# Logical Methods in Combinatorics 236605 (2009/10)

## Homework to Lecture 1 and 2

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(i) Topologies and preorders.

(ii) Counting 1-regular graphs (= perfect matchings).

(iii) Check monotonicity, hereditarity and definability of selected graph properties.

(iv) Forbidden (induced) subgraphs

(v) **MSOL**-definability of regular languages

(vi) 3-colorability

# Toplogies and preorders, I

A *preorder* $R$ is a
transitive and reflexive binary relation.

If additionally $R(a,b) \wedge R(b,a) \Rightarrow a = b$, $R$ is a *partial order*.

$PreO_n$ is the number of preorders on the set $\{1, \ldots, n\}$.

**Theorem:**
The number of preorder on a finite set equals the number of topolgies, i.e., $PreO_n = T_n$.

**Complete and verify the following proof sketch.**

# Toplogies and preorders, II

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**Proof:** (From topology to preorder)

Let $S = \{1, \ldots, n\}$.

Let $Cl : P(S) \rightarrow P(S)$ be any function. closure operation coming from some topology.

Define $cl : S \rightarrow P(S)$ by $cl(a) = Cl(\{a\})$.

Define $R_{Cl}(a, b)$ by $a \in cl(b)$.

**Claim:** $R_{Cl}$ is a preorder iff $Cl$ is the closure operation from some topolgy.

This uses that $S$ is finite.

# Toplogies and preorders, III

**Proof:** (From preorder to topology)

Let $R$ be a preorder on $S$.

We define a function $cl_R : S \to P(S)$ by

$$cl_R(b) = \{a \in S : R(a, b).$$

As for finite sets, the topolgy is defined, once we now the closures of the singletons, we define for $X \subseteq S$

$$Cl(X) = \bigcup_{a \in X} cl(a)$$

To verify that this works we use the previous claim .

# Counting perfect matchings

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Let $\mathcal{P}$ consist of graphs which are 1-regular
(disjoint union of non-connected edges, perfect matchings).

(i) $\mathcal{P}$ is not hereditary. **Verify!**

(ii) $\mathcal{P}$ is $FOL$-definable. **Write the formula explicitly!**

(iii) **Compute and verify the density** $f_{\mathcal{P}}(m)$. and $sp(\mathcal{P})$.

(iv) Compute and verify the fractions and probabilities $\mu_m(\mathcal{P})$ and $\mu(\mathcal{P})$.

## Monotonicity and Definability

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Check for monotonicity, hereditarity and definability in $\mathbf{FOL}, \mathbf{MSOL}$ and $\mathbf{SOL}$:

(i) Connected graphs, graphs of diameter $d$;

(ii) regular graphs, $r$-regular graphs;

(iii) Graphs with an Eulerian or Hamiltonian path (circuit);

(iv) Planar graphs,

(v) bipartite graphs, $r$-colorable graphs.

# Forbidden subgraphs

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Choose your favorite graph $H$.

Consider the classes of graphs $Forb(H)$ and $iForb(H)$ of graphs with no subgraph, resp. induced subgraph isomorphic to $H$.

- For which $H$ are these classes monotone, hereditary?

  Try to formulate a general answer based on properties of $H$.

- Show that for each $H$ both are $FOL$-definable.

- Try to compute the density function and asymptotic probabilities for various choices of $H$.

- Find the optimal results in the literature.

## $\mathbf{MSOL}$ on words (Homework)

**Theorem:**[Büchi, Elgot, Trakhtenbrot, 1961]

A class of binary words is:
recognizable by a finite
(non-deterministic) automaton
iff it is $\mathbf{MSOL}$-definable
(iff it is regular).

**Example:** $(101 \vee 1001)^*$
101 1001 101 101 1001 1001 101........

**Exercise:** Find the $\mathbf{MSOL}$-formula.

## $k$-colorability

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**Homework:**

(i) Compute the speed of $3COL$.

(ii) Is $3COL$ of the form $Forb_{ind}(H)$?

(iii) Is $3COL$ **FOL**-definable?

(iv) Define $k$-colorability and show, that for fixed $k$, it is **MSOL**-definable.