Outline of Lecture 2

First Order Logic and Second Order Logic

Basic summary and Toolbox

- Vocabularies and structures
- Isomorphisms and substructures
- First Order Logic **FOL**, a reminder.
- Completeness and Compactness
- Second Order Logic **SOL** and $\mathbf{SOL}^n$ and Monadic Second Order Logic **MSOL**.
- Definability

Vocabularies

We deal with (possibly many-sorted) relational structures.

Sort symbols are

$$U_\alpha : \alpha \in \mathbb{N}$$

Relation symbols are

$$R_{i,\alpha} : i \in Ar, \alpha \in \mathbb{N}$$

where Ar is a set of *arities*, i.e. of finite sequences of sort symbols.

In the case of one-sorted vocabularies, the arity is just of the form $\langle U, U, \ldots_n \ldots, U \rangle$ which will denoted by n.

A **vocabulary** is a *finite* set of *finitary* **relation symbols**, usually denoted by τ, τ_i or σ.

τ-structures

Structures are **interpretations of vocabularies**, of the form

$$\mathcal{A} = \langle \mathcal{A}(U_\alpha), \mathcal{A}(R_{i,\alpha}) : U_\alpha, R_{i,\alpha} \in \tau \rangle$$

with $\mathcal{A}(U_\alpha) = A_\alpha$ sets, and, for $i = (U_{j_1}, \ldots, U_{j_r})$ $\mathcal{A}(R_{i,\alpha}) \subseteq A_{j_1} \times \ldots \times A_{j_r}$.

Graphs: $\langle V; E \rangle$ with vertices as domain and edges as relation.
$\langle V \sqcup E, R_G \rangle$ with two sorted domain of vertices and edges and incidence relation.

Labeled Graphs: As graphs but with unary predicates for vertex labels and edge labels depending whether edges are elements or tuples.

Binary Words: $\langle V; R_<, P_0 \rangle$ with domain
lineraly ordered by $R_<$ and colored by P_0, marking the zero's.

τ-structures: General relational structures.

τ-Substructures

Let $\mathcal{A}, \mathcal{B}$ be two τ-structures.

$\mathcal{B}$ is a **substructure** of $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{A}$, if

(i) $B \subseteq A$, and

(ii) for all $R_{i,\alpha} \in \tau$ and all $(b_1, \ldots, b_i) \in B^i$ we have $(b_1, \ldots, b_i) \in \mathcal{B}(R_{i,\alpha})$ iff $(b_1, \ldots, b_i) \in \mathcal{A}(R_{i,\alpha})$.

Exercise: Adapt the above definition two many-sorted structures.

Example:

- For graphs of the form $\langle V, E \rangle$ (one-sorted), where E is a binary relation, substructures are **induced subgraphs**.
- For graphs of the form $\langle V, E; R \rangle$ (two-sorted), where $R \subseteq V \times E$ is a binary relation, substructures are **subgraphs**.

Isomorphisms between τ-structures

Let $\mathcal{A}, \mathcal{B}$ be two one sorted τ-structures with universe A, B respectively. A function $f : A \to B$ is an isomorphism if

(i) f is one-one and onto (a bijection).

(ii) For every relation symbol $R_{i,\alpha} \in \tau$ and every $\bar{a} = (a_1, \ldots, a_i) \in A^i$ we have $\bar{a} \in \mathcal{A}(R_{i,\alpha})$ iff $f(\bar{a}) = f(a_1, \ldots, f(a_i) \in \mathcal{B}(R_{i,\alpha})$

Exercise: Write down the definition of isomorphism in the many-sorted case.

How to prove that

two countable τ-structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic?

We play the following infinite game between
Player I (spoiler) and II (duplicator):

(i) In move $2n + 1$ Player I picks an element $a_{2n+1} \in A$ and Player II picks an element $b_{2n+1} \in B$.

(ii) In move $2n$ Player I picks an element $b_{2n} \in B$ and Player II picks an element $a_{2n} \in A$.

(iii) Let $A_0 \subseteq A$ and $B_0 \subseteq B$ be the sets of elements chosen by the players. Player II wins if the map $g : A_0 \to B_0$ defined by $g(a_i) = b_i$ is an isomorphism of the induced substructures $\mathcal{A}_0$ and $\mathcal{B}_0$.

Note:

Player I can choose again elements already chosen by any of the players.

A_0 and B_0 can be proper subsets of A and B.

Cantor's Theorem

Theorem: Two countable τ-structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff Player II has a winning strategy.

Proof: Assume $f : A \to B$ is an ismorphism. Then Player II has a winning strategy by answering using f.

- If Player I played as move $2n+1$ the element a_{2n+1}, II answers $f(a_{2n+1})$.
- If Player I played as move $2n$ the element b_{2n}, II answers $f^{-1}(b_{2n})$.

Conversely, assume Player II has a winning strategy, and let $A = (a_i : i \in \mathbb{N})$ and $B = (b_i : i \in \mathbb{N})$ be enumerations of A and B.

Player I can help building an isomorphisms f by always choosing the smallest element in A and B respectively, which was not yet chosen. Q.E.D.

Properties of a τ-structure

A **property** of τ-structures is a class $\mathcal{P}$ of τ-structures which is **closed under τ-isomorphisms**.

- All *finite* τ-structures.
- All $\{R_{2,0}\}$-structures where $R_{2,0}$ is interpreted as a *linear order*.
- Al finite 3-dimensional matchings $3DM$, i.e. all $\{R_{3,0}\}$-structures with universe A where the interpretation of $R_{3,0}$ contains a subset $M \subseteq A^3$ such that no two triples of M agree in any coordinate.
- All binary words which are palindroms.

A τ-structure $\mathcal{A}$ has property $\mathcal{P}$ iff $\mathcal{A} \in \mathcal{P}$.

Properties of linear orders

We look at one-sorted structures $\langle A, R \rangle$ with one binary relation R.

Sample properties are:

- A is **finite**, A is **countable**;

The relation R is

- a **linear order**, with/without **first** or **last** element;
- a **discrete** linear order (if $a \in A$ is not a first (last) element, there is a biggest (smallest) element smaller (larger) than a)
- a **dense** linear order (between any two distinct elements there is a further element);
- a **well-ordering** (every subset of A has a least element);

Countable isomorphic structures

Theorem: Let $\mathcal{A}$ and $\mathcal{B}$ be two countable or finite linear orders.

(i) If both are discrete, have a first but no last element, and are well-orderings, then they are isomorphic, and isomorphic to $\langle \mathbb{N}, <^{nat} \rangle$.

(ii) There are no uncountable discrete well-orderings.

(iii) If both are dense, have no first nor last element, then they are isomorphic, and isomorphic to $\langle \mathbb{Q}, <^{nat} \rangle$.

Exercise:

(i) Find many non-isomorphic countably infinite well-orderings,

(ii) Find many non-isomorphic uncountable dense orderings of the same cardinality.

First Order Logic **FOL**(τ):

For **one-sorted** structures of the form $\mathcal{A} = \langle V, R_1^V, \ldots R_M^V \rangle$ and $\tau = \{R_1' \ldots, R_M\}$

Variables: $u, v, w, \ldots, u_\alpha, v_\alpha, w_\alpha, \ldots, \alpha \in \mathbb{N}$ ranging over elements of the domain V.

R_j a $\rho(j)$-ary relation symbol whose interpretation is R_j^V.

Atomic formulas: $R_j(\bar{u})$, $u = v$.

Connectives: $\wedge, \vee, \neg$,

Quantifiers: $\forall v, \exists v$

Exercise: Write down the definition of **FOL**(τ) in the many-sorted case.

Exercise: Given two isomorphic τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, show that for every $\phi \in$ **FOL**(τ) without free variables we have $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

What we know about **FOL** (from Sets and Logic, 234293)

Basic Notions

Σ is a (possibly infinite) set **FOL**(τ) formulas, ϕ, ψ are **FOL**(tau) formulas.

(i) A τ-structure $\mathcal{A}$ **satisfies** Σ, denoted by $\mathcal{A} \models \Sigma$.

(ii) Σ is **satisfiable** iff there is a τ-structure $\mathcal{A}$ such that $\mathcal{A} \models \Sigma$.

(iii) **Logical consequence** (a semantic notion) $\Sigma \models \phi$.
In every τ-strаucture $\mathcal{A}$ we have, if $\mathcal{A} \models \Sigma$, then also $\mathcal{A} \models \phi$.

(iv) **Logical equivalence**: $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$.

(v) **Provability** (a syntactic notion) $\Sigma \vdash \phi$
The details of the proof system are not important here

(vi) **Soundness** of provability: $\Sigma \vdash \phi$ implies $\Sigma \models \phi$.

(vii) **Completeness** of provability: $\Sigma \models \phi$ implies $\Sigma \vdash \phi$.

What we know about **FOL** (from Sets and Logic, 234293)

Completeness and Compactness

Completeness Theorem: $\Sigma \models \phi$ iff $\Sigma \vdash \phi$.

Compactness Theorem: Let Σ be an infinite set of **FOL**(τ) formulas. Σ is satisfiable iff every **finite subset** $\Sigma_0 \subset \Sigma$ is satisfiable.

Löwenheim-Skolem Theorem: If Σ is countable or finite and there is an **infinite** $\mathcal{A}$ with $\mathcal{A} \models \Sigma$, there is a **countable** $\mathcal{B}$ with $\mathcal{B} \models \Sigma$.

Löwenheim-Skolem-Tarski-Mal'cev Theorem: For any Σ, if there is an **infinite** $\mathcal{A}$ with $\mathcal{A} \models \Sigma$, there are models $\mathcal{B}$ of **any infinite cardinality** with $\mathcal{B} \models \Sigma$.

What we (should) know about **FOL**

from Sets and Logic (234293), or Logic 2 (23xxxx)

Computability questions

A set Σ of formulas of **FOL**(τ) is **complete** if for every $\phi \in$ **FOL**(τ) without free variables, either $\Sigma \models \phi$ or $\Sigma \models \neg\phi$.

Let Σ be a recursive (=computable) set of formulas of **FOL**(τ).

- The set of consequences of Σ

  $$Con(\Sigma) = \{\phi \in \mathbf{FOL}(\tau) : \Sigma \models \phi\}$$

  is recursively enumerable (= semi-computable).

- If additionally, Σ is complete, $Con(\Sigma)$ is recursive (= computable).

- **The Curch-Turing Theorem:** $Con(\emptyset)$ is **not** recursive.

Monadic Second Order Logic $\mathbf{MSOL}(\tau)$:

Additionally we have, in the **one-sorted** case

Variables: $X, Y, Z, \ldots$ ranging over subsets of V.

Atomic formulas: $u \in X, v \in Y, \ldots$

Quantifiers: $\forall X, \exists X$.

Second Order Logic $\mathbf{SOL}^n(\tau)$ and $\mathbf{SOL}(\tau)$:

We extend (one-sorted) $\mathbf{MSOL}(\tau)$ by the following features:

Variables: $X^m, Y^m, Z^m, \ldots$ for $m \leq n$

Atomic formulas: $(u_1, \ldots, u_m) \in X^m, \ldots$

Quantifiers: $\forall X^m, \exists X^m$.

For fixed m this gives us $\mathbf{SOL}^m$, and $\mathbf{SOL} = \bigcup_n \mathbf{SOL}^n$

Clearly we have syntactically, and hence in expressing power

$$\mathbf{MSOL}(\tau) \subseteq \mathbf{SOL}^2(\tau) \subseteq \mathbf{SOL}(\tau)$$

In $\mathbf{SOL}^2$ we can quantifier over **arbitrary sets of pairs of vertices**,

Isomorphic structures are indistinguishable in $\mathbf{SOL}$.

Exercise: Given two isomorphic τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, show that for every $\phi \in \mathbf{SOL}(\tau)$ without free variables we have $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

$\mathcal{L}(\tau)$-Definability

$\mathcal{L}(\tau)$ stands for a subset of $\mathbf{SOL}(\tau)$. Recall that an $\mathcal{L}(\tau)$-sentence is an $\mathcal{L}(\tau)$-formula without free variables.

Given a a class of τ-structures K, we say that K is **$\mathcal{L}(\tau)$-definable** if there is a $\mathcal{L}(\tau)$-sentence θ such that for every τ-structure $\mathcal{A}$

$$\mathcal{A} \models \theta \text{ iff } \mathcal{A} \in K.$$

We write $Mod_{\mathcal{L}(\tau)}(\theta)$ for the class of τ-structures $\mathcal{A}$ such that $\mathcal{A} \models \theta$.

Proving definability

The class of τ-structures of finite even cardinality, $EVEN(\tau)$, is definable in Second Order Logic:

- Let $\tau_1 = \{R, S, P\}$ with R, S binary and P unary, none of them in τ.
- We write a $FOL(\tau_1)$-formula $\phi_{bij}(R, P)$ which says that R is a bijection between P and its complement.
- We write a $FOL(\tau_1)$-formula $\psi_{inj}(S)$ which says that S is a proper injection of the domain into itself.
- Now the required formula is

$$\exists R \exists P \phi_{bij}(R, P) \land \forall S \neg \psi_{inj}(S)$$

MSOL on words (Homework)

Theorem:[Büchi, Elgot, Trakhtenbrot, 1961]

A class of binary words is:
recognizable by a finite
(non-deterministic) automaton
iff it is **MSOL**-definable
(iff it is regular).

Example: $(101 \vee 1001)^*$
101 1001 101 101 1001 1001 101........

Exercise: Find the **MSOL**-formula.

Definability of order properties

We look again at one-sorted structures $\langle A, R \rangle$ with one binary relation R.

The following are **FOL**-definable:

- R is a **linear order**, with/without **first** or **last** element;
- R is a **discrete** linear order (if $a \in A$ is not a first (last) element, there is a biggest (smallest) element smaller (larger) than a)
- a **dense** linear order (between any two distinct elements there is a further element);

Using compactness we can prove that following are **not** **FOL**-definable:

- A is **finite**, A is **countable**;
- a **well-ordering** (every subset of A has a least element);

Problems with (Monadic) Second Order Logic $\mathbf{SOL}$ ($\mathbf{MSOL}$)

- $\mathbf{MSOL}$ is **not compact**:

  The class of infinite discrete well-orderings is $\mathbf{MSOL}$-definable and has (up to isomorphisms) one one model.

- $\mathbf{MSOL}$ has **no complete provability system**:

  The **Peano axioms** are expressible in $\mathbf{MSOL}$ and characterize the structure $\langle \mathbb{N}, +, \times, 0, 1 \rangle$ up to isomorphims. If there were a complete provability system, the set of $\mathbf{MSOL}(\tau_{arith})$-sentences true in $\langle \mathbb{N}, +, \times, 0, 1 \rangle$ would be computable.
  But this contradicts **Gödel's First Incompleteness Theorem**.

- There is a satisfiable sentence of $\mathbf{MSOL}(\tau_{arith})$
  which has **no countable or finite models**:

  $\langle \mathbb{R}, +, \times, 0, 1 \rangle$ can be characterized up to isomorphims by saying it is an ordered field which is archimedian and Dedekind complete.

From a mathematical point of view this is **good**!

But it means we need **other tools**, to be developed in the sequel.

Graph properties

For graph properties you may consult

- B. Bollobas, Modern Graph Theory, Springer, 1998
- R. Diestel, Graph Theory, Springer, 3rd edition, 2005
- A. Brandstädt and V.B. Le and J. Spinrad, Graph Classes: A survey, SIAM Monographs on Discrete Mathematics and Applications, 1999

Most graph properties are $\mathbf{SOL}$-definable.

Many of them are $\mathbf{MSOL}$-definable.

Few are $\mathbf{FOL}$-definable.

Forbidden induced subgraphs $Forb_{ind}(H)$.

P_n is a path on n vertices.

The following graph properties are **FOL**-definable:

- For H any simple graph, let $Forb_{ind}(H)$ class of finite graphs which have no induced copy of H.

- Cographs were first defined inductively: The class of cographs is the smallest class of graphs which contains the single vertex graph E_1 and is closed under disjoint unions and (loopfree) graph complement.

  [Corneil, Lerchs and Stewart Burlingham, 1981]
  A graph G is a **cograph** if and only if there is no induced subgraph of G isomorphic to a P_4.

- A graph G is P_4-**sparse** if no set of 5 vertices induces more than one P_4 in G.

  Cliques and Cographs are P_4-sparse.

  There is also a characterization of P_4-sparse graphs with forbidden induced subgraphs.

The speed of $Forb_{ind}(H)$

Prömel and Steger, 1992

Theorem: If H is an induced subgraph of P_4 then the speed of the H-free graphs is bounded above by

$$n^{n+o(n)}.$$

Otherwise the speed is bounded below by

$$2^{(\frac{1}{4}+o(1))n^2}.$$

Corollary: If H is an induced subgraph of P_4 then

$$\mu(Forb_{ind}(H)) = 0$$

Problem: Can we determine $\mu(Forb_{ind}(H)) = 0$ for the remaining cases?

3-Colorability

The class of 3-colorable graphs $3COL$ is hereditary and definable by a **MSOL**-formula

$$\exists X_1, X_2, X_3 \phi_{partition}(X_1, X_2, X_3) \wedge \bigwedge_{i=1}^{3} \phi_{color}(X_i)$$

where

- $\phi_{partition}(X_1, X_2, X_3)$ says that X_1, X_2, X_3 form a partition of the vertices and
- $\phi_{color}(X_i)$ says that there are no edges between two vertices in X_i.

Note that all the second order variables are unary and $\phi_{partition}$ and ϕ_{color} are first order formulas over $\tau = \{E, X_1, X_2, X_3\}$.

Homework

Homework:

(i) Compute the speed of $3COL$.

(ii) Is $3COL$ of the form $Forb_{ind}(H)$?

(iii) Is $3COL$ **FOL**-definable?

(iv) Define k-colorability and show, that for fixed k, it is $\mathbf{MSOL}$-definable.