Lecture 13

What we have done in Lecture 12:

- We computed graph polynomials of recursively defined families of graphs
- We introduced the notion of definability of graph polynomials.
- We made the notion of recursively defined families of graphs precise.
- We stated a general theorem.
- We sketched its proof using the
  bilinear form of the Feferman-Vaught Theorem for graph polynomials.

Outline of Lecture 13

- The History of the Feferman-Vaught Theorem
- Proofs using Pebble Games
- Proofs using reduction sequences
- Tree-width of graphs
- Courcelle's Theorem
- Generalization of the Feferman-Vaught Theorem to graph polynomials

Algorithmic Uses of the Feferman-Vaught Theorem

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Vocabularies, structures and theories

A reminder

- Let τ vocabulary (or a *similarity type* as Tarski used to call it) given by a set of relation symbols, but no function symbols nor constants.
- $FOL(\tau)$ denotes the set of τ-formulas in First Order Logic.
- $SOL(\tau)$ and $MSOL(\tau)$ denote the set of τ-formulas in Second Order and Monadic Second Order Logic.
- For a class of τ-structures K $Th_{FOL}(K)$ is the set of sentences of $FOL(\tau)$ true in all $\mathfrak{A} \in K$.
- We write $Th_{FOL}(\mathfrak{A})$ for $K = \{\mathfrak{A}\}$.

  Similarly, $Th_{SOL}(K)$ and $Th_{MSOL}(K)$ for SOL and $MSOL$.

A. Tarski and E.W. Beth

Tarski published four short abstracts on model theory in 1949 (Bulletin of the AMS, vol. 55) and had sent his manuscript for

Contribution to the theory of models, I

to E.W. Beth for publication.

Inspired by these, E.W. Beth published two papers on model theory.
In one of them he showed that

Theorem 1 (Beth 1952)

Let $\mathfrak{A}, \mathfrak{B}$ *be linear orders,* $\mathfrak{C} = \mathfrak{A} \sqcup_{<} \mathfrak{B}$ *their ordered disjoint union. Then* $Th_{FOL}(\mathfrak{C})$ *is uniquely determined by* $Th_{FOL}(\mathfrak{A})$ *and* $Th_{FOL}(\mathfrak{B})$*.*

We know how to prove this using Pebble Games.

In Tarski's school it was asked in early 1950

Let $\mathfrak{A}, \mathfrak{B}$ be two τ-structures,
$\mathfrak{A} \times \mathfrak{B}$ their cartesian product and $\mathfrak{A} \sqcup \mathfrak{B}$ their disjoint union.

> Assume we are given $Th_{FOL}(\mathfrak{A})$ and $Th_{FOL}(\mathfrak{B})$.
>
> What can we say about $Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$?
>
> What happens in the case of infinite sums and products?

This triggered many landmark papers.

It also lead to the study of ultraproducts.

Tarski's pupils dealing with this question

1938 Andrzei Mostowski
A. Mostowski, On direct product theories, JSL 17 (1952), pp. 1-31

1952 Anne Morel
T.E. Frayne, A.C. Morel and D.S. Scott, Reduced direct products, Fundamenta Mathematicae 51 (1962), pp.195-228

1954 Robert Vaught

1957 Solomon Feferman
S. Feferman and R.L. Vaught, The first order properties of algebraic systems, Fundamenta Mathematicae 47 (1959), pp. 57-103

1961 Jerome Keisler
Many papers exploiting ultraproducts

Feferman and Vaught answered

but Tarski did not really appreciate it

Theorem 2 (Feferman and Vaught, 1959)
Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures.

$Th_{FOL}(\mathfrak{A} \times \mathfrak{B})$ *and* $Th_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$ *are uniquely determined by* $Th_{FOL}(\mathfrak{A})$ *and* $Th_{FOL}(\mathfrak{B})$*.*

- For $MSOL$ still true for $\mathfrak{A} \sqcup \mathfrak{B}$.
  Ehrenfeucht, Läuchli, Shelah, Gurevich
- Not true for SOL
- By combining it with transductions and interpretations true for a wide variety of **generalized products**.
- Also true for infinite generalized sums and products provided the index structures are sufficiently $MSOL$ indistinguishable.

Ehrenfeucht's proof Theorem 2

Use Ehrenfeucht-Fraïssé Games !

This gives actually more: Let $q \in \mathbb{N}$ and $FOL^q(\tau)$ denote the set sentences of $FOL(\tau)$ of quantifier rank at most q.
Put $Th^q_{FOL}(\mathfrak{A}) = Th_{FOL}(\mathfrak{A}) \cap FOL^q(\tau)$.

Theorem 3 (Feferman and Vaught, 1959)
$Th^q_{FOL}(\mathfrak{A} \times \mathfrak{B})$ and $Th^q_{FOL}(\mathfrak{A} \sqcup \mathfrak{B})$ are uniquely determined by $Th^q_{FOL}(\mathfrak{A})$ and $Th^q_{FOL}(\mathfrak{B})$.

For $MSOL$ still true for $\mathfrak{A} \sqcup \mathfrak{B}$.

Feferman and Vaught's proof

Use Reduction Sequences !

Here one proves by induction, say for disjoint union

Theorem 4 (Feferman and Vaught, 1959)
For every formula $\phi \in FOL^q(\tau)$

(i) one can compute a sequence of formulas

$$\langle \psi_1^A, \ldots \psi_m^A, \psi_1^B, \ldots \psi_m^B \rangle \in FOL^q(\tau)^{2m}$$

(ii) and a boolean function $B_\phi : \{0,1\}^{2m} \rightarrow \{0,1\}$ *such that*

$$\mathfrak{A} \sqcup \mathfrak{B} \models \phi$$

iff

$$B_\phi(b_1^A, \ldots b_m^A, b_1^B, \ldots b_m^B) = 1$$

where $b_j^A = 1$ *iff* $\mathfrak{A} \models \psi_j^A$ *and* $b_j^B = 1$ *iff* $\mathfrak{B} \models \psi_j^B$

Similarly for $MSOL$

Proof of Theorem 4, I

We exemplify the construction of such a reduction sequence in the case of ordered graphs $G = \langle V, E, < \rangle$ with V the set of vertices, E a binary, not necessarily symmetric, edge relation and $<$ a linear ordering of the vertices.

We are given

$$G_1 = \langle V_1, E_1, <_1 \rangle \text{ and } G_2 = \langle V_2, E_2, <_2 \rangle$$

and their ordered sum

$$G = G_1 \oplus_< G_2$$

with $V = V_1 \sqcup V_2$, $E = E_1 \sqcup E_2$ and the order is defined by

$$< = <_1 \sqcup <_2 \sqcup (V_1 \times V_2)$$

We construct reduction sequences and boolean functions by induction.

As we have free variables in the inductive construction, we assume that

$$z : Variables \to V = V_1 \sqcup V_2$$

is an assignment of the variables.

Proof of Theorem 4, II

There are three types of atomic formulas:

$$E(u,v)$$

$$u < v$$

$$\approx v$$

Using the defintion of ordered sum we get

For $E(u,v)$
Reduction sequence: $\langle E_1(u,v), E_2(u,v)\rangle$
Boolean function: $b_1^1 \vee b_1^2$.

Here only the cases where $z(u)$ and $z(v)$ are both in V_1 or both in V_2 are relevant.

Proof of Theorem 4, III

Recall that the upper index in b_i^j refers to the structure in which we check and the lower index to the reduction formula.

For $u \approx v$

Reduction sequence: $\langle u \approx_1 v, u \approx_2 v \rangle$
Boolean function: $b_1^1 \vee b_1^2$.

Again, only the cases where $z(u)$ and $z(v)$ are both in V_1 or both in V_2 are relevant.

For $u < v$

Reduction sequence: $\langle u <_1 v, u \approx_1 u, u <_2 v, v \approx_2 v \rangle$.
Boolean function: $b_1^1 \vee b_1^2 \vee (b_2^1 \wedge b_2^2)$.

Here, the relevant cases are $z(u)$ and $z(v)$ are both in V_1 or both in V_2, or $z(u)$ is in V_1 and $z(v)$ is in V_2.

Proof of Theorem 4, IV

Let

$$\Phi = \langle \phi_1^A, \ldots \phi_m^A, \phi_1^B, \ldots \phi_m^B \rangle$$

and

$$\Psi = \langle \psi_1^A, \ldots \psi_n^A, \psi_1^B, \ldots \psi_n^B \rangle$$

be reduction sequences for ϕ and ψ and $B_\phi(\bar{b})$ and $B_\psi(\bar{b}')$ the corresponding boolean functions with disjoint variables.

$(\phi \wedge \psi)$

Reduction sequence: $\langle \Phi, \Psi \rangle$.
Boolean function: $B_\phi(\bar{b}) \wedge B_\psi(\bar{b}')$.

$\neg \phi$

reduction sequences: Φ.
Boolean function: $\neg B_\phi(\bar{b})$.

Each application of a propositional connectives results in linear growth of the reduction sequence. The case of quantifcation is considerably more complicated.

Proof of Theorem 4, V

Let B_1 be the disjunctive normal form of $B_\phi(\bar{b})$ with $B_1 = \bigvee_{j \in J} C_j$ with

$$C_j = \left(\bigwedge_{i \in J(i,A,pos)} b_i^A \wedge \bigwedge_{i \in J(i,A,neg)} \neg b_i^A \wedge \bigwedge_{i \in J(i,B,pos)} b_i^B \wedge \bigwedge_{i \in J(i,B,neg)} \neg b_i^B \right)$$

Now let

$$\theta_j^A = \exists x \left(\bigwedge_{i \in J(i,A,pos)} \phi_i^A(x) \wedge \bigwedge_{i \in J(i,A,neg)} \neg \phi_i^A(x) \right) \quad \text{and} \quad \theta_j^B = \exists y \left(\bigwedge_{i \in J(i,B,pos)} \phi_i^A(y) \wedge \bigwedge_{i \in J(i,B,neg)} \neg \phi_i^A(y) \right)$$

Finally we put

$$B_\exists(\bar{c}) = \bigvee_{j \in J} (c_j^A \vee c_j^B)$$

where $c_j^A = 1$ iff $\mathfrak{A} \models \theta_j^A$ and $c_j^B = 1$ iff $\mathfrak{B} \models \theta_j^B$.

With this notation, and $m(J) = \mid J \mid$, it is easy to verify that for

$\exists x \phi$

Reduction sequence: $\langle \theta_1^A, \ldots \theta_{m(J)}^A, \theta_1^B, \ldots \theta_{m(J)}^B \rangle$.

Boolean function: $B_\exists(\bar{c})$.

Proof of Theorem 4, VII

For $MSOL$ the proof is similar. Again, we need that every $X \subseteq V$ has a *unique* decomposition $X = X_1 \sqcup X_2$ with $X_i \subseteq V_i$.

The additional clauses in the induction are:

$u \in X$

Reduction sequence: $\langle u \in X_1, u \in X_2 \rangle$.

Boolean function: $b_1^1, \vee b_2^1$.

Proof of Theorem 4, VIII

$\exists X \phi$

With the same notation is in the case of first order existential quantification we first put

$$\theta_j^A = \exists X_1 \left(\bigwedge_{i \in J(i,A,pos)} \phi_i^A(X_1) \wedge \bigwedge_{i \in J(i,A,neg)} \neg \phi_i^A(X_1) \right)$$

and

$$\theta_j^B = \exists X_2 \left(\bigwedge_{i \in J(i,B,pos)} \phi_i^A(X_2) \wedge \bigwedge_{i \in J(i,B,neg)} \neg \phi_i^A(X_2) \right)$$

and we get:
Reduction sequence:
$\langle \theta_1^A, \ldots \theta_{m(J)}^A, \theta_1^B, \ldots \theta_{m(J)}^B \rangle$.
Boolean function: $B_\exists(\bar{c})$.

Q.E.D.

Proof of Theorem 4, Comments

- The quantification step can be simultaneously performed for a

  block of existential quantifiers.

- Hence, if we have a formula ϕ in prenex normal form, the time complexity of computing the reduction sequence, and its length, is an

  iterated exponential of the number of quantifier alternations,

  rather than of the quantifier rank.

Research problems

- Find, if possible, a

  better algorithms

  for computing the reduction sequences of quantified formulas, which avoids the computation of the disjunctive normal forms.

- Find

  sharp upper and lower bounds

  for the complexity of computing the reduction sequences.

There was some progress in these questions.

Papers by J. Flum, M. Grohe, S. Kreutzer and more.

Beyond disjoint unions and ordered sums:

Operations on coloured, (un)-directed graphs.

We look at possibly coloured, (un)-directed graphs.

Instead for the disjoint union we can prove Theorem 3 and 4 for the following operations:

- Concatenation of words, $v \circ w$.

- Joining two trees at a new common root, $T_1 \bullet T_2$.

- H-sums of graphs:

  For $i = 1, 2$, let $G_i = \langle V(G_i), E(G_i) \rangle$
  and $V(G_1) \cap V(G_2) = V(H)$ and $E(H) = E(G_1) \cap V(H)^2 = E(G_2) \cap V(H)^2$.

  Then $G = G_1 \oplus_H G_2$ is given by $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

  and similarly for edge and vertex coloured graphs

$MSOL$-smooth operations

We generalize the previous operations
to operations satisfying Theorem 3 or 4:

Definition 5

(i) *A n-ary operation $\mathcal{O}$ on τ-structures is* $MSOL$-**smooth** *if for every $q \in \mathbb{N}$ and every $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$*

$$Th^q_{MSOL}(\mathcal{O}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n))$$

depends only on $Th^q_{MSOL}(\mathfrak{A}_i)$ for $1 \leq i \leq n$.

Note: Same q in condition and conclusion!

(ii) *$\mathcal{O}$ is* **effectively** $MSOL$-**smooth** *if there is an algorithm which computes for every $\phi \in MSOL(\tau)$ a* **reduction sequence**, *i.e. a sequence of formulas as described in theorem C.*

Examples of $MSOL$-smooth operations

- The disjoint union is effectively $MSOL$-smooth.
- Quantifier free $MSOL$-**transductions** are effectively $MSOL$-smooth.

The Fuse operation, I

Let $\mathfrak{A}$ be a τ-structure with universe A. We denote the interpretations of the symbols in τ by $P_1^A, \dots, P_m^A, R_1^A, \dots, R_n^A$ respectively.

Definition 6
Let ρ be the maximal arity of the relation symbols in τ, and $\mathfrak{A}$ be a τ-structure. Assume $P_i^A \neq \emptyset$. The structure $Fuse_i(\mathfrak{A}) = \mathfrak{B}$ is defined as follows

(i) $B = (A - P_i^A) \cup \{p_i\}$ *where p_i is an element not in A.*

(ii) $P_i^B = \{p_i\}$

(iii) *For* $j \neq i$ $P_j^B = (P_j^A \cap B) \cup \{p_i\}$ *if* $P_i^A \cap P_j^A \neq \emptyset$.
Otherwise, $P_j^B = P_j^A \cap B = P_j^A$.

(iv) *For binary relation symbols R_j,*
$R_j^B = (R_j^A \cap B^2) \cup \{(b, p_i) : b \in B \wedge \exists x \in P_i^A$ *with* $(b, x) \in R_j^A\}$
$\cup \{(p_i, b) : b \in B \wedge \exists x \in P_i^A$ *with* $(x, b) \in R_j^A\}$
$\cup \{(p_i, b) : b \in B \wedge \exists x \in P_i^A$ *with* $(x, b) \in R_j^A\}$

The Fuse operation, II

Definition 6 continued:

For relations of arity ≥ 3 we have:

(iv) For r-ary relation symbols R_j,
$R_j^B = (R_j^A \cap B^r) \cup \bigcup_{I \subseteq \{1,\ldots,r\}} X_I$, where
$X_I = \{vec_I(\bar{b}, \bar{p}_i) : \bar{b} \in B^{r-|I|} \wedge \exists \bar{x} \in (P_i^A)^{|I|}$ with $vec_I(\bar{b}, \bar{x}) \in R_j^A\}$.
Here $vec_I(\bar{c}, \bar{d}) \in C^r$ is the shuffling of $\bar{c} \in C^{r-|I|}$ and $\bar{d} \in C^{|I|}$, without changing the order of the coordinates of $\bar{c}$ respectively $\bar{d}$.

If $P_i^A = \emptyset$, $Fuse_i(\mathfrak{A}) = \mathfrak{A}$.

We note that this definition is of the form $\Phi^\star_{Fuse_i}$ for a many-sorted translation scheme of quantifier rank $\rho - 1$. Hence we have

Proposition 7
$Th^m_{MSOL}(Fuse_{P_i}(\mathfrak{A}))$ depends only on $Th^{m+\rho-1}_{MSOL}(\mathfrak{A})$.

Furthermore, $\theta \in Th^m_{MSOL}(Fuse_{P_i}(\mathfrak{A}))$ can be computed by checking whether $\Phi^\sharp_{Fuse_i}(\theta) \in Th^{m+\rho-1}_{MSOL}(\mathfrak{A})$.

The Fuse operation, III

It is a somewhat surprising fact that Proposition 7 can be improved if we have only unary and binary relation symbols.

Proposition 8
Assume τ contains only unary and binary relation symbols.

Then for every τ-structure $\mathfrak{A}$ the theory $Th^m_{MSOL}(Fuse_{P_i}(\mathfrak{A}))$ depends only on the theory $Th^m_{MSOL}(\mathfrak{A})$.

Proof of Proposition 8

Assume we have a winning strategy W of the Ehrenfeucht-Fraïssé games for $Th^m_{MSOL}(\mathfrak{A})$ for m moves.

We define a winning strategy W' for $Th^m_{MSOL}(Fuse_{P_i}(\mathfrak{A}))$. Assume k moves for W' have been defined.

(i) Assume p_i, respectively p'_i was not yet chosen. If in the move $(k+1)$ player I chooses an element, say $a \in A - P_i^A$, player II answers with the same element as prescribed by W for move $(k+1)$.

(ii) If I chooses p_i or p'_i then II replies always with p'_i respectively p_i, independently of the previous choices.

(iii) Assume p_i, respectively p'_i was already chosen in a previous move and now it is move $(k+1)$. If in the move $(k+1)$ player I chooses an element, say $a \in A - P_i^A$, player II answers with the same element as prescribed by W for move k, disregarding the choice of p_i and p'_i.

(iv) For set moves $U \subseteq A - P_i^A$ we again use W.

(v) For set moves U containing p_i, respectively p'_i we split accordingly.

Proof of Proposition 8, continued

We have to show that W' is indeed a winning strategy.

- W.l.o.g. we can assume that p_i and p'_i were chosen only once and in fact in the last move.

  This is so, as W' is not affected by the choice of p_i and p'_i.

- The only way, W' could not be a partial isomorphism is that for some R_j, a_k, a'_k chosen in move k we have $(a_k, p_i) \in R_j^A$ but $(a'_k, p'_i) \notin R_j^{A'}$.

  But then we can show that W is not a winning strategy by chosing as the last move some a_m such that $(a_k, a_m) \in R_j^A$ for which II has no reply, as no such a'_m exists.

$Q.E.D$

$MSOL$-inductive classes (graph grammars)

Definition 9

(i) *A class K of τ-structures is* **$MSOL$-inductiv** *if it is defined inductively using a finite set of $MSOL$-smooth operations.*

(ii) *K is* **effectively $MSOL$-inductiv** *if it is defined inductively using a finite set of effectively $MSOL$-smooth operations.*

Effectively $MSOL$-inductive classes of structures

Examples

- Words $\Sigma^\star$ are defined inductively by
  (i) the empty word is a word
  (ii) one letter words are words
  (iii) words are closed under concatenation

- Coloured trees (forests) are defined similary:
  (i) one leave trees are trees
  (ii) trees are closed under root joining
  (iii) forests are closed under disjoint unions

- Series-parallel (SP) graphs are defined by
  (i) one edge graphs are SP.
  (ii) SP graphs are closed under disjoint unions
  (iii) SP graphs are closed under H-sums for all H with at most two vertices.

More effectively $MSOL$-inductive classes

- Graphs of **tree width at most** k TW_k can be defined inductively by looking at vertex coloured graphs with at most $k+1$ colours:
  (i) All graphs with at most $k+1$ vertices are in TW_k.
  (ii) TW_k is closed under disjoint union.
  (iii) TW_k is closed under renaming of colours.
  (iv) TW_k is closed under **fusion**, i.e. contraction of all vertices of a specific colour into one vertex.

- Similarly, for graphs of **clique width at most** k CW_k
  (i) All graphs with at most 1 vertex are in CW_k.
  (ii) CW_k is closed under disjoint union.
  (iii) CW_k is closed under renaming of colours.
  (iv) CW_k is closed under adding all possible edges between to sets of differently coloured edes.

Open problem:

- Are there more examples of $\mathbf{MSOL}$-smooth operations?
- Are there $MSOL$-smooth operations which are not effectively $MSOL$-smooth ?
- Are there $MSOL$-inductive classes K which are not effectively $MSOL$-inductive ?

Decidable theories, I

The following were shown to be decidable by Büchi and Rabin respectively:

- The $MSOL$ theory of words
- The $MSOL$ theory of trees

We can use theorem C to show that the following $MSOL$ theories are decidable.

- The $MSOL$-theory SP-graphs
- The $MSOL$-theory graphs of bounded tree width

Decidable theories, II

We can generalize this to

Theorem 10 (Courcelle and Makowsky, 2001)
Let K *be* $MSOL$*-inductiv using disjoint unions, fusions and quantifier free* $MSOL$*-transductions.*
Then $Th_{MSOL}(K)$ *is decidable.*

Proof idea:

One shows that an $MSOL$-inductive class K is always an $MSOL$-**transduction** of a class of trees.

Then one applies Rabin's theorem for trees.

Seese 1991 showed it for K of bounded tree width.

Model checking, I

Model checking is the problem to check

$$\mathfrak{A} \models \phi$$

for $\mathfrak{A}$ a finite τ structure and $\phi \in SOL(\tau)$.

We measure the problem in the size of $\mathfrak{A}$ and ϕ (combined case) or for specific ϕ.

Theorem 11 (M. and Pnueli, 1996)
Even for $MSOL$ there are ϕ such that the problem is arbitrarily high in the polynomial hierarchy.

Theorem 12 (Vardi, 1982)
The combined problem is $\mathbf{PSpace}$-complete even for FOL.

Model checking, II

We want to do model checking on $MSOL$-inductive classes K.

We can represent the structure $\mathfrak{A}$ by its **relational table**

or by a **parse term** $t_{\mathfrak{A}}$ which displays why it is in K.

In general, finding $t_{\mathfrak{A}}$ is $\mathbf{NP}$-hard.

Theorem 13 (Courcelle and Makowsky, 2001)
Let K be an $MSOL$-inductive class of τ-structures and $\phi \in MSOL(\tau)$.

Given a parse term $t_{\mathfrak{A}}$ for $\mathfrak{A}$, then the problem of deciding

$$\mathfrak{A} \models \phi$$

can be decided in linear time (in the size of $t_{\mathfrak{A}}$).

Proof of Theorem 13

$MSOL^q(\tau)$ is finite for every finite relational τ.

If all the operations are effectively $MSOL$-smooth, there is an algorithm which computes for every ϕ the look-up table (using Theorem 4).

Otherwise, we don't have such an algorithm, but still for each ϕ the look-up table for complete $MSOL^q(\tau)$-types is finite (using Theorem 3).

Now we can compute along $t_{\mathfrak{A}}$, in the style of *dynamic programming*.