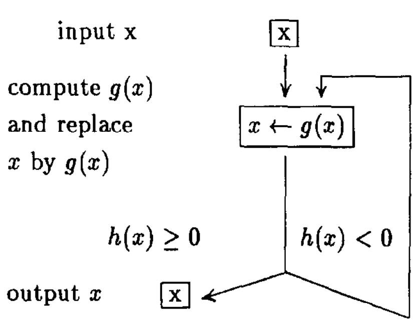

Fig. 4.

Is Q_M a Turing RE open set?

4. Decidable Julia sets

First we give a brief review on the definition of the Julia set of a polynomial in the complex plane. The reader is referred to [5] for more details. For a polynomial g, we say a point $z_0 \in \mathbf{C}$ is periodic (of period n) if $g^n(z_0) = z_0$ for some positive integer n. It is attracting (respectively repelling) if in addition, $|(g^n)'(z_0)| < 1$ (respectively $|(g^n)'(z_0)| > 1$), where $(g^n)'(z_0)$ is the derivative of the nth iterate of g at z_0. If z_0 is attracting of period n, then there is a neighborhood U of z_0 that is attracted into itself under g, that is, $g^n(U) \subset U$; and so if the orbit $z, g(z), g^2(z), \ldots$ of a point z eventually enters U, it will asymptotically approach z_0. Such points are said to be in the basin (of attraction) of z_0. The basin of g is the union of all such basins. For a polynomial g with degree $g \geqslant 2$, the Julia set of g is the closure of the repelling periodic points of g. If a Julia set J is disconnected, then it has uncountably many components. Thus it cannot be a BSS RE closed set, for a BSS RE set can only have countably many components. Moreover it has been shown in [2] that a BSS RE Julia set of a rational map $g : C_\infty \to C_\infty$ is either

(a) empty; and g is a rotation, or a constant; or
(b) a point; and g is fractional linear but not a rotation; or
(c) a real analytic arc; or
(d) a real analytic Jordan curve; or
(e) the whole sphere C_∞.

Thus, most Julia sets are not BSS RE over R and therefore not decidable in the BSS model. In contrast, we prove in this section that the Julia set of any computable hyperbolic polynomial is decidable in the Turing model. A polynomial is hyperbolic if all of its periodic points are hyperbolic, where a periodic point p of period n is hyperbolic if $|(P^n)'(p)| \neq 1$. (Computable polynomials are dense in the set of polynomials and hyperbolic polynomials are open and nonempty in the set of polynomials of each degree.) A polynomial root finding algorithm proposed by Kim [8] will be used.

Let

$$P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

be a computable complex polynomial (i.e. all coefficients of P are computable numbers) of degree at least two with $a_d \neq 0$. Let P^k be the kth iteration of P, then P^k is a polynomial of degree d^k. Since P is computable, the sequence $\{P^k\}$ is computable. As a consequence the sequence $\{(P^k)'\}$ of derivatives of P^k's is computable as well. Since the algorithm proposed in [8] applies only to polynomials with coefficients satisfying $a_d = 1$ and $|a_j| \leqslant 1$ for $0 \leqslant j \leqslant d-1$, the following lemma is needed in our later work.

Lemma 4.1. *Let*

$$P_d(1) = \left\{ f(z) = \sum_{j=0}^{d} a_j z^j, a_d = 1, |a_j| \leqslant 1 \right\}.$$

Then, for the computable sequence $\{P^k\}$ of iterates of P, there are two computable sequences $\{\alpha_k\}$ and $\{\beta_k\}$ of complex numbers such that

$$\bar{P}^k(z) = \alpha_k P^k(\beta_k z) \in P_{d^k}(1),$$

and $\{\bar{P}^k(z)\}$ is a computable sequence of polynomials.

Proof. Assume that

$$P^k(z) = a_{d^k}(k)z^{d^k} + a_{d^k-1}(k)z^{d^k-1} + \cdots + a_1(k)z + a_0(k).$$

Let

$$\beta_k = \max\left\{1, \frac{a_j(k)}{a_{d^k}(k)},\ 0 \leqslant j \leqslant d^k - 1\right\}$$

and

$$\alpha_k = \frac{1}{a_{d^k}(\beta_k)^{d^k}}.$$

Then

$$\bar{P}^k(z) = \alpha_k P^k(\beta_k z) \in P_{d^k}(1).$$

It is obvious that $\{\alpha_k\}$ and $\{\beta_k\}$ are two computable sequences of complex numbers, and $\{\bar{P}^k(z)\}$ is a computable sequence of polynomials. The proof is complete. □

We turn now to the main theorem of this section.

Theorem 4.1. *The Julia set of a computable hyperbolic polynomial g is a Turing RE closed set.*

Proof. If the degree of g is at most 1, the Julia set of g is either empty or a round circle with computable center and radius, and is therefore a Turing RE closed set. Thus we can assume that the degree, d, of g is at least two. In this case, the Julia set of g is the same as the closure of the repelling periodic points of g. Recall that a subset of C_∞ is Turing RE closed if it is the closure of a computable sequence of complex numbers. Therefore, it suffices to prove that the sequence of the repelling periodic points of g is computable. Accordingly, our plan for the proof goes as follows. We first show that the sequence of the periodic points of g is computable. This is Lemma A below. Then we prove that the set of the repelling periodic points is computable. We remark that Lemma A holds not only for computable hyperbolic polynomials, but for all computable polynomials.

Lemma A. *If g is a computable polynomial, then the sequence of the periodic points of g is computable.*

Proof. For $k \in N$, let

$$P_k(z) = g^k(z) - z.$$

Then the sequence $\{P_k\}$ is a computable sequence of polynomials, and the degree of P_k is d^k. It is obvious that the set of the periodic points of g is the same as the set of the roots of $\{P_k\}$. Let

$$\bar{P}_k(z) = \alpha_k P_k(\beta_k z),$$

where α_k and β_k are defined as in Lemma 4.1. Then $\beta_k z$ is a root of P_k if z is a root of $\bar{P}_k$. Let

$$\xi_k = \{\xi_1^{(k)}, \ldots, \xi_{d^k}^{(k)}\}$$

be the set of the roots of $\bar{P}_k$. We use $\{\xi_k\}$ to denote the sequence

$$\{\xi_1^{(1)}, \ldots, \xi_d^{(1)}, \xi_1^{(2)}, \ldots, \xi_{d^2}^{(2)}, \xi_1^{(3)}, \ldots\}.$$

Our objective is to prove that the sequence $\{\xi_k\}$ is a computable sequence. To do this we construct in the following a computable double sequence $\{\boldsymbol{x}_{kn} : n \in N, 1 \leqslant k \leqslant n\}$ such that for all $n \in N, 1 \leqslant k \leqslant n$, and $1 \leqslant j \leqslant d^k$,

$$\boldsymbol{x}_{kn} = \{x_1^{kn}, \ldots, x_{d^k}^{kn}\}$$

and

$$|x_j^{kn} - \xi_j^{(k)}| < \frac{1}{2^n}.$$

It is easy to see that the sequence $\{\boldsymbol{x}_{kn} : n \in N, 1 \leqslant k \leqslant n\}$ converges to $\{\xi_k\}$ uniformly and effectively, which implies in turn that $\{\xi_k\}$ is a computable sequence of complex numbers. Then the sequence $\{\beta_k \xi_k\}$ of the periodic points of g is computable as well (recall that $\beta_k \geqslant 1$ for all $k \in N$). Now the construction.

Construction of $\{\boldsymbol{x}_{kn} : n \in N, 1 \leqslant k \leqslant n\}$: We use the algorithm proposed in [8] to construct the sequence. For $n, k \in N$ with $1 \leqslant k \leqslant n$, we compute
(1) $\mu_{kn} = (1/4d^k \times 2^n)^{d^k}$;
(2) $h_{kn} = 1/16(d^k)^3 \times 1/(2 + 3\log 2d^k + 2n \log 2)$;
(3) $S_{kn} = 4(d^k)^4[9(\log d^k)^2 + 4n^2(\log 2)^2 + 23n \log 2 \log d^k]$.
Then begin with $t = 1$, $1 \leqslant t \leqslant d^k$, we compute
(4) $z_{knt} = (z_{knt1}, \ldots, z_{knt(d^k)})$, where $z_{kntj} = 5d^k \mathrm{e}^{2\pi\mathrm{i}(t+jd^k)/(d^k)^2}$;
(5) $x_{kntj} = N_{g_{knt}, h_{kn}}^{s_{kn}}(z_{nktj})$ where

$$N_{g_{knt}, h_{kn}}(z_{nktj}) = z_{nktj} - h_{kn} \frac{g_{knt}(z_{nktj})}{g'_{knt}(z_{kntj})},$$

$$g_{knt}(z_{kntj}) = \bar{P}_k(z_{kntj}) - \mu_{kn} \mathrm{e}^{2\pi\mathrm{i}t/d^k},$$

and $N_{g_{knt}, h_{kn}}^{s_{kn}}(z_{kntj})$ denotes the s_{kn}th iteration of $N_{g_{knt}, h_{kn}}(z_{nktj})$.

(6) Finally, we compute $|\bar{P}_k(z)-\prod_{j=1}^{d^k}(z-x_{kntj})|$, where $|\sum_{i=0}^{d} a_i z^i - \sum_{i=0}^{d} b_i z^i| = \sum_{i=0}^{d} |a_i - b_i| 2^i$, and test whether

$$\left|\bar{P}_k(z) - \prod_{j=1}^{d^k} (z - x_{kntj})\right| < \mu_{kn}.$$

If $|\bar{P}_k(z) - \prod_{j=1}^{d^k}(z - x_{kntj})| < \mu_{kn}$, then outputs $\boldsymbol{x}_{kn} = (x_{knt1}, \ldots, x_{knt(d^k)})$. Otherwise return to the computations (4), (5), and (6) for $t+1$.

It is proved in [8] that there is at least one t with $1 \leqslant t \leqslant d^k$ such that $|\bar{P}_k(z) - \prod_{j=1}^{d^k}(z - x_{kntj})| < \mu_{kn}$, and if the inequality is satisfied, then for all $1 \leqslant j \leqslant d^k$,

$$|x_{kntj} - \xi_j^{(k)}| < \frac{1}{2^n}.$$

In other words, for every input $n, k \in N$ with $1 \leqslant k \leqslant n$, the computation halts and outputs $\boldsymbol{x}_{kn} = (x_1^{kn}, \ldots, x_{d^k}^{kn}) = (x_{knt1}, \ldots, x_{knt(d^k)})$ with $|x_j^{kn} - \xi_j^{(k)}| < 1/2^n$. The construction of $\{\boldsymbol{x}_{kn}\}$ is thus completed. It is obvious from the construction that the sequence $\{\boldsymbol{x}_{kn}\}$ has the properties we asked for. The proof of the lemma is complete. □

In Lemma B below, we prove that the set of the repelling periodic points of g is a computable sequence. Thus, as the closure of it, the Julia set of g is a Turing RE closed set.

Lemma B. *Let g be a computable hyperbolic polynomial and let*

$$\{\beta_k \xi_j^{(k)} : |(g^k)'(\beta_k \xi_j^{(k)})| > 1\} \tag{$*$}$$

be the set of the repelling periodic points of g. Then $(*)$ is a computable subsequence of $\{\beta_k \xi_k\}$; here $\{\beta_k \xi_k\}$ is the set of the periodic points of g.

Proof. Since the sequence $\{g^k\}$ of the iterates of g is computable, the sequence $\{(g^k)'\}$ of the derivative of g^k is computable. Thus, there exists a recursive function $e : N \times N \to N$ such that for all $k, n \in N$,

$$|z_1 - z_2| < \frac{1}{2^{e(k,n)}} \text{ implies } |(g^k)'(z_1) - (g^k)'(z_2)| < \frac{1}{2^n}. \tag{$**$}$$

Let $a(k,n) = \beta_k + e(k,n)$. Since $\{\beta_k\}$ is a computable sequence, a is a recursive function from $N \times N$ to N, and $|z_1 - z_2| < \beta_k / 2^{a(k,n)}$ would imply $|z_1 - z_2| < 1/2^{e(k,n)}$. In Lemma A, we have shown that the computable double sequence $\{\beta_k \boldsymbol{x}_{kn} : n \in N, 1 \leqslant k \leqslant n\}$ converges uniformly and effectively to $\{\beta_k \xi k\}$, and for all $n \in N, 1 \leqslant k \leqslant n$, and $1 \leqslant j \leqslant d^k$,

$$|x_j^{kn} - \xi_j^{(k)}| < \frac{1}{2^n}. \tag{$***$}$$

Combining $(**)$ and $(***)$, we have

$$|(g^k)'(\beta_k x_j^{ka(k,n)}) - (g^k)'(\beta_k \xi_j^{(k)})| < \frac{1}{2^n}$$

which implies

$$|(g^k)'(\beta_k \xi_j^{(k)})| < |(g^k)'(\beta_k x_j^{ka(k,n)})| + \frac{1}{2^n}$$

and

$$|(g^k)'(\beta_k \xi_j^{(k)})| > |(g^k)'(\beta_k x_j^{ka(k,n)})| - \frac{1}{2^n}.$$

We now compute

$$|(g^k)'(\beta_k x_j^{ka(k,n)})| - \frac{1}{2^n} \quad \text{and} \quad |(g^k)'(\beta_k x_j^{ka(k,n)})| + \frac{1}{2^n}.$$

If $|(g^k)'(\beta_k x_j^{ka(k,n)})| - 1/2^n > 1$ for some $n \in N$, then $\beta_k \xi_j^{(k)}$ is a repelling periodic point; on the other hand, if $|(g^k)'(\beta_k x_j^{ka(k,n)})| + 1/2^n < 1$ for some $n \in N$, then $\beta_k \xi_j^{(k)}$ is an attracting periodic point. Since g is a hyperbolic polynomial, this determining process will halt on some $n \in N$. For if it is not, we then have, for all $n \in N$,

$$|(g^k)'(\beta_k x_j^{ka(k,n)})| - \frac{1}{2^n} \leqslant 1$$

and

$$|(g^k)'(\beta_k x_j^{ka(k,n)})| + \frac{1}{2^n} \geqslant 1,$$

which would imply that $\lim_{n \to \infty} |(g^k)'(\beta_k x_j^{ka(k,n)})| = |(g^k)'(\beta_k \xi_j^{(k)})| = 1$. This contradicts to the assumption that g is a hyperbolic polynomial. Therefore, the subsequence

$$\{\beta_k \xi_j^{(k)} : |(g^k)'(\beta_k \xi_j^{(k)})| > 1\}$$

is computable. The proof is complete. □

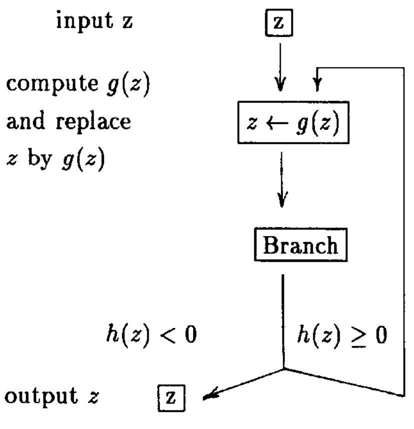

Fig. 5.

For a polynomial g, the set of its attracting periodic points is finite. In [2], it is proved that the basin of attraction of g is a BSS RE set over R (Proposition 3), which is the halting set Q_M of the following machine M (see Fig. 5), where h is a real polynomial (of two real variables) with the property that $h(z) < 0$ if and only if z belongs to a finite union of discs around the attracting periodic points which is contracted into itself by g.

If g is a computable polynomial, then attracting points of g are computable, and the polynomial h can be constructed as a computable real polynomial. This implies, by Theorem 3.2, that the basin of g is a Turing RE open set.

If, in addition, g is a hyperbolic polynomial, then the Julia set of g is the complement of the basin of attraction of g. Proposition 3 of [2] together with Theorem 3.2 imply that the basin of g is a Turing RE open set, while Theorem 4.1 says the Julia set of g is a Turing RE closed set. Accordingly the following result follows.

Theorem 4.2. *The Julia set of any computable hyperbolic polynomial is decidable in the Turing computing model.*

References

[1] E.A. Bishop, D.S. Bridges, Constructive Analysis, Springer, Berlin, 1985.
[2] L. Blum, M. Shub, S. Smale, On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines, Bull. Amer. Math. Soc.(NS) 21 (1989) 1–46.
[3] N. Cutland, Computability, Cambridge Univ. Press, Cambridge, 1980.
[4] R. Engelking, General Topology, Polish Scientific Publishers, 1977.
[5] K. Falconer, Fractal Geometry, Wiley, New York, 1990.
[6] H. Friedman, K. Ko, Computational complexity of real functions, Theoret. Comput. Sci. 20 (1982) 323–352.
[7] A. Grzegorczyk, Computable functionals, Fundam. Math. 42 (1955) 168–202.
[8] M.H. Kim, Topological complexity of a root finding algorithm, J. Complexity 5 (1989) 331–344.
[9] G. Kreisel, D. Lacombe, Ensembles Récursivement measurables et ensembles récursivement ouverts oufermés, C.R. Acad. Sci. Paris 245 (1957) 1106–1109.
[10] M.B. Pour-El, J.I. Richards, Computability in Analysis and Physics, Springer, Berlin, 1989.
[11] J.I. Richards, Q. Zhou, Computability on open and closed subsets of euclidean spaces, preprint.