On Asymptotic Probabilities of Monadic Second Order Properties

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Abstract

We propose a new, general and easy method for proving nonexistence of asymptotic probabilities of monadic second–order sentences in classes of finite structures where first–order extension axioms hold almost surely.

1 Introduction

1.1 The problem

In this paper we discuss logical problems of *random structure theory.*

Let us consider a class of finite structures over some fixed signature, equipped with a probability space structure. This probability is usually assumed to be only *finitely* additive. Then we draw one structure at random and ask:

- how does the drawn structure look like?
- does the drawn structure have some particular property?

Those questions are typical in random structure theory. To turn to the logical part of it, look at the drawn structure through the logical glasses: we can only notice properties definable in some particular logic. Then new questions become natural:

- does every property we can observe have a probability (is it measurable)?
- if so, what is the value of this probability?
- is it possible to compute this probability, and what is the complexity of the computation?

It becomes clear from the above that the random structure theory is closely connected to combinatorics, finite model theory, mathematical logic and computer science. An exposition of the logical part of the random structure theory may be found in a nice survey of Compton [1].

One of the problems we would like to pursue in this paper is the following: It was empirically observed that first–order 0–1 laws sometimes extend to fixpoint 0–1 laws, sometimes to monadic second–order 0–1 laws, but almost never to both of these logics at the same time. There was no theoretical explanation of this fact so far. One of our main intentions, except presenting some new results and new proofs of some already known results, is to make a first step towards such explanation. It will follow from uniform proof method we propose for nonconvergence results in monadic logic.

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1.2 Definitions

Throughout the paper we are dealing with first–order and monadic second–order (monadic, for short) logics over some fixed, finite signature σ (with equality). We assume that σ contains exclusively relation symbols, and therefore functions are represented as restricted relations.

Formulas of first–order logic are built from atomic formulas of the form $R(\vec{x})$, where R is a symbol in σ and $\vec{x}$ is a vector of variables of length equal to arity of R, and using usual connectives: $\wedge$, $\vee$, $\neg, \rightarrow$, and quantifiers $\forall$, $\exists$.

In formulas of monadic logic set variables are allowed to occur in atomic formulas of the form $x \in X$. Also set quantification, denoted $\forall$ and $\exists$, is allowed.

We use uppercase letters, or two–letter uppercase abbreviations, to denote set variables, and lowercase ones to denote first–order variables. Therefore it will always be clear what kind of quantification we have in mind.

Let $\boldsymbol{\mathcal{A}}$ be a set of finite structures $\mathbf{A}$ over the signature σ, such that the universe $|\mathbf{A}|$ of $\mathbf{A}$ is some initial segment of natural numbers. Let $\boldsymbol{\mathcal{A}}(n)$ be a subset of $\boldsymbol{\mathcal{A}}$ containing all structures $\mathbf{A}$ with carrier set (of cardinality) $|\mathbf{A}| = n = \{0, \ldots, n-1\}$. To avoid pathological cases we assume that for each positive $n \in \omega$ the set $\boldsymbol{\mathcal{A}}(n)$ is nonempty.

Let for $n = 0, 1, \ldots$ μ_n be a probability distribution on $\boldsymbol{\mathcal{A}}(n)$. We write $\boldsymbol{\mu}$ for $\{\mu_n\}_{n\in\omega}$, and call $\boldsymbol{\mu}$, somehow loosely, also a distribution. The pair $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$ is an object of our study in this paper.

For any subset $D \subseteq \boldsymbol{\mathcal{A}}$ we define

$$\mu_n(D) = \mu_n(D \cap \boldsymbol{\mathcal{A}}(n)).$$

If $D = \{\mathbf{A} \in \boldsymbol{\mathcal{A}} \mid \mathbf{A} \models \varphi\}$ for some sentence φ of the logic under consideration, then we write $\mu_n(\varphi)$, instead of $\mu_n(D)$. We are interested in asymptotic properties of $\mu_n(\varphi)$, and especially whether the limit $\mu(\varphi) = \lim_{n\to\infty} \mu_n(\varphi)$ exists, for sentences φ of the logic under consideration. If it exists, we call it an *asymptotic probability of* φ. If this is the case for every sentence of the logic L, we say that the *convergence law* holds (for L and $\boldsymbol{\mu}$). If, in addition, every sentence has probability either 0 or 1, we say that the *0–1 law* holds. Sometimes, instead of writing $\mu(\varphi) = 1$, we say that φ *holds almost surely (a.s. in short).*

The following family of examples of probability distributions on the class $\boldsymbol{\mathcal{G}}$ of all finite graphs $\mathbf{G} = \langle |\mathbf{G}|, E \rangle$ was first introduced and studied by Erdős and Rényi in [3]. Let $p = p(n)$ be a function from ω into the real interval [0,1]. Then we define the probability model $\boldsymbol{\mathcal{G}}(n,p) = \langle \boldsymbol{\mathcal{G}}, \boldsymbol{\mu}^p \rangle$ with $\boldsymbol{\mathcal{G}}$ being the class of all undirected finite graphs, and $\mu_n^p(\{\mathbf{G}\}) = p^e(1-p)^{\binom{n}{2}-e}$, where e is the number of edges in $\mathbf{G}$. Equivalently, one obtains a random graph $\mathbf{G} \in \boldsymbol{\mathcal{G}}(n,p)$ as a result of the following experiment: For every pair of vertices in $\{0, \ldots, n-1\}$, independently of other pairs, one tosses a coin with outcomes 1 (edge) with probability $p(n)$ and 0 (non-edge) with probability $1 - p(n)$. After $\binom{n}{2}$ tosses the random graph is constructed.

Another example is the following:

Let $\boldsymbol{\mathcal{A}}$ be arbitrary class of finite structures, satisfying the conditions we formulated at the beginning. We then define the *uniform labelled probability distribution on* $\boldsymbol{\mathcal{A}}$. Namely, we set

$$\mu_{|\mathbf{A}|}(\{\mathbf{A}\}) = \begin{cases} 1/|\boldsymbol{\mathcal{A}}(n)| & \text{if } \mathbf{A} \in \boldsymbol{\mathcal{A}}(n), \\ 0 & \text{if } \mathbf{A} \notin \boldsymbol{\mathcal{A}}(n). \end{cases}$$

E.g., if $p = const. = 1/2$ then the $\boldsymbol{\mu}^p$ and the uniform labelled distributions on $\boldsymbol{\mathcal{G}}$ coincide.

Unlabelled distribution is the one in which equivalence classes of the isomorphism relation are equiprobable rather than structures themselves.

For a finite structure $\mathbf{A} \in \boldsymbol{\mathcal{A}}(n)$ we define a first–order quantifier–free formula $[\{x_0, \ldots, x_{n-1}\} \simeq \mathbf{A}]$, called the *diagram of* $\mathbf{A}$. Let v be a valuation such that $v(x_i) = i$ for $i = 0, \ldots, n-1$. Then we define $[\{x_0, \ldots, x_{n-1}\} \simeq \mathbf{A}]$ to be the conjunction of all formulas in the set

$$\big\{R(x_{i_1}, \ldots, x_{i_k}) \mid R \in \sigma,\ 0 \le i_1, \ldots, i_k < n,\ \mathbf{A}, v \models R(x_{i_1}, \ldots, x_{i_k})\big\} \cup$$

$$\big\{\neg R(x_{i_1}, \ldots, x_{i_k}) \mid R \in \sigma,\ 0 \le i_1, \ldots, i_k < n,\ \mathbf{A}, v \models \neg R(x_{i_1}, \ldots, x_{i_k})\big\}.$$

1.3 Organization of the paper

The paper is organized as follows: In the second section we give a new proof of a classical result of Kaufmann and Shelah [6], stating that there are monadic second–order properties without asymptotic probability with respect to uniform, labelled probability on the class of finite graphs. In the third section we show how the proof can be improved to work for so called *sparse* random graphs $\mathcal{G}(n,p)$ with $p = p(n)$ bounded away from 1 and satisfying $p(n) \geq n^{-\alpha}$ for some $\alpha < 1$. The fourth section is devoted to adaptation of the same proof to the case of random (uniform, labelled) partial orders, and random K_{m+1}–free graphs. In the fifth section we discuss interrelations of monadic second–order and fixpoint logics, which can be deduced from our results. In the last, sixth section, we present some final remarks.

2 The Kaufmann and Shelah Theorem

In this section we give a new proof of the following, classical result of Kaufmann and Shelah.

Theorem 1 ((Kaufmann and Shelah [6])) *Then there are monadic second–order sentences without asymptotic uniform labelled probability in the class of finite graphs $\mathcal{G}$.*

Proof: First short description of the main idea: consider a pair $(\mathbf{G},\mathbf{H})$ of graphs $\mathbf{G} \subseteq \mathbf{H}$ with $|\mathbf{G}| = \{0,\ldots,k-1\}$ and $|\mathbf{H}| \setminus |\mathbf{G}| = \{k,\ldots,\ell-1\}$. From now on writing $(\mathbf{G},\mathbf{H})$ we tacitly make an assumption that $\mathbf{G} \subseteq \mathbf{H}$.

Denote by $Ext(\mathbf{G},\mathbf{H})$ the following sentence, usually called an *extension axiom*:

$$\forall x_0,\ldots,x_{k-1}\ \left([\{x_0,\ldots,x_{k-1}\} \simeq \mathbf{G}] \rightarrow \left(\exists x_k,\ldots,x_{\ell-1}\ [\{x_0,\ldots,x_{\ell-1}\} \simeq \mathbf{H}]\right)\right),$$

i.e., the one expressing *"every copy of* $\mathbf{G}$ *extends to a copy of* $\mathbf{H}$*".*

It is well known since the paper of Fagin [4] that for arbitrary two graphs $\mathbf{G} \subseteq \mathbf{H}$ the equality $\mu(Ext(\mathbf{G},\mathbf{H})) = 1$ holds. In particular $\mu(Ext(\emptyset,\mathbf{G})) = 1$ and $\mu(Ext(\mathbf{G},\mathbf{H})) = 1$. Then, assuming additionally that $|\mathbf{H}|$ is of much greater cardinality than $|\mathbf{G}|$, the graph of the function $n \mapsto \mu_n(Ext(\mathbf{G},\mathbf{H}))$ looks like the one on figure 1, at the end of the paper.

It is important that $\mu_n(Ext(\mathbf{G},\mathbf{H})) = 1$ for $n < |\mathbf{G}|$, and $\mu_n(Ext(\mathbf{G},\mathbf{H})) = 1 - \mu_n(Ext(\emptyset,\mathbf{G}))$ for $n < |\mathbf{H}|$.

Then it is natural to expect that a sentence expressing $\bigwedge_{i\in\omega} Ext(\mathbf{G}_i,\mathbf{H}_i)$, with cardinalities of $(\mathbf{G}_i,\mathbf{H}_i)$ growing very fast with i, should have no asymptotic probability, as we may naturally expect

$$\mu_n(\bigwedge_{i\in\omega} Ext(\mathbf{G}_i,\mathbf{H}_i)) \approx \min_{i\in\omega} \mu_n(Ext(\mathbf{G}_i,\mathbf{H}_i)),$$

as in the figure 2 at the end of the paper.

Now let us be more precise.

Let M be a deterministic, one tape Turing Machine that accepts numbers in unary expansion as its arguments, and always halts.

Define

$$size_M(m) = m + space_M(m) \cdot time_M(m),$$

where $time_M(m)$ denotes the number of steps of computation of M on input m, and similarly for $space_M$.

We construct a monadic formula $\varphi(X)$ with the property that whenever $\mathbf{G} \models \varphi(X)$ is true, the cardinality of X is equal to $size_M(m)$ for some m.

Indeed, take $\varphi(X)$ to be:

$$\exists M, U\ \exists EC, OC, ER, OR\ \tilde{\varphi}(M, U, EC, OC, ER, OR),$$

where $\tilde{\varphi}$ is the conjunction of the following conditions:

1. $M, U \subseteq X$
2. $EC, OC, ER, OR \subseteq U$

3. $M \cap U = \emptyset,\ M \cup U = X$,

4. $EC \cap OC = \emptyset,\ EC \cup OC = U$,

5. $ER \cap OR = \emptyset,\ ER \cup OR = U$,

6. $\langle U, E\rangle$ is a square grid, such that EC and OC (ER and OR, resp.) are unions of disjoint chains, which are even and odd columns (rows, resp.) of the grid,

7. E is a bijection from M onto the first $|M|$ elements in the first row of the grid,

8. there are no other edges, except possible edges between vertices of M.

X extends M – we call extensions of that shape *grid extensions.* A figure showing grid extension is to be found at the end of the paper. Now it is routine to express, adding more set variables that correspond to letters in tape cells, control states and head positions that the grid with these sets represents successful computation of M on input $1\ldots1$ of length $|M|$.

Now similarly, for another Turing machine N we can write a formula $\gamma(X, Y)$ with the property that whenever $\mathbf{G} \models \gamma(X, Y)$, then $|Y \setminus X|$ is equal to $size_N(|X|)$. It is to be done by treating X exactly as M in φ above, but without quantifying it.

Now let the function $g : \omega \to \omega$ be defined as follows:

$$g(m) = 1 + \begin{pmatrix} \text{least } n > g(m-1) \text{ such that } \mu_n(\bigwedge Ext(\mathbf{G}, \mathbf{H})) \geq \\ 1 - 1/m, \text{ where we conjunct over all grid exten-} \\ \text{sions } (\mathbf{G}, \mathbf{H}) \text{ with } |\mathbf{H}| \leq m \end{pmatrix} \tag{1}$$

Clearly g is recursive and strictly increasing.

Let a machine N compute some space constructible function $h > g$ in the way that $h(m) = space_N(m)$, taking unary strings as inputs and producing unary strings as outputs.

Now let M be a one tape deterministic Turing Machine that takes unary input strings and outputs also unary strings. Moreover, let M compute a total function f that for $m > 0$ satisfies:

$$f(m) > g(size_N(size_M(m-1))).$$

Let us see how to make a monadic sentence without asymptotic probability from N and M.

Let $\varphi(X)$ and $\gamma(X, Y)$ be formulas constructed for M and N, respectively.

Consider sentence

$$\mathbf{Ext} \equiv \forall X\ (\varphi(X) \to (\exists Y\ \gamma(X, Y))).$$

Observe that indeed **Ext** is equivalent to the infinitary conjunction of the form $\bigwedge_{m\in\omega} Ext(\mathbf{G}_m, \mathbf{H}_m)$, where $\mathbf{G}_m$ is a grid extension of $\emptyset$ with $size_M(m)$ elements, and $\mathbf{H}_m$ is a grid extension of $\mathbf{G}_m$ with $size_N(size_M(m))$ elements.

We claim that Ext has no limiting probability.

First let $n = g(size_M(m)) - 1$ for some m. Then, by construction of g,

$$\mu_n(Ext(\emptyset, \mathbf{H})) \geq 1 - 1/m$$

for all grid extensions $(\emptyset, \mathbf{H})$ with $|\mathbf{H}| \leq size_M(m)$. Therefore in random $\mathbf{G}$ with n elements there is a choice of X to satisfy $\varphi(X)$ of cardinality $|X| = size_M(m)$, with probability $\geq 1 - 1/m$. But in this case there is no choice of Y to satisfy $\gamma(X, Y)$ since then it would be $|Y \setminus X| \geq g(size_M(m)) > n$, which is impossible. Therefore $\mu_n(\mathbf{Ext}) \leq 1/m$.

In this part of the computation we essentially check that the function $n \mapsto \mu_n(Ext(\mathbf{G}_m, \mathbf{H}_m))$ *will have* a minimum with value close to 0 in n.

Secondly, let $n = g(size_N(size_M(m-1)))$ for some m. Then, as $size_M(m) > n$, all X's that may satisfy $\varphi(X)$ are of cardinalities $size_M(0), \ldots, size_M(m-1)$. But $\mu_n(\bigwedge Ext(\mathbf{G}, \mathbf{H})) \geq 1 - 1/m$, where we conjunct over all grid $(\mathbf{G}, \mathbf{H})$ with $|\mathbf{H}| \leq size_N(size_M(m-1))$, hence for every X that satisfies $\varphi(X)$ there is a choice of Y to satisfy $\gamma(X, Y)$, with probability no less than $1 - 1/m$. Therefore $\mu_n(\mathbf{Ext}) \geq 1 - 1/m$.

In this part of the computation we essentially check that $\mu_n(Ext(\mathbf{G}_m, \mathbf{H}_m))$ *will not have* a minimum with value close to 0 before $\mu_{n'}(\bigwedge_{i=0}^{m-1} Ext(\mathbf{G}_i, \mathbf{H}_i))$ becomes close to 1 for some $n' > |\mathbf{H}_{m-1}|$.

We immediately infer that **Ext** has no asymptotic probability. □

3 Sparse random graphs

We next show how to improve our proof of Kaufmann and Shelah Theorem to obtain the following, much more general result:

Theorem 2 *Let $p : \omega \longrightarrow [0,1]$ be a function such that $n^{-\alpha} \leq \beta$ for some $0 < \alpha, \beta < 1$, and all large n. Then there exist monadic second–order sentences without asymptotic probability with respect to random graphs $\mathcal{G}(n,p)$.*

It should be stressed that Spencer and Shelah [10] proved a first–order 0–1 law for random graphs $\mathcal{G}(n, n^{-\alpha})$ with α *irrational.* Therefore our nonconvergence result is almost optimal.

Proof: We will describe step by step the necessary changes of our proof of Kaufmann and Shelah Theorem to make it work for sparse random graphs. First we give informal description of the changes. Precise formulations follow it, and are stated as separate definitions and lemmas.

1. It is not true that $\mu^p(Ext(\mathbf{G},\mathbf{H})) = 1$ holds for arbitrary grid extension $(\mathbf{G},\mathbf{H})$. Therefore we find an improved notion of extensions, called *k–improved grid extensions,* for which the equality holds. It is still possible to encode computations of Turing machines in such extensions.

2. We change the function *size* suitably.

3. We define the function g by equality similar to (1):

$$g(m) = 1 + \begin{pmatrix} \text{least } n > g(m-1) \text{ such that } \mu_n(\bigwedge Ext(\mathbf{G},\mathbf{H})) \geq \\ 1 - 1/m, \text{ where we conjunct over all } k\text{–improved grid} \\ \text{extensions } (\mathbf{G},\mathbf{H}) \text{ with } |\mathbf{H}| \leq m \end{pmatrix}$$

   As g defined above need not be recursive, we show that there exists a recursive function $\widetilde{g}$ such that $\widetilde{g} \geq g$.

Precise formulations of the points above follow:

1. Extension axioms Let for a graph $\mathbf{G}$ the symbol $e(\mathbf{G})$ denote the number of edges in $\mathbf{G}$, and $v(\mathbf{G})$ the number of vertices of $\mathbf{G}$. We say that the pair $(\mathbf{G},\mathbf{H})$ is *safe for exponent* α iff for every S with $\mathbf{G} \subset S \subseteq \mathbf{H}$:

$$(v(S) - v(\mathbf{G})) - \alpha \cdot (e(S) - e(\mathbf{G})) > 0.$$

Now we cite the theorem:

Theorem 3 ((Ruciński and Vince [9])) *If $(\mathbf{G},\mathbf{H})$ is safe for exponent $\alpha < 1$ and $0 < \beta < p(n) \geq n^{-\alpha}$ for some constant β, then $\mu^p(Ext(\mathbf{G},\mathbf{H})) = 1$.* □

According to the above result we construct k–improved grid extensions, safe for arbitrary given exponent $\alpha > 0$. They are obtained from standard grid extensions by adding many new vertices: we add k intermediate vertices on edges between vertices of $X \setminus M$, and $2k$ intermediate vertices on edges connecting elements of M with elements of $X \setminus M$. The added vertices we call *new,* in contrary to the *old* ones. (See figure 4 at the end of the paper.)

The only thing we need is to show that choosing k large enough we can make such extension safe for any given $0 < \alpha < 1$. Of course all necessary relations are still monadic definable.

Lemma 1 *If positive natural k is chosen so that $\alpha < 2k/2k+1$, then the k–improved grid extension is safe for exponent α.*

Proof: Let (M,X) be an improved grid extension, and let $M \subset S \subseteq X$. Let us call a subset H of $S \setminus M$ a *hole* if it induces a cycle in S inside of which there is no other vertex of S ("inside" should be understood to refer to the situation on figures 3 and 4). Let us call a *root* an old vertex in S that is connected by some path made of new vertices in S to some vertex in M. Now let v denote the number of vertices in $S \setminus M$, e the number of edges in S, excluding edges between vertices inside M, h the number of holes in S, and r the number of roots in S.

Then, as it is easy to see,

$$e - v \leq r + h. \tag{2}$$

Observe that each hole in S has at least $4k$ new vertices on its boundaries, and each new vertex in S cannot be counted in this way more than twice. Each root is connected to an element in M by a path of $2k$ new vertices, which do not lie on a boundary of any hole, and, moreover, are different for different roots. Therefore we get

$$v \geq 2kr + 2kh. \tag{3}$$

Taking (2) and (3) together we get

$$\frac{v}{e} = \frac{v}{v + (e - v)} \geq \frac{2k(r+h)}{2k(r+h) + (r+h)} = \frac{2k}{2k+1} > \alpha,$$

so indeed $v - \alpha \cdot e > 0$, as desired. □

2. The function *size***.** Once we have the notion of k–grid extension, the definition of the function *size* is modified in obvious way.

3. The functions g **and** $\tilde{g}$**.** We show that g defined in the proof of theorem 2 is *majorized* by a recursive function. Our estimates generally come from the fact that there exist rational numbers and $0 < \alpha, \beta < 1$ such that $n^{-\alpha} \leq p(n) \leq \beta$ holds for all sufficiently large n.

First of all, we reduce our attention to k–improved (we will omit this word in the sequel) grid extensions, which are safe for exponent α.

Now we modify our random graph structure. We allow three valued $\{0, \frac{1}{2}, 1\}$ logic for edge existence. Now between two given vertices there may exist: non-edge, half-edge and edge. Moreover, we suitably change the probability space. For every pair of vertices from $\{0, \ldots, n-1\}$, independently of other pairs, we toss a three-sided die, which gives outcomes: 0 with probability β, 1 with probability $n^{-\alpha}$ and $\frac{1}{2}$ with probability $1 - \beta - n^{-\alpha}$. After $\binom{n}{2}$ tosses we obtain random three valued graph. This defines a probability space $\langle \boldsymbol{\mathcal{G}}^3, \boldsymbol{\mu}^{\alpha,\beta} \rangle$, where $\boldsymbol{\mathcal{G}}^3$ stands for the class of finite three valued graphs.

The informal idea is as follows: the half-edge should be understood as "not yet decided: there is an edge or not". Usually a $\boldsymbol{\mu}^{\alpha,\beta}$*–random graph in* $\boldsymbol{\mathcal{G}}^3$ *contains many such undecided places. But it is possible that the places that are already decided allow one to verify that whatever will be decided about half-edges later, extension axioms are satisfied. We want to show that this possibility holds with large probability.*

The class $\boldsymbol{\mathcal{G}}$ of standard, two valued graphs we denote $\boldsymbol{\mathcal{G}}^2$, to avoid confusion. It is natural that one may assume $\boldsymbol{\mathcal{G}}^2 \subseteq \boldsymbol{\mathcal{G}}^3$, and therefore it is clear what it means that two graphs $\mathbf{G} \in \boldsymbol{\mathcal{G}}^3$ and $\mathbf{H} \in \boldsymbol{\mathcal{G}}^2$ are isomorphic, etc. In particular, μ^p is also a probability distribution in $\boldsymbol{\mathcal{G}}^3$.

Now it is routine, following the proof of theorem 3 presented in [9], to prove also the following:

Lemma 2 *Let pair* $(\mathbf{G}, \mathbf{H})$ *with* $\mathbf{G}, \mathbf{H} \in \boldsymbol{\mathcal{G}}^3$*, but without half-edges in* $\mathbf{H}$*, except between vertices of* $\mathbf{G}$*, be safe for exponent* α*. Definition of safety ignores edges inside* $\mathbf{G}$*, so the above makes sense. Then with asymptotic probability one every subgraph of a random three-valued graph* $\mathbf{K}$ *that is isomorphic to* $\mathbf{G}$ *can be extended to a subgraph isomorphic to* $\mathbf{H}$*.* □

Let $(\mathbf{G}, \mathbf{H})$ be as in the above lemma. The property that every isomorphic copy of $\mathbf{G}$ extends to a copy of $\mathbf{H}$ we denote by $Ext(\mathbf{G}, \mathbf{H})$, and call it also an extension axiom. Note that formally $Ext(\mathbf{G}, \mathbf{H})$ is not a first–order sentence, unless $\mathbf{G}, \mathbf{H} \in \boldsymbol{\mathcal{G}}^2$.

Now we define the function $\tilde{g}$ by

$$\tilde{g}(m) = 1 + \begin{pmatrix} \text{least } n > \tilde{g}(m-1) \text{ such that } \mu_n^{\alpha,\beta}(\bigwedge Ext(\mathbf{G}, \mathbf{H})) \geq \\ 1 - 1/m, \text{ where we conjunct over all } k\text{–improved grid} \\ \text{extensions } (\mathbf{G}, \mathbf{H}) \text{ with } |\mathbf{H}| \leq m, \text{ with } \mathbf{G}, \mathbf{H} \in \boldsymbol{\mathcal{G}}^3 \text{ like} \\ \text{in the last lemma} \end{pmatrix}$$

It can be easily observed that $\tilde{g}$ is recursive.

Lemma 3 *For every natural number* n

$$\tilde{g}(n) \geq g(n).$$

Proof: It suffices to prove the following:

For all n such that $n^{-\alpha} \leq p(n) \leq \beta$ the inequality

$$\mu_n^{\alpha,\beta}(\bigwedge Ext(\mathbf{G},\mathbf{H})) \leq \mu_n^p(\bigwedge Ext(\mathbf{G},\mathbf{H}))$$

holds, where in both sides we conjunct over all k–improved grid extensions $(\mathbf{G},\mathbf{H})$ with $|\mathbf{H}| \leq m$, possibly with $\mathbf{G} \in \mathcal{G}^3$.

We fix n satisfying $n^{-\alpha} \leq p(n) \leq \beta$.

For $\mathbf{K} \in \mathcal{G}^3(n)$ and $i = 2,3$ let $Cl^i(\mathbf{K})$ denote the set of all graphs $\mathbf{G} \in \mathcal{G}^i(n)$ such that if there is edge (non-edge, resp.) between u and v in $\mathbf{K}$, then there is edge (non-edge, resp.) between u and v in $\mathbf{G}$. In particular, always $Cl^2(\mathbf{K}) = Cl^3(\mathbf{K}) \cap \mathcal{G}^2(n) \subseteq Cl^3(\mathbf{K})$.

Let $\mathbf{K}' \in \mathcal{G}^3$ be any graph obtained from $\mathbf{K} \in \mathcal{G}^3$ by replacing some of its half-edges by non-edges or edges, in arbitrary way. Then it is not difficult to observe that if $\bigwedge Ext(\mathbf{G},\mathbf{H})$ is true in $\mathbf{K}$, then so is in $\mathbf{K}'$.

By the above observation it becomes clear that if $\mathbf{K} \models \bigwedge Ext(\mathbf{G},\mathbf{H})$, then $\mathbf{K}' \models \bigwedge Ext(\mathbf{G},\mathbf{H})$ for every $\mathbf{K}' \in Cl^i(\mathbf{K})$.

Moreover, it is easy to observe that for two graphs $\mathbf{K},\mathbf{K}' \in \mathcal{G}^i$, the sets $Cl^i(\mathbf{K})$ and $Cl^i(\mathbf{K}')$ are either disjoint, or one is included in the other.

Let $\mathcal{K} = \{\mathbf{K}_0,\ldots,\mathbf{K}_m\}$ be the set of all graphs in $\mathcal{G}^3(n)$ in which the sentence $\bigwedge Ext(\mathbf{G},\mathbf{H})$ is true. Then maximal sets among $Cl^3(\mathbf{K}_j)$, $j = 0,\ldots,m$ partition $\mathcal{K}$ into disjoint subsets.

Now, in order to finish the proof, it suffices to check that always

$$\mu_n^p(Cl^2(\mathbf{K}_j)) \geq \mu_n^{\alpha,\beta}(Cl^3(\mathbf{K}_j)).$$

Indeed, if we look at a graph as a sequence of outcomes in $\binom{n}{2}$ die tosses, $Cl^i(\mathbf{K}_j)$ is the set of such sequences that have 0's and 1's in some specified places, and "anything" in other places. Now it is easier to get 1 in a specified place according to μ_n^p than according to $\mu_n^{\alpha,\beta}$, and similarly for 0. Obtaining "anything" is equally easy.

Now the claim easily follows. □

It is left for the reader to verify that the changes described above guarantee that our proof of Kaufmann and Shelah Theorem still works for $\mathcal{G}(n,p)$, and therefore that the proof of theorem 2 is finished. □

4 Partial orders and K_{m+1}–free graphs

First we improve our proof of Kaufmann and Shelah Theorem to make it work for random partial orders. The theorem we obtain is not new – it has been proved by Compton, as reported in [2].

Theorem 4 ((Compton [2])) *There are monadic second–order sentences without asymptotic probability with respect to uniform labelled probability distribution on the class of all finite partial orders.*

Proof: The only problem in our case is to find another representation of computations of Turing machines, as the one found for graphs cannot be embedded in random partial orders. To do so we present now part of the first–order description of a random partial order (i.e., a part of the complete axiomatization of the almost sure theory), after cf. [2]. We assume $\leq$ to be the symbol of the ordering relation, subject to random choice.

With labelled asymptotic probability 1 a partial order will have no chains of length greater than 3. Thus, almost every partial order can be partitioned into 3 levels: L_0, the set of minimal elements, L_1, the set of elements immediately succeeding elements in L_0, and L_2, the set of elements immediately succeeding elements in L_1.

First we add three unary relations to the signature for the levels L_0, L_1, and L_2. Now we formulate extension axioms for random partial orders.

Let $x_1, \ldots, x_m,\ y_1, \ldots, z_k$ and $y_1, \ldots, y_\ell$ be variables, all of them different. Let $S \subseteq \{1, \ldots, m\} \times \{1, \ldots, \ell\}$ be arbitrary. The following formula is an extension axiom, and holds a.s. in a random partial order:

$$(\forall x_1, \ldots, x_m \in L_1)\ (\forall z_1, \ldots, z_k \in L_0)$$
$$\left[(\bigwedge_{1 \le i < j \le m} x_i \neq x_j \wedge \bigwedge_{1 \le i < j \le k} z_i \neq z_j) \rightarrow\right.$$
$$((\exists y_1, \ldots, y_\ell \in L_0)\ \bigwedge_{1 \le i < j \le \ell} y_i \neq y_j \wedge \bigwedge_{1 \le i \le k,\ 1 \le j \le \ell} z_i \neq y_j \wedge$$
$$\left.\bigwedge_{(i,j) \in S} x_i \ge y_j \wedge \bigwedge_{(i,j) \notin S} x_i \not\ge y_j)\right].$$

A formula resulting from the above by interchanging places of L_0 and L_1, and changing $\geq$ into $\leq$, is also an axiom.

Now we would like to represent, in a way definable in first–order logic, graphs in random partial orders so that all extension axioms for graphs are a.s. true about these representations.

Namely, a graph $\mathbf{G}$ on vertex set $U_0 \subseteq L_0$ is represented by a 4–tuple of subsets of a random partial order $\langle U_0, U_1, U_2, U_3 \rangle$, such that:

1. $U_0, U_2 \subseteq L_0$; $U_1, U_3 \subseteq L_1$;
2. $\leq$ is a bijection from U_0 into U_1 (denoted φ_0);
3. $\geq$ is a bijection from U_1 into U_2 (denoted φ_1);
4. $\leq$ is a bijection from U_2 into U_3 (denoted φ_2);
5. for $u, v \in U_0$ let $\varphi_2(\varphi_1(\varphi_0(u))) \geq v$ iff there is an edge from u to v in $\mathbf{G}$.

We leave for the reader easy verification that the graph extension axioms of the form $Ext(\mathbf{G}, \mathbf{H})$ hold a.s., when we look at their representations in random partial orders. Now our proof of Kaufmann and Shelah result applies here, and the thesis follows immediately. □

In contrast to the previous result, the following is, to the best of author's knowledge, a new one. A $K_{m+1}-$ free graph is a one having no subgraph isomorphic to K_{m+1}, the complete graph on $m+1$ vertices.

Theorem 5 *There are monadic second–order sentences without asymptotic probability with respect to uniform labelled probability distribution on the class of all finite $K_{m+1}-$ free graphs, for $m \geq 2$.*

Proof: Once more we need a part of the first–order description of a random $K_{m+1}-$ free graph, this time provided by Kolaitis, Prömel, and Rothschild [7, 8].

A *spindle* connecting two vertices x and y in a $K_{m+1}-$ free graph is a subgraph isomorphic to K_{m-1} such that all its vertices are adjacent to both x and y. It appears that with labelled asymptotic probability 1 the relation of being connected by a spindle is an equivalence relation of index m, and no two vertices in an equivalence class of this relation are adjacent.

Suppose that the edge relation is denoted by E and that $L_0, \ldots, L_{m-1}$ are the equivalence classes for the spindle connection relation in a random $K_{m+1}-$ free graph.

Let $x_1, \ldots, x_m,\ y_1, \ldots, z_k$ and $y_1, \ldots, y_\ell$ be variables, all of them different. Let $S \subseteq \{1, \ldots, m\} \times \{1, \ldots, \ell\}$ be arbitrary. The following formula is an extension axiom, and holds a.s. in a random $K_{m+1}-$ free graph:

$$(\forall x_1, \ldots, x_m \in L_1)(\forall z_1, \ldots, z_k \in L_0)$$
$$\left[(\bigwedge_{1 \le i < j \le m} x_i \neq x_j \wedge \bigwedge_{1 \le i < j \le k} z_i \neq z_j) \rightarrow\right.$$
$$((\exists y_1, \ldots, y_\ell \in L_0) \bigwedge_{1 \le i < j \le \ell} y_i \neq y_j \wedge \bigwedge_{1 \le i \le k,\ 1 \le j \le \ell} z_i \neq y_j$$
$$\left.\bigwedge_{(i,j) \in S} E(x_i, y_j) \wedge \bigwedge_{(i,j) \notin S} \neg E(x_i, y_j))\right].$$

A formula resulting from the above by interchanging places of L_0 and L_1 is also an axiom.

Now we can easily observe, that almost identical representation as in random partial orders is appropriate here. The equivalence classes L_2 to L_{m-1} are ignored in our construction. □

5 Monadic second–order logic vs. fixpoint logic in the theory of asymptotic probabilities

In this section we shall investigate connections between behaviours of asymptotic probabilities in fixpoint logic and in monadic logic. It was empirically observed that first–order 0–1 laws sometimes extend to fixpoint 0–1 laws, sometimes to monadic second–order 0–1 laws, but almost never to both of these logics at the same time.

First of all, in order to support the above claim, we recall some results concerning behaviour of asymptotic probabilities in fixpoint and monadic second–order logics. Results for monadic second–order logic can be found, together with references, in [1, 2], while their fixpoint counterparts are to be found in [11, 12]. In all the cases below the first–order 0–1 law holds.

1. Graphs and uniform, labelled or unlabelled probabilities: the fixpoint 0–1 law holds, nonconvergence in monadic logic;
2. Partial orders and uniform, labelled probabilities: the fixpoint 0–1 law holds, nonconvergence in monadic logic;
3. Graphs that are unions of cycles and uniform, unlabelled probabilities: the monadic 0–1 law holds, nonconvergence in fixpoint logic;
4. Random graphs $\mathcal{G}(n,p)$ with recursive $p = p(n)$ such that either for all $\varepsilon > 0$ $n^{-1-\varepsilon} \ll p(n) \ll n^{-1}$ or $n^{-1} \ll p(n) \ll n^{-1}\log n$: the monadic 0–1 law holds, nonconvergence in fixpoint logic;
5. Equivalence relations and uniform, labelled probabilities: both monadic and fixpoint 0–1 laws hold.

The last entry above is essentially degenerated: it is known that both fixpoint and monadic logics collapse to first–order logic on equivalence relations. We will return to this observation later.

As we saw in previous sections, the fact that appropriate extension axioms hold with asymptotic probability one is a strong premise against the convergence law for monadic logic. At the same time, almost all known proofs of 0–1 laws for fixpoint logic are based on extension axioms. Therefore it seems that the 0–1 law for fixpoint logic is a strong premise against the convergence law for monadic logic. As far as we know, this is the first step towards explanation of the observation we mentioned at the beginning of this section.

Of course it would be nice to have some theorem of the form:

"If both fixpoint and monadic 0–1 laws hold for $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$*, then ...".*

At first glance it seems promising that *"both fixpoint and monadic logics almost surely collapse to first–order logic in* $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$*"* could be placed as a thesis in the above. Unfortunately, this is not true. Appropriate counterexample is constructed in [12], and cited also in [13]. Only the fixpoint logic is mentioned there, but the 0–1 law for monadic logic can be proved in a way similar to that for the fixpoint logic. So the hypothetic theorem–explanation has to be modified. We suggest to add restriction to *recursive distributions* only, the latter being defined in [13]. The above mentioned counterexample is nonrecursive. To the best of author's knowledge, no known recursive distribution violates the suggested formulation, which follows:

Conjecture 1 *Let* $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$ *be such that the relation* $I_{\boldsymbol{\mu}} = \{\langle \mathbf{A}, q\rangle \mid \mu_{|\mathbf{A}|}(\{\mathbf{A}\} \geq q\} \subseteq \boldsymbol{\mathcal{A}} \times \mathbf{Q}$ *is recursive.*

If both fixpoint and monadic 0–1 laws hold for $\langle \boldsymbol{\mathcal{A}}, \boldsymbol{\mu} \rangle$*, then both fixpoint logic and monadic second–order logic a.s. collapse to first–order logic.*

For recursive distributions we have a tool which may provide a handle for attacking the conjecture. Namely, it is proved in [13], that if fixpoint 0–1 law holds for a recursive distribution, then fixpoint logic is a.s. bounded, and hence a.s. collapses to first–order logic. Similar property, even for recursive distributions, is not true for monadic logic.

6 Final remarks

6.1 Further results

It can be shown that all our results for labelled distributions are true for unlabelled ones, as well. In fact, in all cases we considered, except the one of sparse random graphs, the extension axioms that hold a.s. for labelled distribution, are also known to hold a.s. for the unlabelled one (see [1]).

Our proof method also applies to: arbitrary classes of relational structures over arbitrary similarity type with at least one at least binary relation symbol (but without constants or functions) with either uniform labelled or uniform unlabelled probabilities, uniform labelled or unlabelled $d-$ complexes for $d > 0$, and most of classes given by parametric conditions with uniform labelled or unlabelled probabilities. Descriptions of these classes, together with references, are to be found in [1].

6.2 Existential monadic logic

Kaufmann in [5] showed that 0–1 law for uniform labelled random graphs fails even for existential monadic logic. Our proof method can also be applied in this case. However, the suitable changes are greater than in previously presented cases.

Their informal description is as follows: replace the subformula $\exists Y\ \gamma(X,Y)$ in the sentence **Ext** by

$$\exists y\ \forall x\ (x \in X \rightarrow E(x,y)).$$

The modified **Ext** is then universal monadic second–order sentence. The existential sentence we then obtain negating the modified **Ext**.

To apply our proof method we have to show the following:

> for every pair $(\mathbf{G},\mathbf{H})$ such that $|\mathbf{H}| - |\mathbf{G}| = 1$ and the only vertex of $|\mathbf{H}| \setminus |\mathbf{G}|$ is incident to all vertices of $\mathbf{G}$, the graph of the function $n \mapsto \mu_n(Ext(\mathbf{G},\mathbf{H}))$ has the shape presented on figure 1 at the end of the paper.

The proof can be based on well known combinatorial estimates for random graphs. Then our proof method can be used.

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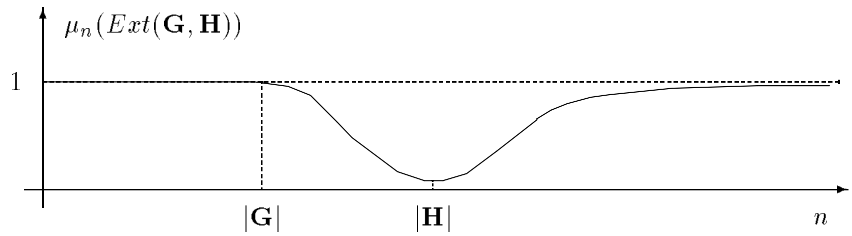

Figure 1: Graph of the function $n \mapsto \mu_n(Ext(\mathbf{G},\mathbf{H}))$.

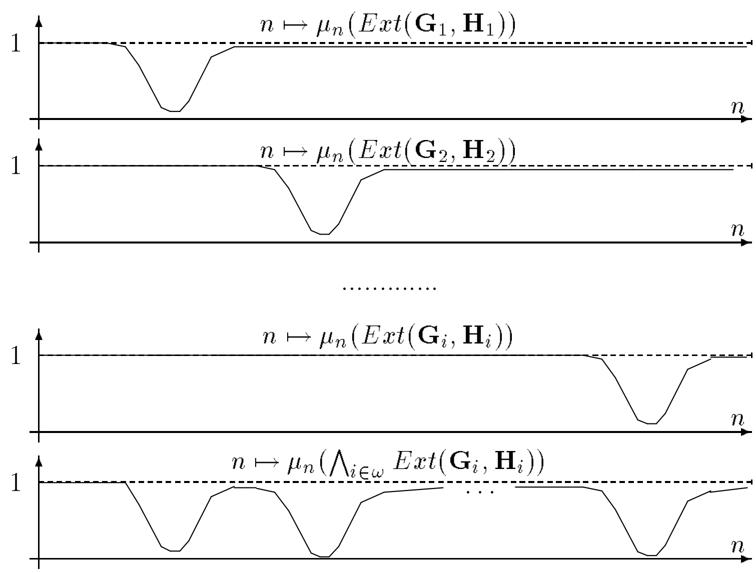

Figure 2: Infinite conjunction of extension axioms without asymptotic probability

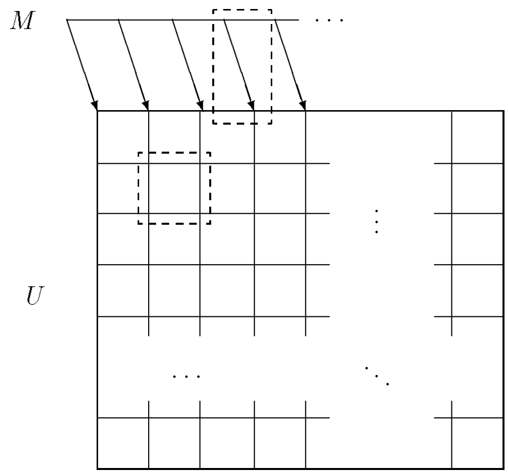

Figure 3: A grid extension.

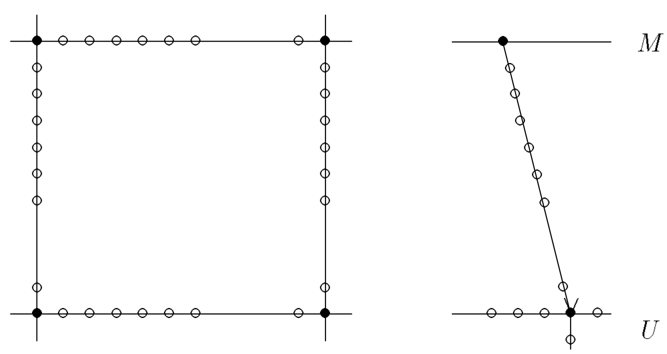

Figure 4: Details of Fig. 3 *after* improvement. New vertices are denoted by circles, and old ones by filled circles. There are k new vertices on each side of the square. There are $2k$ new vertices on each path from a vertex in M to a root.