Lecture 4

(Mostly after [Lib04], chapters 11 and 12.)

What did we do so far?

We proved the 0-1 Law for FOL.

This was done in the following steps.

- We defined the extension axioms $EA = \{EA_n : n \in \mathbb{N}\}$.
- We showed that for each n, the probability $\mu_{EA_n} = 1$.
- We showed that EA is an $\aleph_0$-categorical theory without finite models.
- We used Vaught's test to show that EA is complete and decidable.

The infinitary logic $\mathcal{L}_{\infty,\omega}$

We define the formulas of $\mathcal{L}_{\infty,\omega}(\tau)$ inductively:

- $FOL(\tau)$-formulas are $\mathcal{L}_{\infty,\omega}(\tau)$-formulas.
- $\mathcal{L}_{\infty,\omega}(\tau)$ is closed under binary conjunction $\wedge$, binary disjunction $\vee$, and unary negation $\neg$.
- $\mathcal{L}_{\infty,\omega}(\tau)$ is closed under existential ($\exists$) and universal ($\forall$) quantification of first order variables.
- If Φ is a set (of any cardinality) of $\mathcal{L}_{\infty,\omega}(\tau)$-formulas with all free variables from a fixed set $\{v_1, \ldots, v_k\}$ of free variables, then

  $$\psi_1(v_1, \ldots, v_k) = \bigwedge_{\phi \in \Phi} \phi \quad \text{and} \quad \psi_1(v_1, \ldots, v_k) = \bigvee_{\phi \in \Phi} \phi$$

  are $\mathcal{L}_{\infty,\omega}(\tau)$-formulas. with all free variables in $\{v_1, \ldots, v_k\}$.

Expressive power of $\mathcal{L}_{\infty,\omega}$
(An exercise, will be discussed on the black board)

Prove the following

- The class of finite sets is $\mathcal{L}_{\infty,\omega}$-definable for $\tau = \emptyset$.
- The class of finite sets of even cardinality is $\mathcal{L}_{\infty,\omega}$-definable for $\tau = \emptyset$.
- The class of finite graphs is $\mathcal{L}_{\infty,\omega}$-definable for $\tau = \{R_{2,0}\}$.
- The class of connected (finite or infinite) graphs is $\mathcal{L}_{\infty,\omega}$-definable for $\tau = \{R_{2,0}\}$.
- **Exercise:** None of the above is $FOL(\tau)$-definable.
  This was shown in 234293 (Logic and Sets).
  The tools for the exercise will be developed in this lecture.

$\mathcal{L}_{\infty,\omega}$ is too expressive.

Theorem 1

(i) Let K be any class of finite τ-structures closed under τ-isomorphisms. K is Then $\mathcal{L}_{\infty,\omega}(\tau)$-definable.

(ii) Let K be any class of finite or countable τ-structures closed under τ-isomorphisms. K is Then $\mathcal{L}_{\infty,\omega}(\tau)$-definable.

Proof:
We prove (i) by showing that for every finite τ-structure $\mathfrak{A}$ there is a $FOL(\tau)$-sentence $\phi_{\mathfrak{A}}$ such that $\mathfrak{A} \models \phi_{\mathfrak{A}}$ and if $\mathfrak{B} \models \phi_{\mathfrak{A}}$ then $\mathfrak{B} \cong_\tau \mathfrak{A}$.

(ii) follows from the Engeler-Scott theorem:
For every countable structure $\mathfrak{A}$ there is countable $\mathcal{L}_{\infty,\omega}(\tau)$-sentence, which characterizes it among countable structures up to isomorphisms.

Conclusion: $\mathcal{L}_{\infty,\omega}$ is too expressive.

Restricting the total number of variables.

We fix a set of variables $Var_k = \{v_1, \ldots, v_k\}$.

We denote by

(i) $FOL^k(\tau)$ the set of $FOL(\tau)$-formulas where all the variables (free and bound) are in Var_k.

(ii) $\mathcal{L}^k_{\infty,\omega}$ the set of $\mathcal{L}_{\infty,\omega}$-formulas where all the variables (free and bound) are in Var_k.

(iii) $\mathcal{L}^\omega_{\infty,\omega} = \bigcup_k^\omega \mathcal{L}^k_{\infty,\omega}$.

Exercise:
Discuss the cases $k = 0, 1$ for various τ.
How many variables are there in EA_n?
How many variables do we need to express that "there are at least k different elements"?

Negational Normal Form (NNF) and Prenex Normal Form (PNF) (An Exercise)

A formula is in **NNF**, if all the **negations** occur only in front of atomic formulas.

A formula is in **PNF** if all the **quantifications** are performed before any other boolean operation.

- Prove: Every formula $\phi \in FOL(\tau)$ is equivalent to a formula $\psi \in FOL(\tau) \cap NNF$.
- Is the same true for $\phi \in FOL^k(\tau)$?
- Prove: Every formula $\phi \in FOL(\tau)$ is equivalent to a formula $\psi \in FOL(\tau) \cap PNF$.
- Is the same true for $\phi \in FOL^k(\tau)$?
- What happens for $\mathcal{L}_{\infty,\omega}$ and $\mathcal{L}^k_{\infty,\omega}$?

Expressive power of $\mathcal{L}^{\omega}_{\infty,\omega}$: Connectedness

We want to say "there is an E-path from x to y.

We write $\bigvee_{n \geq 1} \phi_n(x, y)$ with

(i) $\phi_1(x, y) = E(x, y)$

(ii) $\phi_{n+1}(x, y) = \exists z_n \, (E(x, z_n) \wedge \phi_n(z_n, y))$
$\phi_n(x, y)$ has $n + 1$ variables.

Can we do better?

(i) $\psi_1(x, y) = E(x, y)$

(ii) $\psi_{n+1}(x, y) = \exists z(E(x, z) \wedge (\exists x(z = x \wedge \phi_n(x, y))$
$\psi_n(x, y)$ has 3 variables.

(iii) Now we write $\psi = \bigvee_{n \geq 1} \phi_n(x, y)$, hence $\psi \in \mathcal{L}^{3}_{\infty,\omega}$.

Expressive power of $\mathcal{L}^{\omega}_{\infty,\omega}$: $\exists^{=k} x \phi(x)$

Equality only:

$\exists^{\geq k} x \phi(x) = \exists x_1 \exists x_2 \ldots \exists x_k \left(\bigwedge_{i \neq j} \neg x_i = x_j \right)$.

$\exists^{\leq k} x \phi(x) = \forall x_0 \forall x_1 \ldots \forall x_k \left(\bigvee_{0 \leq i,j \leq k} x_i = x_j \right)$.

$\exists^{=k} x \phi(x) = \left(\exists^{\leq k} x \phi(x) \wedge \exists^{\geq k} x \phi(x) \right)$.

It is in $\mathcal{L}^{k+1}_{\infty,\omega}$. We would like to prove that this is best possible.

Linear orders:

$\psi_1(x) = (x = x)$.

$\psi_{n+1}(x) = \exists y \left((x > y) \wedge \exists x (x = y \wedge \psi_n(x)) \right)$.

$\psi_n(x)$ is true in linear orders, if x has at least $n-1$ predecessors.

Let $C \subseteq \mathbb{N}$. The sentence

$$\Psi_C = \bigvee_{n \in C} \left(\exists x \psi_n(x) \wedge \neg \exists x \psi_{n+1} \right)$$

says, for linear orders, that its cardinality is in C. $\Psi_C \in \mathcal{L}^2_{\infty,\omega}$.

The axioms of linear order are in FOL^3.

The most important tool

Pebble games on τ-structures.

Pebble games, I

- Given two τ-structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$.
- Two players: I (spoiler) and II (duplicator).
- k pairs of pebbles $(p_0^1, p_1^1), \ldots (p_0^k, p_1^k)$.
- Length of the game: $n \in \mathbb{N}$ or ∞.
- Move number m:
  - Player I choses $i \in \{0, 1\}$ and places a pebble p_i^j on an element of $\mathfrak{A}_i$.
  - Player II responds by placing the pebble p_{1-i}^j on an element of $\mathfrak{A}_{1-i}$.

The game of length n is denoted by $PG_k^n(\mathfrak{A}_0, \mathfrak{A}_1)$.

The game of length that continues for ever is denoted by $PG_k^\infty(\mathfrak{A}_0, \mathfrak{A}_1)$.

Pebble games, II

- After each round m of the game the pairs of pebbles placed on the structure define a relation $F_m \subseteq A_0 \times A_1$:
  $(a_0^j, a_1^j) \in F_m$ iff p_0^j is placed on $a_0^j \in A_0$ and p_1^j is placed on $a_1^j \in A_1$.

- Player II (duplicator) wins the game if in each round played F_m is a (partial) τ-isomorphism.

- Player II (duplicator) has a winning strategy for the game if he can ensure that after each round played F_m is a (partial) τ-isomorphism.

- In the case of game length n we write $\mathfrak{A}_0 \equiv_k^n \mathfrak{A}_1$.

- In the case of game length ∞ we write $\mathfrak{A}_0 \equiv_k^\infty \mathfrak{A}_1$.

The classical **colorblue Ehrenfeucht-Fraïssé Game for FOL** of length is n $PG_n^n(\mathfrak{A}_0, \mathfrak{A}_1)$ with $\mathfrak{A}_0 \equiv_n^n \mathfrak{A}_1$.

Pebble games, III

An exercise in winning strategies

Pure sets ($\tau = \emptyset$).

What are the cardinalities of sets S_0, S_1 such that $S_0 \equiv_k^n S_1$ or $S_0 \equiv_k^\infty S_1$?

Linear orders.

Let L_0, L_1 be any two linear orders with at least two comparable elements.
Let P_m^{dir} the directed path and C_m^{dir} the directed circle of size m.
Discuss the two and three pebble games on these structures.

Extension axioms.

Assume G_0, G_1 are two graphs in which for all $m \leq n \leq k \in \mathbb{N}$ the exioms $EA_{n,m}$ hold. Then $G_0 \equiv_k^\infty G_1$.

Pebble games, IV

Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be two τ-structures.

We say $\mathfrak{A}_0 \sim_k^{\infty,\omega} \mathfrak{A}_1$ if they agree on all sentences of $\mathcal{L}^k_{\infty,\omega}$.

We say $\mathfrak{A}_0 \sim_{k,n}^{\infty,\omega} \mathfrak{A}_1$ if they agree on all sentences of $\mathcal{L}^k_{\infty,\omega}$ of quantifier rank n.

Theorem 2

(i) $\mathfrak{A}_0 \equiv_k^n \mathfrak{A}_1$ *iff* $\mathfrak{A}_0 \sim_{k,n}^{\infty,\omega} \mathfrak{A}_1$

(ii) $\mathfrak{A}_0 \equiv_k^\infty \mathfrak{A}_1$ *iff* $\mathfrak{A}_0 \sim_k^{\infty,\omega} \mathfrak{A}_1$

The proof will be given in the next lecture.