Lecture 6

What have done so far?

- We have given a proof of the easy direction of Theorem 1 (Lecture 4).
- We did the case for **FOL**, $\mathbf{MSOL}$, when there are more pebbles than moves.
- We left it as an **exercise**, to the case with a fixed number of pebbles, i.e. for the logics $\mathbf{FOL}^k$, $\mathbf{MSOL}^k$ and $\mathcal{L}^k_{\infty,\omega}$.

Homework for Lecture 5

- Show that the class of 2-colorable graphs is definable in $\mathcal{L}^3_{\infty,\omega}$.

  Hint: Look at the revised slides of Lecture 5!

- Show that, up to logical equivalence, in $\mathcal{L}^3_{\infty,\omega}$ contains only finitely many formulas of **fixed** quantifier rank n.

  Moreover, all these formulas are equivalent to formulas of $\mathbf{FOL}^k$ of the same quantifier rank.

- Prove the easy directions of Theorem 1 (Lecture) for fixed k.

Constructing Game Sentences

- They are called **Hintikka sentences** or **Scott sentences** depending on the taste of the authors and the context.
- Intuitively, a Hintikka sentence is a maximally expressive, satisfiable sentence of some fixed quantifier rank q.
- A Scott sentence is a sentence which characterizes a countable or finite structure up to isomorphism.
- If a Scott sentence has quantifier rank q, then it is a Hintikka sentences of quantifier rank q.
- The converse is not true, but Hintikka sentences
- We shall discuss Hintikka sentences and formulas for **FOL**.
The cases for $\mathbf{FOL}^k$, $\mathbf{MSOL}$ and $\mathbf{MSOL}^k$ are analoguous.
- The case $\mathcal{L}^k_{\infty,\omega}$ needs a bit more care.

Hintikka formulas, I

τ is a finite, relational vocabulary.

We denote by $Fm_{k,q}^{\mathbf{MSOL}}(\tau)$ the set of $\mathbf{MSOL}(\tau)$ formulas such that the variables are among

$$x_1, \ldots, x_k, U_1, \ldots, U_k$$

and each formula has quantifier rank atmost q.

Similarly with $Fm_{k,q}^{\mathbf{FOL}}(\tau)$.

Definition:
ϕ and ψ are (finitely) equivalent if the have the same (finite) models.
Free variables are uninterpreted constants

Note: There are, up to logical equivalence infinitely many formulas in three variables (use repeated quantification).

The boolean algebra $Fm_{k,q}(\tau)$, I

Proposition:
There are, up to (finite) equivalence, only finitely many formulas in $Fm_{k,q}(\tau)$.

If ϕ and ψ have only infinite models, they are finitely equivalent (**false**).
There are fewer formulas for finite equivalence.
The number of equivalence classes is growing very fast.

Proposition:
$Fm_{k,q}(\tau)$ is closed under conjunction $\wedge$,
disjunction $\vee$ and negation $\neg$,
i.e. it forms a finite *boolean algebra*.

The boolean algebra $Fm_{k,q}(\tau)$, II

The formula $\exists x(x \neq x)$ is the *minimal element*.

The formula $\exists x(x = x)$ is the *maximal element*.

A formula ϕ is an *atom*, if

- it is not (finitely) equivalent to $\exists x(x \neq x)$,
- but for each ψ either $\phi \wedge \psi$ is equivalent to ϕ or to $\exists x(x \neq x)$.

Hintikka formulas, II

We denote by $\mathcal{B}_{k,q}(\tau)$ and $\mathcal{B}^f{}_{k,q}(\tau)$ the
finite boolean algebra of $Fm_{k,q}^{MSOL}(\tau)$
up to equivalence and finite equivalence, resp.
The elements are denoted by $\bar{\phi}$.

The set of atoms in $\mathcal{B}_{k,q}(\tau)$ and $\mathcal{B}^f{}_{k,q}(\tau)$ is denoted by $\mathcal{H}_{k,q}(\tau)$ and $\mathcal{H}^f{}_{k,q}(\tau)$.

The formulas ϕ with $\bar{\phi} \in \mathcal{H}_{k,q}(\tau)$ ($\bar{\phi} \in \mathcal{H}^f{}_{k,q}(\tau)$) are called *Hintikka formulas*.

Hintikkika formulas, III

Proposition:

(i) Every sentence $\phi \in Fm_{k,q}(\tau)$ is equivalent to the disjunction of a unique set of (k, q)- Hintikka sentences $\bigvee_i h_{i(\phi)}$, with $\bar{h}_{i(\phi)} \in \mathcal{H}_{k,q}(\tau)$.

Not computable from k, q, τ and ϕ alone.

(ii) For every k, q, τ and τ-structure $\mathcal{A}$ there is a unique Hintikka sentence $h_{k,q}(\mathcal{A}) \in Fm_{k,q}(\tau)$ such that $\mathcal{A} \models h_{k,q}(\mathcal{A})$.

(iii) Furthermore, if $\mathcal{A}$ is finite, $h_{k,q}(\mathcal{A})$ is computable from k, q, τ and $\mathcal{A}$.

But only highly ineffective algorithms are known.

Hintikka formulas, IV

Theorem:(Ehrenfeucht-Fraïssé)

For two τ-structures $\mathcal{A}_1$ and $\mathcal{A}_2$ the following are equivalent:

(i) II has a winning strategy in the game with n moves and k point pebbles and
k set pebbles.

(ii) $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the same
sentences of $Fm_{k,m}(\tau)$.

(iii) $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the same unique
(up to equivalence) (k, m)-Hintikka sentence.

We have shown already (1) $\Rightarrow$ (3).
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (2) follows from the
properties of Hintikka formulas.
We are left with (3) $\Rightarrow$ (1).

Constructing the Hintikka sentence, I

Assume we have more pebbles than moves.

Let $\mathcal{A}$ be a finite τ-structure and $a_1, a_2, \ldots, a_s$ elements $\mathcal{A}$.

We define a formula $\phi(v_1, \ldots, v_s)^m_{\bar{a}}$

such that

$$\mathcal{A}, \bar{a} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$$

and whenever

$$\mathcal{B}, \bar{b} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$$

then player II has a winning strategy in the game for **FOL** for m more moves starting with $\mathcal{A}, \bar{a}$ and $\mathcal{B}, \bar{b}$.

$\phi(v_1, \ldots, v_k)^q_{\bar{a}}$ (i.e. $k = s, q = m$) will be a Hintikka formula for $Fm^{\mathbf{FOL}}_{k,q}(\tau)$.

Constructing the Hintikka sentence, II

$$\phi(v_1, \ldots, v_k)^0_{\bar{a}} :=$$

$$\left(\bigwedge\{R(v_{j_1}, \ldots, v_{j_s}) : R \in \tau, \mathcal{A}, \bar{a} \models R(v_{j_1}, \ldots, v_{j_s})\}\right)$$

$$\wedge$$

$$\left(\bigwedge\{\neg R(v_{j_1}, \ldots, v_{j_s}) : R \in \tau, \mathcal{A}, \bar{a} \models \neg R(v_{j_1}, \ldots, v_{j_s})\}\right)$$

$$\wedge$$

$$\left(\bigwedge\{v_{j_1} = v_{j_2} : j_1, j_2 \leq s \text{ and } \mathcal{A}, \bar{a} \models v_{j_1} = v_{j_2}\}\right)$$

$$\wedge$$

$$\left(\bigwedge\{v_{j_1} \neq v_{j_2} : j_1, j_2 \leq s \text{ and } \mathcal{A}, \bar{a} \models v_{j_1} \neq v_{j_2}\}\right)$$

The formula is finite, provided τ is, and has quantifier rank 0.

Exercise: Look at the example of a linear order with $s = 3$ and $m = 2$.

Assume $a_2 < a_1 = a_3$ in $\mathcal{A}$. Compute the formula.

Constructing the Hintikka sentence, III

$$\phi(v_1, \ldots, v_k)^m_{\bar{a}} :=$$
$$\left(\bigwedge_{a \in A} \exists v_{s+1} \phi(\bar{v}, v_{s+1})^{m-1}_{\bar{a} \cdot a} \right) \wedge$$
$$\left(\forall v_{s+1} \bigvee_{a \in A} \phi(\bar{v}, v_{s+1})^{m-1}_{\bar{a} \cdot a} \right)$$

This is finite, and has qnuatifier rank m.

Exercise: We look at the example of a linear order with $s = 3$ and $m = 2$.

Assume $a_2 < a_1 = a_3$ in $\mathcal{A}$. Compute the formula.

Constructing the Hintikka sentence, IV

We have to verify:

- $\mathcal{A}, \bar{a} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$
- whenever $\mathcal{B}, \bar{b} \models \phi(v_1, \ldots, v_s)^m_{\bar{a}}$
  then player **II** has a winning strategy
  in the game for **FOL** for m more moves
  starting with $\mathcal{A}, \bar{a}$ and $\mathcal{B}, \bar{b}$.

Constructing the Hintikka sentence, V

- We can do *"the same"* for $\mathbf{MSOL}$ and even for SOL^n or SOL.
- How do we have to modify the construction of there are fewer pebbles than moves?
- What happens if play infintely long?

We shall return to these questions later.

Project

Determine the complexity of the games.

E. Pezzoli,
Computational complexity of Ehrenfeucht-Fraïssé games on finite structures,
CSL'1998, LNCS 1584, pp.159-170.