Lecture 10

Last lecture

- We showed a combinatorial proof of the Little Fermat Theorem
- We proved Gessel's Theorem
- We discussed DU-index and the Specker-Blatter Theorem

We did not show all the slides on the Specker-Blatter Theorem

Homework: Read them

Lecture 10 and 11: Outline

We start a new topic !

Graph Polynomials

It does not yet use material from Lectures 1-9.

- The chromatic polynomial
- More graph polynomials
- The best studied graph polynomials from the literature.

The Chromatic Polynomial

and

Its Variations

The (vertex) chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem 1 (G. Birkhoff, 1912) *$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.*

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

Interpretation of $\chi(G, \lambda)$ for $\lambda \notin \mathbb{N}$

What's the point in considering $\lambda \notin \mathbb{N}$?

Theorem 2 (Stanely, 1973)
For simple graphs G, $|\chi(G, -1)|$ counts the number of acyclic orientations of G.

There are also combinatorial interpretations of $\chi(G, -m)$ for each $m \in \mathbb{N}$, which are more complicated to state.

Question: What about $\chi(G, \lambda)$ for each $m \in \mathbb{R} - \mathbb{Z}$?

The Four Color Conjecture

Birkhoff wanted to prove the **Four Color Conjecture** using techniques from real or complex analysis.

Conjecture:(Birkhoff and Lewis, 1946)
If G is planar then $\chi(G,\lambda) \neq 0$ for $\lambda \in [4,+\infty) \subseteq \mathbb{R}$.

For real roots of χ we know:

Jackson, 1993 For simple graphs G we have $\chi(G,\lambda) \neq 0$ for $\lambda \in (-\infty,0)$, $\lambda \in (0,1)$ and $\lambda \in (1,\frac{32}{27})$.

Birkhoff and Lewis, 1946 For planar graphs G we have $\chi(G,\lambda) \neq 0$ for $\lambda \in [5,+\infty)$.

Still open: Are there planar graphs G such that $\chi(G,\lambda) = 0$ for some $\lambda \in (4,5)$?

Thomassen, 1997 and Sokal, 2004 The real roots of all chromatic polynomials are dense in $(1,\frac{32}{27})$; the complex roots are dense in $\mathbb{C}$.

The edge-chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-edge-coloring** is a map

$$c : E(G) \to [\lambda]$$

such that if $(e, f) \in E(G)$ have a common vertex then $c(e) \neq c(f)$.

We define $\chi_e(G, \lambda)$ to be the number of λ- edge-colorings

Fact: $\chi_e(G, \lambda)$ a polynomial in $\mathbb{Z}[\lambda]$.

Let $L(G)$ be the **line graph** of G.
$V(L(G)) = E(G)$ and $(e, f) \in E(L(G))$ iff e and f have a common vertex.

Observation: $\chi_e(G, \lambda) = \chi(L(G), \lambda)$, where $L(G)$ is the line graph of G.

Conclusion: $\chi_e(G, \lambda)$ **is a polynomial** in $\mathbb{Z}[\lambda]$.

Variations on coloring, I

We can count other coloring functions.

- Total colorings

  $f_V : V \to [\lambda_V]$, $f_E : E \to [\lambda_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- Connected components

  $f_V : V \to [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- Pre-coloring extensions

  Given graph $G = (V, E)$ and an equivalence relation R on V and $f_V : V \to [\lambda_V]$, we require that if $(u, v) \in R$ they have the same color, and if $(u, v) \in E - R$ they have different colors.

Fact: The corresponding counting functions are polynomials in λ.

Variations on coloring, II

Encountered at CanaDam-07:

Let $f : V(G) \to [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with atmost λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injectiv on the neighborhood of every vertex.

\- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

\- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

New fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

Variations on coloring, III

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* **path-rainbow:** Let $f : E \to [\lambda]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

New fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ

Rainbow colorings of various kinds arise in computational biology

* **-monochromatic components:** Let $f : V \to [\lambda]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors, if all the connected components of a monochromatic set have size at most t.

New fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$, the number of mcc_t-colorings of G with λ colors, is a polynomial in λ. but not in t.

mcc_t colorings were first studied in:

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *Journal of Combinatorial Theory, Series B*, 87:231–243, 2003.

Variations on coloring, IV

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [\lambda]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs and $n = 2$ we get the **acyclic colorings**, studied in XXX-india.

Theorem: Let $\chi_{\mathcal{P},n}(G, \lambda)$ be the number of colorings of G with λ colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then Let $\chi_{\mathcal{P},n}(G, \lambda)$ is a polynomial in λ.

Variations on colorings, V: coloring relations

Let $G = (V, E)$. Here we look at an example where the coloring is a relation $R \subseteq V \times [k]$ rather than a function $f : V \to [k]$.
We denote by C_v the set $\{c \in [k] : (v, c) \in R\}$.

Let $a, b \in \mathbb{N}$. An (a, b)-coloring relation with k colors is a relation $R \subseteq V \times [k]$ such that

- For each $v \in V$ there are at most a-many colors $c \in [k]$ such that $(v, c) \in R$.
- If $(u, v) \in E$ then $C_u \neq C_v$ and there are at most b-many distinct elements $c_1, \ldots, c_b$ in $C_u \cap C_v$.

Exercise:

- Compute the number of (a, b)-coloring relations of the complete graphs K_n for various $a, b, k \in \mathbb{N}$.
- Is the number (a, b)-coloring relations with k colors of a graph G a polynomial in a, b or k?
- Look at the corresponding definitions with "at most" replaced by "at least" or "exactly".

Variations on colorings, VI: Two kinds of colors.

Let $G = (V, E)$.
Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$.
The colors in A are called proper colorings.
Our coloring is a function $f : V \to [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.
- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in k_1 (proper).

Theorem 3 (K. Dohmen, A. Pönitz and P. Tittman, 2003)
This gives us a polynomial $P(G, k_1, k)$ in k_1 and k.

Hypergraph colorings, I

Given hypergraph $H = (V, E)$ with $E \subset \wp(V)$, and a set of λ colors $[\lambda]$. Let $f : V \to [\lambda]$.

- f is a **weak hypergraph coloring**, if for each $e \in E$ with at least two vertices, there are $u \neq v$ and $u, v \in e$ with $f(u) \neq f(v)$.

- f is a **strong hypergraph coloring**, if for each $e \in E$ and for all $u, v \in e$ with $u \neq v$ we have $f(u) \neq f(v)$.

- f is a **conflict free hypergraph coloring**, if f is a weak hypergraph coloring and for each $e \in E$ there is $u \in e$ such that for all $v \in e, v \neq u$ we have $f(v) \neq f(u)$.

  This was introduced in Even, G., Lotker, Z., Ron, D., and Smorodinsky, S. (2003), K. Aardal, S. van Hoesel, A. Koster, C. Mannino, and A. Sassano (2003), cf. also J. Pach, E. Tardos, (2009)

Hypergraph colorings, II

Let $\chi_{h-weak}(H,\lambda)$, $\chi_{h-strong}(H,\lambda)$ and $\chi_{h-cf}(H,\lambda)$, denote the number of weak, strong and conflict free hypergraph colorings of G with λ colors, respectively.

Theorem: The counting functions

(i) $\chi_{h-weak}(H,\lambda)$,

(ii) $\chi_{h-strong}(H,\lambda)$ and

(iii) $\chi_{h-cf}(H,\lambda)$,

are hypergraph polynomials in λ.

Proof: For (i) and (ii) one can mimick Birkhoff's proof for graphs.

We shall give a **uniform proof** of these statements later on.

Hypergraph colorings, III

Given hypergraph $H = (V, D, E)$ with two kinds of hyperedges, $D, E \subset \wp(V)$, and a set of λ colors $[\lambda]$. Let $f : V \to [\lambda]$.

f is a **strong/weak mixed hypergraph coloring**, if

- for (V, E) the function f is a strong/weak hypergraph coloring, and
- for every $d \in D$ and for every $u, v \in d$ we have $f(u) = f(v)$.

Theorem: The number $\chi_{mixed}(H, \lambda)$ of mixed hypergraph colorings with λ colors is a polynomial in λ.

Proof: This was shown in

Vitaly I. Voloshin
Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, AMS 2002

Our **uniform proof** applies also to this case.

Digression:

Typical theorems about the chromatic polynomial

Main reference:

[DKT] F.M. Dong, K.M. Koh and K.L. Teo
Chromatic polynomials and the chromaticity of graphs
World scientific, 2005

Expressive power of $\chi(G, \lambda)$, I

We denote by $k(G)$ the number of connected components, $b(G)$ the number of blocks (2-connected components), and $\chi(G)$ the chromatic number of G, respectively.

Theorem 4 (DKT, 3.2.1.)
Let G_1, G_2 be two graphs with $\chi(G_1, \lambda) = \chi(G_2, \lambda)$. Then

(i) $V(G_1) = V(G_2)$ *and* $E(G_1) = E(G_2)$.

(ii) $\chi(G_1) = \chi(G_2)$.

(iii) $k(G_1) = k(G_2)$, *in particular* G_1 *is connected iff* G_2 *is connected.*

(iv) $b(G_1) = b(G_2)$, *in particular* G_1 *is 2-connected iff* G_2 *is 2-connected.*

(v) G_1 *is bipartite iff* G_2 *is bipartite.*

Expressive power of $\chi(G, \lambda)$, II

We denote by $n_H(G)$ and $i_H(G)$ the number of subgraphs and induced subgraphs of G, respectively, which are isomorphic to H.
K_k is the clique (complete graph) on k vertices. C_k is the cycle on k vertices.
The girth $g(G)$ is the smallest k such that $n_{C_k}(G) \neq 0$.

Theorem 5 (DKT, 3.2.1.)
Let G_1, G_2 be two graphs with $\chi(G_1, \lambda) = \chi(G_2, \lambda)$. Then

(i) $n_{C_3}(G_1) = n_{C_3}(G_2)$

(ii) $g(G_1) = g(G_2)$.

(iii) $i_{C_4}(G_1) - 2n_{K_4}(G_1) = i_{C_4}(G_2) - 2n_{K_4}(G_2)$

(iv) $n_{C_k}(G_1) = n_{C_k}(G_2)$, *provided* $g = g(G_1) = g(G_2) \leq k \leq \lceil \frac{3g}{2} \rceil - 2$

Normal forms of $\chi(G, \lambda)$, I: Power form

As $\chi(G, \lambda)$ is a polynomial, we can write it as

$$\chi(G, \lambda) = \sum_{i}^{|V(G)|} a_i(G)\lambda^i$$

in power form.

For the disjoint union we note that

Proposition 6

$$\chi(G_1 \sqcup G_2, \lambda) = \chi(G_1, \lambda) \cdot \chi(G_2, \lambda).$$

Question: Is there a combinatorial interpretation of the $a_i(G)$?

Normal forms of $\chi(G, \lambda)$, II: Factorial form

We define $\lambda_{(i)} = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - i + 1)$.

We write $\chi(G, \lambda)$

$$\chi(G, \lambda) = \sum_{i}^{|V(G)|} b_i(G) \lambda_{(i)}$$

There is a combinatorial interpretation of $b_i(G)$:

Theorem 7 *$b_i(G)$ is the number of partitions of V into i non-empty independent sets.*

Normal forms of $\chi(G,\lambda)$, II: Factorial form (continued)

We define a an operation $\circ$ on the $\lambda_{(i)}$ by $\lambda_{(i)} \circ \lambda_{(j)} = \lambda_{(i+j)}$ and extend it naturally to polynomials in $\lambda_{(i)}$.

The join of two graphs G_1, G_2, $G_1 \bowtie G_2$,
is obtained by taking the disjoint union and
adding all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 8

$$\chi(G_1 + G_2, \lambda) = \left(\sum_i^{|V(G)|} c_i(G_1)\lambda_{(i)} \circ \sum_i^{|V(G)|} c_i(G_2)\lambda_{(i)} \right)$$

Normal forms of $\chi(G,\lambda)$, III: Binomial Form

We note that $\frac{\lambda_{(i)}}{i!} = \binom{\lambda}{i}$.

We write $\chi(G,\lambda)$

$$\chi(G,\lambda) = \sum_{i}^{|V(G)|} c_i(G)\binom{\lambda}{i}$$

There is a combinatorial interpretation of $c_i(G)$:

Theorem 9 *$c_i(G)$ is the number of proper colorings of G with exactly i colors.*

Corollary 10 $\frac{b_i(G)}{i!} = c_i(G)$

Exercise: Give a direct proof of the corollary!

Planar graphs revisited

Theorem 11 (P.J. Heawood, 1890)
Every planar graph is 5-colorable. Hence $\chi(G,5) \neq 0$ *for* G *planar.*

Theorem 12 (G. Birkhoff and D. Lewis, 1946)
$\chi(G,a) \neq 0$ *for* G *planar and* $a \in \mathbb{R}, a \geq 5$.

Note that this is much stronger than the 5-color theorem.

Theorem 13 (K. Appel and W. Haken, 1977)
Every planar graph is 4-colorable. Hence $\chi(G,4) \neq 0$ *for* G *planar.*

Problem 14
Find an analytic proof of the 4-color theorem.

Conjecture 15 (G. Birkhoff and D. Lewis, 1946)
For G *planar, there are no real roots of* $\chi(G,a)$ *for* $4 \leq a \leq 5$.

Real roots of $\chi(G, \lambda)$

We note that $\chi(G, 0) = 0$ always, and $\chi(G, 1) = 0$ any graph with at least one edge.

Theorem 16 (D. Woodall, 1977) *Let G be any graph.*

- *There are no negative real roots of $\chi(G, \lambda)$.*
- *There are no real roots of $\chi(G, \lambda)$ in the open interval $(0, 1)$.*

Theorem 17 (B. Jackson, 1993)

- *There are no real roots of $\chi(G, \lambda)$ in the semi-open interval $(1, \frac{32}{27}]$.*
- *For any $\epsilon > 0$ there is a graph G_ϵ such that $\chi(G_\epsilon, \lambda)$ has a root in $(\frac{32}{27}, \frac{32}{27} + \epsilon)$.*

Theorem 18 (S. Thomassen, 1997) *For any real numbers a_1, a_2 with $\frac{32}{27} \leq a_1 < a_2$ there exists a graph G such that $\chi(G, \lambda) = 0$ for some $a \in (a_1, a_2)$.*

Subgraph expansions

Let G be a graph with $k(G)$ connected components.
Let $S \subset E(G)$ and denote by $\langle S \rangle$ the subgraph generated by S in G.

- The **rank** $r(G)$ is defined as $r(G) = \mid V(G) \mid -k(G)$.
- The **corank** $s(G)$ is defined as $s(G) = \mid E(G) \mid - \mid V(G) \mid +k(G)$.
- The **rank polynomial** of a graph is defined by

$$R(G; X, Y) = \sum_{S \subseteq E(G)} X^{r(\langle S \rangle)} Y^{s(\langle S \rangle)}$$

Theorem 19 (H. Whitney, 1932)

(i) $\chi(G, X) = \sum_{S \subseteq E(G)} (-1)^{|S|} X^{|V(G)|-r(\langle S \rangle)}$

(ii) $\chi(G, X) = X^{|V|} R(G, -X^{-1}, -1)$

The complexity of the chromatic polynomial, I

Let us look at the chromatic polynomial $\chi(G, \lambda)$.

- $\chi(G, \lambda)$ has integer coefficients,
  and for $\lambda \geq 0$ non-negative values,
  hence evaluating it at $\lambda = a, a \in \mathbb{N}$ is in $\sharp\mathbf{P}$.
- For $a = 0, 1, 2$ evaluating $\chi(G, \lambda)$ is in $\mathbf{P}$.
- For integer $a \geq 3$ evaluating $\chi(G, \lambda)$ is $\sharp\mathbf{P}$-complete.
- What about evaluating $\chi(G, \lambda)$ for $\lambda = b$ with
  $b \in \mathbb{Z}$, $b \leq 0$?
  $b \in \mathbb{R}$ or $b \in \mathbb{C}$?

Given evaluations of $\chi(G, \lambda)$ at $\mid V(G) \mid +1$ many points, we can compute the coefficients of $\chi(G, \lambda)$ efficiently.

The complexity of the chromatic polynomial, II

Theorem 20 *(N. Linial, 1986)*

For any two points $a, b \in \mathbb{C} - \mathbb{N}$
there is a **polynomial time algebraic reduction**
from the evaluation of $\chi(G, a)$ *to the evaluation of* $\chi(G, b)$.

Hence they are all **equally difficult**.

Proof: We note that

$$\chi(G \bowtie K_n, \lambda + n) = n_\lambda \cdot \chi(G, \lambda).$$

We use this to compute sufficently many points of $\chi(G, \lambda)$, and then use Lagrange interpolation. Q.E.D.

End of digression on typical theorems about the chromatic polynomial

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Parametrized Numeric Graph Invariants

Bounded numeric invariants

In graph theory it is often customary to look at numeric invariants which bounded by a function $b : G \to \mathbb{N}$.

- $k(G)$: the number of connected components of G;
  $k(G, \lambda)$: the number of connected components of G of size λ.

- $cl(G)$: the number of cliques of G;
  $cl(G, \lambda)$: the number of cliques of G of size λ.

- $indep(G, \lambda)$: the number of independent sets of G of size λ.

- $v(G, \lambda)$: the number of vertex covers of G of size λ.

- $m(G, \lambda)$: the number of matchings of G of size λ.

Obviously, these functions are not polynomials in λ,
because they vanish for large enough λ.

Pngi's: Parametrized numeric graph invariants

Let $\mathcal{K}$ denote a class of finite (colored) graphs
(hypergraphs, or structures over some fixed vocabulary).

A **parametrized numeric graph invariant (pngi)** is a function $\alpha(G, \lambda)$

$$\mathcal{K} \times \mathbb{N} \to \mathbb{N}$$

such that, for each $\lambda \in \mathbb{N}$ and G_1 isomorphic to G_2
we have that $\alpha(G_1, \lambda) = \alpha(G_2, \lambda)$.

Let $\alpha(G, \lambda)$ and Let $\beta(G, \lambda)$ be two pngi's.
Clearly, we can form new such invariants by forming

- $\alpha(G, \lambda) + \beta(G, \lambda), \quad \alpha(G, \lambda) \cdot \beta(G, \lambda), \quad 2^{\alpha(G,\lambda)}$
- If $\alpha(G, \lambda) = 0$ for all large enough λ,

$$\beta(G, \lambda) = \sum_n \alpha(G, n)\lambda^n$$

If $\alpha(G, \lambda) \in \mathbb{Z}[\lambda]$ is a polynomial, we speak of **graph polynomials**.

The behaviour of parametrized numeric graph invariants

The pngi's of the form $\alpha(G, \lambda)$ we have seen so far show the following behaviour:

- For each graph there is $b_G \in \mathbb{N}$ such that $\alpha(G, \lambda) \leq \lambda^{b_G}$.
- For each $n \in \mathbb{N}$ we have $\alpha(G, n) \in \mathbb{N}$.
- There is $n_G \in \mathbb{N}$ such that
  either $\alpha(G, n) = 0$ for all $n \geq n_G$
  or $\alpha(G, n)$ is not decreasing for all $n \geq n_G$.

Coloring Properties

A Model-Theoretic View

Enter logic: Model theory

Our framework is as follows:

- Let $\mathfrak{M}$ be a finite τ-structure with universe M.
- Let $k \in \mathbb{N}$ and $[k] = \{0, \ldots, k-1\}$.
- Let $\mathfrak{M}_k$ be the two-sorted τ' structure $\langle \mathfrak{M}, [k] \rangle$.
- Let F be an r-ary function symbol with interpretations in $\mathfrak{M}_k$ of the form $f : M^r \to [k]$.

Coloring properties, I

We denote relation symbols by **bold-face letters**, *and their interpretation by the corresponding* roman-face letter.

Let $\tau_R = \tau_1 \cup \{\mathbf{R}\}$, where is $\mathbf{R}$ is a two-sorted relation symbol of arity $r = s + t$.

A class of τ_R- structures $\mathcal{P}$ is a **coloring property** if

Extension Property: Let $\mathcal{M}$ be fixed. Then $\mathcal{M}_k$ is a substructure of $\mathcal{M}_n$ for each $n \geq k$. Let R_0 be a fixed relation on $\mathcal{M}_k$. If $\langle \mathcal{M}_k, R_0 \rangle \in \mathcal{P}$ and $n \geq k$ then also $\langle \mathcal{M}_n, R_0 \rangle \in \mathcal{P}$.

Isomorphism Property: $\mathcal{P}$ is closed under τ_R-isomorphisms.

This implies the permutation property:

Permutation Property: Let $R \subseteq M^s \times [k]^t$ be a fixed relation on $\mathcal{M}_k$. For π is a permutation of $[k]$, We define $R_\pi = \{(\bar{m}, \pi(\bar{a})) \in M^\times [k]^t : (\bar{m}, \bar{a}) \in R\}$.

Then $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$ iff $\langle \mathcal{M}_k, R_\pi \rangle \in \mathcal{P}$.

We refer to $\mathbf{R}$ and its interpretations R as **coloring predicates**.

Coloring properties, II

(i) A coloring property is **bounded**, if for every $\mathcal{M}$ there is a number N_M such that for all $k \in \mathbb{N}$ the set of colors

$$\{x \in [k] : \exists \bar{y} \in M^m R(\bar{y}, x)\}$$

has size at most N_M.

(ii) A coloring property is **range bounded**, if its range is bounded in the following sense: There is a number $d \in \mathbb{N}$ such that for every $\mathcal{M}$ and $\bar{y} \in M^m$ the set $\{x \in [k] : R(\bar{y}, x)\}$ has at most d elements.

Clearly, if a coloring property is range bounded, it is also bounded.

Coloring properties, III

Let ϕ be a sentence of some logic $\mathcal{L}$.

$\mathcal{L}$ could be first order logic, second order logic, infinitary logic, or some fragment thereof.

(i) $\phi(\mathbf{R})$ is a **coloring formula**, if the class of its models, which are of the form of the form $\langle \mathcal{M}, [k], R \rangle$, is a **coloring property**.

(ii) Let $\mathcal{P}$ be a bounded coloring property. A relation $R_M \subset M^m \times [k]$ is a **generalised $k - \mathcal{P}$-coloring** if $\langle \mathcal{M}_k, R \rangle \in \mathcal{P}$.

(iii) We denote by

$$\chi_{\mathcal{P}}(\mathcal{M}, k)$$

the number of generalised $k - \mathcal{P}$-coloring R on $\mathcal{M}$.

If $\mathcal{P}$ is defined by $\phi(\mathbf{R})$ we also write

$$\chi_{\phi(R)}(\mathcal{M}, k).$$

Generalized multi-colorings, I

To construct also graph polynomials in several variables, we extend the definition to deal with several color-sets, and also call them generalized chromatic polynomials.

Let $\mathcal{M}$ be a τ-structure with universe M.

We say an $(\alpha+1)$-sorted structure

$$\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R \rangle$$

for the vocabulary $\tau_{\alpha,R}$ with

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

is a **generalized coloring** of $\mathcal{M}$ for colors $\bar{k}^\alpha = (k_1, \ldots, k_\alpha)$.

By abuse of notation,

$m_i = 0$ is taken to mean the color-set k_i is not used in R.

Generalized multi-colorings, II

A class of generalized multi-colorings $\mathcal{P}$ is a coloring property if it satisfies the following conditions:

Isomorphism property : $\mathcal{P}$ is closed under $\tau_{\alpha,R}$-isomorphisms.

Extension property : For every $\mathcal{M}$, $k_1 \leq k_1', \ldots, k_\alpha \leq k_\alpha'$, and R,
if $\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R\rangle \in \mathcal{P}$ then $\langle \mathcal{M}, [k_1'], \ldots, [k_\alpha'], R\rangle \in \mathcal{P}$.

Non-occurrence property : Assume

$$R \subset M^m \times [k_1]^{m_1} \times \ldots \times [k_\alpha]^{m_\alpha}$$

with $m_i = 0$, and

$$\langle \mathcal{M}, [k_1], \ldots, [k_\alpha], R\rangle \in \mathcal{P},$$

then for every $k_i' \in \mathbb{N}$,

$$\langle \mathcal{M}, [k_1], \ldots, [k_i'], \ldots, [k_\alpha], R\rangle \in \mathcal{P}.$$

The **boundedness conditions** are the obvious adaptions.

Main result, A

Generalized chromatic polynomials

Main result, A

THEOREM A: If $\phi(\mathbf{R})$ is an $\mathcal{L}$-sentence and defines a bounded coloring propert then

$$\chi_\phi(\mathfrak{M}, k_1, \ldots k_\alpha) \in \mathbb{Z}[k_1, \ldots k_\alpha]$$

is indeed a polynomial in $k_1, \ldots k_\alpha$.

We shall call polynomials obtained like this $\mathcal{L} - MG$-polynomials.

MG-polynomial for model theoretic growth polynomial

(as studied by B. Zilber in his work on categoricity).

Corollary: Taking $\mathcal{L}$ to be (monadic) second order logic,
this covers **all** the previous examples,
and allows us to construct **infinitely many more** MG-polynomials.

A theorem with an elementary generic proof

suggested simplification by A. Blass

We prove something a bit stronger (for the case of $\alpha = 1$, i.e., **one** color set):

THEOREM A': For every $\mathcal{M}$ the number $\chi_{\phi(R)}(\mathcal{M}, k)$ is a polynomial in k of the form

$$\sum_{j=0}^{d\cdot|M|^m} c_{\phi(R)}(\mathcal{M}, j)\binom{k}{j}$$

where $c_{\phi(R)}(\mathcal{M}, j)$ is the number of generalised $k-\phi$-colorings R with a fixed set of j colors.

In the light of this theorem we call $\chi_{\phi(R)}(\mathcal{M}, k)$

also a *generalised chromatic polynomial*.

Proof

We first observe that any generalised coloring R uses at most

$$N = d\cdot \mid M \mid^m$$

of the k colors.

For any $j \leq N$, let $c_{\phi(R)}(\mathcal{M}, j)$ be the number of colorings, with a fixed set of j colors, which are generalised vertex colorings and use all j of the colors.

Next we observe that any permutation of the set of colors used is also a coloring.

Therefore, given k colors, the number of vertex colorings that use exactly j of the k colors is the product of $c_{\phi(R)}(\mathcal{M}, j)$ and the binomial coefficient $\binom{k}{j}$. So

$$\chi_{\phi(R)}(\mathcal{M}, k) = \sum_{j \leq N} c_{\phi(R)}(\mathcal{M}, j)\binom{k}{j}$$

The right side here is a polynomial in k, because each of the binomial coefficients is. We also use that for $k \leq j$ we have $\binom{k}{j} = 0$. Q.E.D.

Does the converse of Theorem A hold?

Let G be a graph with $|V(G)| = n$, $P \subseteq V$, and $f : V(G) \to [k]$ such that

- $P = \emptyset$ or $P = V$;
- If $P = \emptyset$ then f is **onto**.
- If $P = V$ then f is **not onto**.

Let $\Phi(P, f)$ be the formula expressing this. $\chi_{\Phi(P,f)}$ counts the pairs (P, f). $\chi_{\Phi(P,f)}(G) = k^n$ counts the pairs (P, f) and is a polynomial in k.

Clearly, if $P = \emptyset$, $\Phi(P, f)$ does **not** have the **extension property**.

Conclusion: There are formulas **without the extension property** which have polynomials as their counting functions.

But $\Phi(P, f)$ counts all functions $f : V \to [k]$, which is the counting function of a different formula (not equivalent) which has the extension property.

How to formulate a converse?

- We shall introduce a notion of **definability** of graph polynomials, where the domain of summation is definable in some formalism.
- We shall show that every definable graph polynomial is the counting function of **bounded coloring property** definable in the corresponding formalism.

Graph polynomials

Prominent graph polynomials

- The chromatic polynomial (G. Birkhoff, 1912)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The characteristic polynomial
  (T.H. Wei 1952, L.M. Lihtenbaum 1956, L. Collatz and U. Sinogowitz 1957)
- The various matching polynomials (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials (I. Gutman and F. Harary 1983)
- The Farrel polynomials (E.J. Farrell, 1979)
- The cover polynomials for digraphs (F.R.K. Chung and R.L. Graham, 1995)
- The interlace-polynomials
  (M. Las Vergnas, 1983, R. Arratia, B. Bollobás and G. Sorkin, 2000)
- The various knot polynomials (of signed graphs)
  (Alexander polynomial, Jones polynomial, HOMFLY-PT polynomial, etc)

Application of graph polynomials

There are plenty of applications of graph polynomials in

- Graph theory proper
- Knot theory
- Chemistry
- Statistical mechanics
- Quantum physics
- Quantum computing
- Biology

Using our framework: The matching polynomial

We want to show that the matching polynomial can be obtained in our framework.

- For a graph $G = (V, E)$ we form a 4-sorted structure

  $$\mathfrak{M}(G) = \langle V, E, \wp(V), \wp(E), \in, R_G \rangle$$

  where $\in$ is the membership relation between elements of V and $\wp(V)$, and elements of E and $\wp(E)$ respectively,
  and R_G is the incidence relation between vertices and edges.

- $\mathfrak{M}(G)_k = \langle V, E, \wp(V), \wp(E), \in, R_G, [k] \rangle$

- The formula $\phi_{matching}(m, f)$ now says:

  (i) $m \in \wp(E)$ is a matching.

  (ii) f is a function $f : m \to [k]$.

Using our framework: The matching polynomial, contd

We replace k by λ.

Now we put $\bar{g}(G, \lambda)$ to be the number of pairs (m, f) such that

$$\langle \mathfrak{M}(G)_\lambda, m, f \rangle \models \phi_{matching}(m, f)$$

- For fixed m there are $\lambda^{|m|}$ many f's satisfying the formula $\phi_{matching}(m, f)$.
- For matchings m with $\mid m \mid = j$ we get $m(G, j)\lambda^j$ many such pairs.
- Hence we get

$$\bar{g}(G, \lambda) = \sum_j m(G, j)\lambda^j = \sum_{\substack{M : M \subseteq E \\ M \text{ is a matching}}} \prod_{e : e \in M} \lambda = g(G, \lambda)$$