# Lecture 8

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## Last lecture:

- We discussed the density function of regular languages with examples.
- We formulated and proved Schützenberger's Theorem, stating that regular languages have density functions which satisfy a linear recurrence relation over $\mathbb{Z}$.
- We discussed density fuctions of relational structures.

# Lecture 8

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This lecture:

- We formulate and prove an alternative to Schützenberger's Theorem, using $\mathbf{MSOL}$ rather than regular languages.
- We continue our discussion of density fuctions of relational structures.

# An alternative proof Schützenberger's Theorem using Hintikka sentences: an Exercise

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**Theorem:** Let $L$ be an $\mathbf{MSOL}$-definable set of words.
Then $d_L(n)$ satisfies a linear recurrence relation.

Proof:

- Assume $L$ is defined by $\phi$. Let $h_\alpha, \alpha \leq \gamma$ be all the Hintikka sentences with quantifier rank $q = qr(\phi)$. $\phi = \bigvee_{\alpha \in A} h_\alpha$.
- Let $L_\alpha$ be the languages associated with $h_\alpha, \alpha \leq \gamma$, and $d_\alpha$ the corresponding density functions.
- Define a matrix $M$ with $m_{\alpha,\beta} = 1$ if a words satisfying $h_\alpha$ has a one letter extensions satisfying $h_\beta$, and $m_{\alpha,\beta} = 0$ otherwise.
- Use this matrix to compute all the $d_\alpha$'s simultaneously.

- Apply Caley's Theorem. Q.E.D.

**Question:** What can we say about the recurrence depth?

# Theorem C revisited, I

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Let $\mathcal{P}$ be a graph property which is $MSOL$-definable. and let $d_{\mathcal{P}}(n)$ be its density function.

- (Specker and Blatter, 1981)
  $d_{\mathcal{P}}(n)$ satisfies modular recurrence relations for each $m$.
- (Specker and Blatter, 1981)
  This remains true with several binary edge relations and unary predicates on the vertices.
- (E. Fischer, 2003) Is false for an $FOL$-definable class with one quaternary relation.

# Relations of bounded degree

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Let $\mathcal{A} = \langle A, \bar{R} \rangle$ be a $\tau$-structure.
We define a symmetric relation $E_A$ on $\mathcal{A}$, and call $\langle A, E_A \rangle$ the **Gaifman-graph of** $\mathcal{A}$.

- Let $a, b \in A$. $(a, b) \in E_A$ iff there exists a relation $R \in \bar{R}$ and some $\bar{a} \in R$ such that both $a$ and $b$ appear in $\bar{a}$ (possibly with other members of $A$ as well).

- For any element $a \in A$, the **degree** of $a$ is the number of elements $b \neq a$ for which $(a, b) \in E_A$.

- We say that $\mathcal{A}$ is of **bounded degree** $d$ if every $a \in A$ has degree at most $d$.

- We say that $\mathcal{A}$ is **connected** if its Gaifman-graph is connected.

- For a class of structures $\mathcal{P}$ we say it is of bounded degree $d$ (resp. connected) iff all its structures are of bounded degree $d$ (resp. connected).

# Theorem C revisited, II

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**Theorem 1 (E. Fischer and J.A. Makowsky, 2002)**
*Let $\mathcal{P}$ be a property of $\tau$-structures, which is $MSOL$-definable.*
*Let $d_{\mathcal{P}}(n)$ be its density function.*

- *If $\mathcal{P}$ is of bounded degree $d$,*
  *the function $d_{\mathcal{P}}(n)$ satisfies a modular recurrence relation for every $m$.*

- *Furthermore, if additionally all the models in $\mathcal{P}$ are connected,*
  *the function $d_{\mathcal{P}}$ satisfies the trivial recurrence relations for every $m$.*

*We have no restrictions on $\tau$, besides not allowing function symbols,*

Theorem C and Theorem 1 remain true if we extend $MSOL$ and allow modular counting quantifiers.

## Ingredients of the proof of Theorem C

- The $DU$-index of a class of structures.
- The Specker-index of a class of structures.
- The $DU$-index of a class of structures $\mathcal{P}$ is always smaller or equal to the Specker index.
- Finite $DU$-index of a class of $\tau$-structures of bounded degree implies modular recurrence relations for all $m$.
- If $\tau$ contains only relation symbols of arity at most 2, finite Specker-index of a class of $\tau$-structures. implies modular recurrence relations for all $m$.
- $MSOL$-definability of $\mathcal{P}$ (even $CMSOL$-definability) implies finite $DU$-index.
- If $\tau$ contains only relation symbols of arity at most 2, the definability assumption implies finite Specker index.

# $DU$-index of $\mathcal{P}$

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We denote by $\mathfrak{A} \sqcup \mathfrak{B}$ the disjoint union of two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$.

We also count the graph on the empty set of vertices as a graph.

Let $\mathcal{P}$ be a class of $\tau$-structures.

(i) We say that $\mathfrak{A}_1$ is $DU(\mathcal{P})$-equivalent to $\mathfrak{A}_2$,
denoted by $\mathfrak{A}_1 \sim_{DU(\mathcal{P})} \mathfrak{A}_2$,
if for every $\tau$-structure $\mathfrak{B}$, $\mathfrak{A}_1 \sqcup \mathfrak{B} \in \mathcal{P}$ if and only if $\mathfrak{A}_2 \sqcup \mathfrak{B} \in \mathcal{P}$.

(ii) The $DU$-index of $\mathcal{P}$ is the number of $DU(\mathcal{P})$-equivalence classes.

(iii) A class of structures $\mathcal{P}$ is a **Gessel class** if for every $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \sqcup \mathfrak{B} \in \mathcal{P}$ iff both $\mathfrak{A} \in \mathcal{P}$ and $\mathfrak{B} \in \mathcal{P}$.

# Basics on the $DU$-index

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- The class of forests is a Gessel class.
- If $\mathcal{P}$ is hereditary and closed under disjoint unions, it is a Gessel class.
- Every Gessel class has $DU$-index at most 2.
- If $\mathcal{P}$ is a class of connected graphs, $\mathcal{P}$ has $DU$-index at most 2, but is not a Gessel class.
- If $\mathcal{P}_1$ and $\mathcal{P}_2$ have finite $DU$-index, so do $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1 \cap \mathcal{P}_2$, and the complement $\bar{\mathcal{P}}_1$.

# Gessel's Theorem (1984)

## (Proof follows in the sequel)

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**Theorem 2 (I. Gessel 1984)**

*If $\mathcal{C}$ is a Gessel class of directed graphs of degree at most $d$ with density function $d_{\mathcal{C}}(n)$ then*

$$d_{\mathcal{C}}(m+n) \equiv d_{\mathcal{C}}(m) \cdot d_{\mathcal{C}}(n) \pmod{\frac{m}{\ell}}$$

*where $\ell$ is the least common multiple of all divisors of $m$ not greater than $d$.*

*In particular, $d_{\mathcal{C}}(n)$ satisfies for every $m \in \mathbb{N}$ the linear recurrence relation*

$$d_{\mathcal{C}}(n) \equiv a^{(m)} d_{\mathcal{C}}(n - d!m) \pmod{m}$$

*where $a^{(m)} = d_{\mathcal{C}}(d!m)$.*

## $DU$-index and pebble games

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We can use pebble games to prove:

**Theorem 3**

*(i) If $\mathcal{P}$ is $FOL$-definable, it has finite $DU$-index.*

*(ii) If $\mathcal{P}$ is $MSOL$-definable, it has finite $DU$-index.*

*(iii) If $\mathcal{P}$ is $CMSOL$-definable, it has finite $DU$-index.*

# Proof of Theorem 3

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The proof uses several steps.
We do it for $FOL$, but for $MSOL$ it works the same, using the corresponding pebble game.

- If $\phi$ defines $\mathcal{P}$ and is of quantifier rank $q$, we look at the equivalence classes $\sim_q^q$.
- Using pebble games we show that, if for graphs $G_1, G_2, H_1, H_2$ with $G_1 \sim_q^q H_1$ and $G_2 \sim_q^q H_2$ we also have $G_1 \sqcup G_2 \sim_q^q H_1 \sqcup H_2$.
- Next we show that $DU(\mathcal{P})$-equivalence classes are closed under $\sim_q^q$-equivalence.
- Counting non-equivalent $FOL(\tau)$-formulas of quantifier rank $q$, we see that there are only finitely many $\sim_q^q$-equivalence classes.
- We conclude that there are only finitely many $DU(\mathcal{P})$-equivalence classes. Hence $\mathcal{P}$ has finite $DU$-index.

Details on the blackboard.

# Many classes of finite $DU$-index.
## Homework

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There are only countably many classes of structures definable by $MSOL$-formulas.

How many classes are there of finite $DU$-index?

(How many are there for which $\mu(\mathcal{P}) = 1$?)

**Proposition 4 (Specker 2002)**

*There are continuum many classes of structures (closed under isomorphisms) with finite $DU$-index.*

Can we do this also for Gessel classes?

## Many classes of finite $DU$-index.
## Solution to homework

There are only countably many classes of structures definable by $MSOL$-formulas.

How many classes are there of finite $DU$-index?

**Proposition 5 (After an idea of Specker, 2002)**

(i) *There are continuum many classes of structures (closed under isomorphisms) with $DU$-index $\leq 2$.*

(ii) *There are continuum many Gessel classes.*

# Solution, I

Let $C_n$ denote the cycle with $n$ vertices.

- Let $A \subseteq \mathbb{N}$, and $Cycle(A) = \{C_n : n \in A\}$.
- Prove that $Cycle(A)$ has $DU$-index at most 2.

  Hint: Use that all graphs of $Cycle(A)$ are connected.
- There are continuum many sets $A$, each giving a different class $Cycle(A)$.

We can play with this idea further.......
We can define silimarly $Path(A), Clique(A)$, and even mix them.

**Question:** Can we do this also for Gessel classes?

**Answer:** The closure $Cl_{DU}(Cycle(A))$ of $Cycle(A)$ under disjoint unions is a Gessel class. For $A, B \subseteq \mathbb{N}, A \neq B$ we have $Cl_{DU}(Cycle(A)) \neq Cl_{DU}(Cycle(B))$.

## Compute $d_{Cycle(A)}(n)$.

**Questions:**

- Compute $d_{Cycle(A)}(n)$.
- For which $A$ does it satisfy a recurrence relation?
- How does its recurrence relation modulo $m$ depend on $A$ and $m$?

# Compute $d_{Cycle(A)}(n)$, continued

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- For each $n$ $d_{Cycle(A)}(n)$ contains, up to (unlabeled) isomorphisms, at most one graph.
- For $C_n$ there are $n!$ many permutations of the labels.
- A permuation $\pi$ of the labels produces the same labeled structure $C_n$ iff $\pi$ is an automorphism of $C_n$.
- There $2n$ such automorphisms (for $n \geq 3$).
- Hence there are $\frac{n!}{2n} = \frac{(n-1)!}{2}$ many labelings.
- Hence

$$d_{Cycle(A)}(n) = \begin{cases} \frac{(n-1)!}{2} & n \in A \\ 0 & \text{else} \end{cases}$$

which is a trivial modular recurrence, independently of $A$ and for each $m$.

# Compute $d_{Path(A)}(n)$.

$Path(A)$ is the class of graphs $P_n$ for $n \in A$.

- $Path(A)$ has $DU$-index 2, as all its members are connected.
- $Path(A)$ is of bounded degree 2.
- We have
$$d_{Path(A)}(n) = \begin{cases} \frac{n!}{2} & n \in A \\ 0 & \text{else} \end{cases}$$
as there are 2 automorphisms of $P_n$.
- $d_{Path(A)}(n)$ satisfies a trivial modular recurrence, independently of $A$.

# Compute $d_{Clique(A)}(n)$.

$Clique(A)$ is the class of graphs $K_n$ for $n \in A$.

- $Clique(A)$ has $DU$-index 2, as all its members are connected.
- $Clique(A)$ is of unbounded degree (for $A$ infinite).
- We have
$$d_{Clique(A)}(n) = \begin{cases} 1 & n \in A \\ 0 & \text{else} \end{cases}$$
as there are $n!$ many automorphisms of $K_n$.
- $d_{Clique(A)}(n)$ satisfies a trivial modular recurrence, provided $A$ is finite, co-finite, or the characteristic function of $A$ ultimately periodic.

  If $A = \mathbb{N}$ it is constant to 1 which is rather simple, but still not trivial in our sense.

# Compute $d_{OneEdge(A)}(n)$.

We denote by $E_n$ the graph with $n$ vertices no edges.
We denote by $OE_n$ the graph with $n$ vertices and exactly one edge.
$OneEdge(A)$ is the class of graphs $OE_n$ for $n \in A$.

- $OneEdge(A)$ has $DU$-index 3.

  To see this we analyse $G_1 \sqcup G_2 \simeq OE_n$.
  this is the case if either $G_1 \simeq OE_m$ and $G_2 \simeq E_{n-m}$ or vice versa. This gives three $DU$-classes.

- $OneEdge(A)$ is of bounded degree 1.

- We have

$$d_{OneEdge(A)}(n) = \begin{cases} \frac{n(n-1)}{2} = \binom{n}{2} & n \in A \\ 0 & \text{else} \end{cases}$$

  as there are $2 \cdot (n-2)!$ many automorphisms of $OE_n$.

- $d_{OneEdge(A)}(n)$ satisfies a non-trivial modular recurrence, provided the characteristic function of $A$ is ultimately periodic.

# Compute $d_{EQ_2CLIQUE(A)}(n)$

We know already that

$$d_{EQ_2CLIQUE(A)}(n) = \begin{cases} \frac{1}{2}\binom{2m}{m} & n = 2m \in A \\ 0 & \text{else} \end{cases}$$

We can now interpret it:

$$\frac{1}{2}\binom{2m}{m} = \frac{1}{2} \cdot \frac{(2m)!}{m! \cdot m!}$$

where $m!$ are the number of automorphisms of the cliques and 2 is the number of mapping between the cliques.

We note that the graphs in $EQ_2CLIQUES$ are

- of unbounded degree,
- there are no modular recurrences for $d_{EQ_2CLIQUE(A)}(n)$,
- What is the $DU$-index of $EQ_2CLIQUES$?

## $DU$-index of $EQ_2CLIQUES(A)$

We have to analyze the relation

$$G_1 \sqcup G_2 \simeq 2K_m$$

This is the case if both $G_1 \simeq K_m$ and $G_2 \simeq K_m$, or one of the graphs is empty.

The $DU$ equivalence classes are

- $\{2K_m : m \in A\}$.
- $\{K_m\}$ for each $m \in A$.
- all the others.

Hence we have infinitely many classes, if $A$ is infinite.

## Passing to the complement graph

For a graph $G = (V, E)$ we denote by $\bar{G}$ the complement graph

$$G = (V, V^2 - E - \{(v, v) : v \in V\})$$

For a classe of graphs $\mathcal{P}$ we denote by $\bar{\mathcal{P}}$ the class

$$\{\bar{G} : G \in \mathcal{P}\}$$

The complement of $2K_m$ is $K_{m,m}$.

**Fact:** The number of labelings of $G$ and $\bar{G}$ is the same.

For $\overline{EQ_2CLIQUES(A)}$ we get a class of connected graphs,
hence of $DU$-index 2,
of unbounded degree,
with no modular recurrence relation for its density function.

# Compute $d_{EQ_2PATH(A)}(n)$

$EQ_2PATH(A)$ is the class of graphs $2P_n$ for $2n \in A$.

- $EQ_2PATH(A)$ has infinite $DU$-index. The argument is the same as with the cliques.
- $EQ_2PATH(A)$ is of bounded degree 2.
- We have
$$d_{EQ_2PATH(A)}(2n) = \begin{cases} \frac{2n!}{4} & n \in A \\ 0 & \text{else} \end{cases}$$
as there are 2 automorphisms of $P_n$ and two automorphisms between the paths.
- $d_{EQ_2PATH(A)}(n)$ satisfies a trivial modular recurrence, independently of $A$.

# Compute $d_{EQ_2CYCLES(A)}(n)$

$EQ_2CYCLES(A)$ is the class of graphs $2C_n$ for $2n \in A$.

- $EQ_2CYCLES(A)$ has infinite $DU$-index. The argument is the same as with the cliques.
- $EQ_2CYCLES(A)$ is of bounded degree 2.
- We have

$$d_{EQ_2CYCLES(A)}(2n) = \begin{cases} \frac{2n!}{n^2} & n \in A \\ 0 & \text{else} \end{cases}$$

  as there are $n$ automorphisms of $P_n$ and two automorphisms between the paths.

- $d_{EQ_2PATH(A)}(n)$ satisfies a trivial modular recurrence, independently of $A$, because $d_{EQ_2CYCLES(A)}(2n)$ s divbisible by $(n-1)!$.

# Regular graphs and grids
# Homework

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Discuss the following:

- Regular graphs of fixed degree $r$.

  They are not connected, but form a Gessel class.
  This class is $FOL$-definable.

- The class of Grids $Grid_{m,n}$.

  They are connected, hence have $DU$-index 2
  and are of bounded degree 4.
  Is the class of grids $MSOL$-definable? (YES)

**Remark:** For both cases Theorem 11 below slide applies.

# HOMEWORK

(i) Given $d, n \in \mathbb{N}$ construct a class of graphs $\mathcal{C}_n$ such that

- $\mathcal{C}_n$ is $MSOL$-definable,
- $\mathcal{C}_n$ is of degree $d$,
- $\mathcal{C}_n$ has $DU$-index $n$.

(or show it does not exist)

(ii) For each $n \in \mathbb{N}$ construct continuum many classes of graphs $\mathcal{D}_n(A)$ with $DU$-index $n$.

## Passing to bounded degree

Let $\mathcal{P}$ be a class of $\tau$-structures and denote by Let $\mathcal{P}_d$ the class of $\tau$-structures in $\mathcal{P}$ of degree at most $d$.

**Lemma 6**
*If $\mathcal{P}$ has finite $DU$-index, so does $\mathcal{P}_d$.*

**Proof:**

The class of structures of degree at most $d$ is a Gessel class, hence of $DU$-index 2.

But the intersection of two classes of finite $DU$-index is again a class of finite $DU$index. Q.E.D.