LECTURE 4

Lecture 11

In Lecture 10 we introduced the notion of a graph polynomial.

- The chromatic polynomial was introduced and many facts about it were presented.
- We proved that there are many, MANY, graph polynomials.
- We have listed many explicit examples:
  Variations on colorings and others.

Homework: Reread the slides of Lecture 10!

Lecture 11
Comparing graph polynomials

- Distinctive power of graph polynomials
- P-equivalence and complete graph polynomials
- Reducibility via coefficients

Comparing graph parameters and graph polynomials

Jointly prepared with E.V. Ravve

Graph parameters and graph polynomials

Let $\mathcal{R}$ be a (possibly ordered) ring or a field.

For a set of indeterminates $\bar{X}$ we denote by $\mathcal{R}[\bar{X}]$ the polynomial ring over $\mathcal{R}$.

A **graph parameter** p is a function from the class of all finite graphs $Graphs$ into $\mathcal{R}$ which is invariant under graph isomorphism.

A **graph polynomial** p is a function from the class of all finite graphs $Graphs$ into $\mathcal{R}[\bar{X}]$ which is invariant under graph isomorphism.

Remark: In most situations in the literature $\mathcal{R}$ is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. The choice of the underlying ring or field may depend on the way we want to represent the graph parameter or graph polynomial.

For the graph parameter $d_{max}(G)$, the maximal degree of its vertices,
$\mathbb{Z}$ suffices, but for $d_{average}(G)$, the average degree of its vertices, $\mathbb{Q}$ is needed.

Equivalence of graph polynomials, I

Let $\mathcal{C}$ be a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 7
We say that Q determines P over $\mathcal{C}$, or
Q is at least as distinctive than P over $\mathcal{C}$, and write $P \preceq^{\mathcal{C}}_{d.p.} Q$
if for all graphs G_1 and G_2 in $\mathcal{C}$,

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.

P and Q are d.p.-equivalent over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{d.p.} Q$,
iff $P \preceq^{\mathcal{C}}_{d.p.} Q$ and $Q \preceq^{\mathcal{C}}_{d.p.} P$

Examples of $P \preceq^{\mathcal{C}}_{d.p.} Q$

(i) (DKT, 3.2.1) The chromatic polynomial $\chi(G,X)$ determines the graph parameters $\mid V(G) \mid$, $\mid E(G) \mid$, $\chi(G)$, $k(G)$, $b(G)$, $g(G)$, etc.

(ii) d_{max} and $d_{average}$ are d.p-incomparable.

(iii) The Tutte polynomial $T(G,X,Y)$ determines $\chi(G,X)$ on connected graphs, but not on all graphs.

(iv) Assume $P(G;X), Q(G;X), U(G,X)$ are three polynomials and $P(G,X) = U(G,X) \cdot Q(G,X)$.
Let $\mathcal{C}_U$ be a class of graphs such that for all $G_1, G_2 \mathcal{C}_U$ we have $U(G_1,X) = U(G_2 : X)$. Then $P \preceq^{\mathcal{C}_U}_{d.p.} Q$.

(v) Let $\mathcal{F}$ be the class of forests. For the characteristic polynomial $char(G,\lambda)$ and the matching polynomial $dm(G,\lambda)$ and we have

$$char \sim^{\mathcal{F}}_{d.p.} dm.$$

Adjoint polynomials

Let $P(G, \lambda)$ be a graph polynomial.
We denote by $\bar{G}$ the complement graph of G.

The adjoint polynomial $\bar{P}(G, \lambda)$ is the polynomial defined by

$$\bar{P}(G, \lambda) =_{def} P(\bar{G}, \lambda)$$

- **Exercise:** $P \preceq^{\mathcal{C}}_{d.p.} \bar{P}$ iff $\bar{P} \preceq^{\mathcal{C}}_{d.p.} P$
- For the Tutte polynomial $T(G, X, Y)$ and $\bar{E}_n = K_n$ we have
  (i) $T(E_m) = T(E_n) = 1$ for all $n \in \mathbb{N}$.
  (ii) $T(K_m) \neq T(K_n)$ for $m \neq n$.
  (iii) Hence the Tutte polynomial and its adjoint are not d.p.-comparable.

P-unique and P-equivalent graphs

Definition 8 *Let $P = P(G; \bar{X})$ a graph polynomial and $\mathcal{C}$ a class of graphs.*

(i) Two graphs G_1 and G_2 are P-equivalent for $\mathcal{C}$ if $P(G; \bar{X}) = P(G_1; \bar{X})$.

(ii) A graph $G \in \mathcal{C}$ is P-unique for $\mathcal{C}$ if for any other graph $G_1 \in \mathcal{C}$ with $P(G; \bar{X}) = P(G_1; \bar{X})$ the graph G_1 is isomorphic to G.

(iii) P is complete for $\mathcal{C}$ if every graph $G \in \mathcal{C}$ is $P(G; \bar{X})$-unique for $\mathcal{C}$.

If $\mathcal{C}$ consists of all graphs we omit $\mathcal{C}$.

Proposition 9 *Let P and Q be graph polynomials such that $P \preceq^{\mathcal{C}}_{d.p.} Q$.*

(i) If G_1 and G_2 are Q-equivalent for $\mathcal{C}$ then they are also P-equivalent for $\mathcal{C}$.

(ii) If G is P-unique for $\mathcal{C}$ then G is Q-unique for $\mathcal{C}$.

(iii) If P is complete for $\mathcal{C}$ then Q is complete for $\mathcal{C}$.

Complete graph polynomials

Are there complete graph polynomials?

The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and Y be indeterminates.
  For a graph $\langle V, E \rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left(\sum_{\sigma \in \mathfrak{S}_n} \prod_{(i,j) \in E} X_{\sigma(i),\sigma(j)} \right)$$

  Here $\mathfrak{S}_n$ is the permutation group of $[n]$.

Challenge: Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.

An “unnatural” graph-complete invariant

Let $g : \mathcal{G} \rightarrow \mathbb{N}$ be a Gödel numbering for labeled graphs of the form $G = \langle [n], E, <_{nat} \rangle$.

We define a graph polynomial using g:

$$\Gamma(G, X) = \sum_{H \simeq G} X^{g(H)}$$

Clearly this is a graph invariant.

But it is **“obviously unnatural”** !

Can we make precise

what a **natural** graph polynomial should be?

χ-equivalent graphs (from [DKT, chapter 5])

(i) The graphs E_n, K_n and $K_{n,n}$ are χ-unique for $n \geq 1$.

(ii) The graphs C_n are χ-unique for $n \geq 3$, $C_i = K_i$ for $i \leq 2$.

(iii) Any two trees on n vertices are χ-equivalent.

In [DKT, chapter 5] many pairs of χ-equivalent graphs are constructed using a method due to R.C. Read (1987) and G.L. Chia (1988).

Research project:

Study P-equivalence for the various generalized colorings of Lecture 10.

$char$-equivalent graphs

From M. Noy, Graphs determined by polynomial invariants (2003)

Let $char(G, x) = \det(x \cdot \mathbf{1} - A_G)$ be the characteristic polynomial of G with adjacency matrix A_G.

(i) The graphs $K_{n,n}$ are $char$-unique.

(ii) The line graphs $L(K_n)$ are $char$-unique for $n \neq 8$.
For $n = 8$ there are three exceptions.

(iii) The line graphs $L(K_{n,n})$ are $char$-unique for $n \neq 4$.
For $n = 4$ there is one exception.

The two matching polynomials

Recall, for $G = (V, E)$ with $\mid V \mid = n$,

$$dm(G, x) = \sum_r (-1)^r m_r(G) x^{n-2r}$$

be the (defect) matching polynomial and

$$gm(G, x) = \sum_r m_r(G) x^r$$

the (generating) matching polynomial.

We have

$$dm(G; x) = x^n gm(G; (-x)^{-2})$$

Graphs equivalent for matching polynomials.

From M. Noy, Graphs determined by polynomial invariants (2003)

- For every graph G we have $gm(G, x) = gm(G \sqcup E_n, x)$ but $dm(G, x) \neq dm(G \sqcup E_n, x)$.

  $dm(P_2, x) = x^2 - 1$ and $dm(P_2 \sqcup E_k, x) = x^3 - x$,
  but $gm(P_2, x) = x^2 - 1$ and $gm(P_2 \sqcup E_k, x) = x^2 - 1$

- $| V(G) | \preceq_{d.p.} dm$, and therefore $gm \preceq_{d.p.} dm$.
  In other words gm is strictly less expressive than dm.

- $gm \sim_{d.p.} dm$ on graphs of a fixed number of vertices.

- The graphs $K_{n,n}$ are dm-unique.

  Are they also gm-unique?

Research project:

Study dm-equivalence and gm-equivalence of graphs further.

T-unique graphs

From A. de Mier and M. Noy, On Graphs determined by the Tutte polynomial (2004)

For a graph $G = (V, E)$ and $A \subseteq E$ we denote by $G[A] = (V, A)$ the spanning subgraph generated by A. We set $r(A) = \mid V \mid - k(G[A])$ and $n(A) = \mid A \mid - r(A)$.

The Tutte polynomial is defined by

$$T(G; X, Y) = \sum_{A \subseteq E} (X-1)^{r(E)-r(A)} (Y-1)^{n(A)}$$

(i) Recall that $\chi \preceq_{d.p.} T$ on **connected** graphs.
Hence the graphs $K_{n,n}$ are T-unique.

(ii) The wheels W_n are T-unique for all $n \in \mathbb{N}$.

Wheels are χ-unique for W_{2n}, W_5 and W_7 are not. In general it is not known (?) whether W_{2n+1} is χ-unique.

(iii) The ladders L_n are T-unique for all $n \geq 3$.

They are only known to be χ-unique for small values of n.

Bollobas-Pebody-Riordan Conjecture:

Almost all graphs are T-unique and even χ-unique

Let us make it more precise:

Let TU (χU) be the graph property:
$G \in TU$ $(G \in \chi U)$ iff G is T-unique (χ-unique),
and $TU(n)$ $(\chi U(n))$ be the density function of TU (χU).

The conjecture for the Tutte polynomial now is

$$\lim_{n \to \infty} \frac{TU(n)}{2^{\binom{n}{2}}} = 1$$

Similar for $\chi(G, \lambda)$.

Is TU (χU) definable in some logic with a $0-1$-law?

B. Bollobás, L. Pebody and O. Riordan, Contraction-Deletion Invariants for Graphs,
Journal of Combinatorial Theory, Serie B, vol. 80 (2000) pp. 320-345.

Almost complete graph invariants

A graph polynomial P is almost complete, if almost all graphs are P-unique.

Research problems:

- Study the definability of the graph property G is P unique for various graph polynomials P.
- Find natural graph polynomials which are almost complete.
- In particular, is the signed Tutte polynomial T_{signed} almost complete for signed graphs.

  A positive answer would be interesting for knot theorists: T_{signed} is intimately related to the Jones polynomial of knot theory.

Comparison of graph polynomials by coefficients

Coefficients of graph polynomials, I: The univariate case

We denote by $\mathbb{Z}^{<\omega}$ the finite sequences of elements of $\mathbb{Z}$.

Let $P(G, X) \in \mathbb{Z}[X]$ and $P(G, X) = \sum_{i=0}^{d(G)} a_i(G) \cdot X^i$ with $a(G)_{d(G)} \neq 0$.

We denote by $cP(G, X)$ the finite sequence $(a_i(G))_{i \leq d(G)} \in \mathbb{Z}^{<\omega}$.

$cP(G, X)$ are the (standard) coefficients of $P(G, X)$, and $d(G)$ is its degree.

c is a one-one and onto function $c : \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$.

Instead of looking at graph polynomials $P : Graphs \xrightarrow{P} \mathbb{Z}[X]$, we can look at the function $cP : Graphs \longrightarrow \mathbb{Z}^{<\omega}$ defined by

$$cP : Graphs \xrightarrow{P} \mathbb{Z}[X] \xrightarrow{c} \mathbb{Z}^{<\omega}$$

Lemma 10
For all graphs G_1, G_2, we have that $P(G_1) = P(G_2)$ iff $cP(G_1) = cP(G_2)$.

Other representations of graph polynomials

Our definition of cP uses the **power form of P**.

We could have used also factorial form or binomial form of P.

- cP denotes the coefficients of P in power form.
- c_1P denotes the coefficients of P in factorial form.
- c_2P denotes the coefficients of P in binomial form.

We note that there are simple algorithms to pass from one representation to another.

Equivalence of graph polynomials, II

Let $\mathcal{C}$ be a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 11
We say that Q determines coefficient-wise P over $\mathcal{C}$ and write $P \preceq^{\mathcal{C}}_{coeff} Q$ if there is a function $F : \mathbb{Z}^{<\omega} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$

$$F(cQ(G)) = cP(G)$$

P and Q are coefficient-equivalent over $\mathcal{C}$, and write $P \sim^{\mathcal{C}}_{coeff} Q$, iff $P \preceq^{\mathcal{C}}_{coeff} Q$ and $Q \preceq^{\mathcal{C}}_{coeff} P$

- If $\mathcal{C}$ consists of all graphs, we omit $\mathcal{C}$.
- The definition also applies to graph parameters $P(G), Q(G) \in \mathbb{Z}$.
- Our definition is invariant under the choice of representations cP, c_1P or c_2P.

An example: F can be arbitrarily complex

Let $P(G,\lambda) = \sum_i a_i(G)\lambda^i$.

Let $P_{exp}(G,\lambda) = \sum_i 2^{a_i(G)}\lambda^i$,

and for $g : \mathbb{N} \to \mathbb{N}$ one-one and onto let $P_g(G,\lambda) = \sum_i a_i(G)\lambda^{g(i)}$.

Clearly,

$$P \sim_{coeff} P_g \sim_{coeff} P_{exp}$$

- If g is not computable, then F showing that $P \sim_{coeff} P_g$ cannot be computable in the **Turing model** of computation.

- Furthermore, F showing that $P \sim_{coeff} P_{exp}$ cannot be computable in the **Blum-Shub-Smale model** of computation.

Theorem 12 $P \preceq^{\mathcal{C}}_{coeff} Q$ *iff* $P \preceq^{\mathcal{C}}_{d.p.} Q$

Proof: $P \preceq^{\mathcal{C}}_{coeff} Q$ implies $P \preceq^{\mathcal{C}}_{d.p.} Q$.

Assume there is a function $F : \mathbb{Z}^{<\omega} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$.

Now let $G_1, G_2 \in \mathcal{C}$ such that $Q(G_1) = Q(G_2)$.

By Lemma 10 we have $cQ(G_1) = cQ(G_2)$.

Hence $F(cQ(G_1)) = F(cQ(G_2))$.

Since for all $G \in \mathcal{C}$ we have $F(cQ(G)) = cP(G)$, we get $cP(G_1) = cP(G_2)$ and, using Lemma 10 again, we have $P(G_1) = P(G_2)$.

Proof: $P \preceq^{\mathcal{C}}_{d.p} Q$ implies $P \preceq^{\mathcal{C}}_{coeff} Q$.

We use the well-ordering principle which equivalent to axiom of choice.

Let $\{F_\alpha : \alpha < \beta\}$ be a well-ordering of all the functions $F : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$.

For $G \in \mathcal{C}$, let $\gamma(G) < \beta$ be the smallest ordinal such that $F_{\gamma(G)}(cQ(G)) = cP(G)$.

Now given $P(G, X) \preceq_{d.p.} Q(G, X)$, we define a function $F_{P,Q} : \mathbb{Z}^{<\omega} \to \mathbb{Z}^{<\omega}$ as follows:

$$F_{P,Q}(cQ(G)) = \begin{cases} F_{\gamma(G)}(cQ(G)) & \text{if } G \in \mathcal{C} \\ 0 & \text{else} \end{cases}$$

Using Lemma 10 and $P(G, X) \preceq_{d.p.} Q(G, X)$, this indeed defines a function.

Finally, as $F_{\gamma(G)}(cQ(G)) = F_{\gamma(G)}(cP(G))$, we get

$$F_{P,Q}(cQ(G)) = cP(G)$$

Q.E.D.

A proof without well-ordering (suggested by Ofer David)

Let S be a set of finite graphs and $s \in \mathbb{Z}^{<\omega}$.
For a graph polynomial P we define:

$P[S] = \{s \in \mathbb{Z}^{<\omega} : cP(G) = s \text{ for some } G \in S\}$ and $P^{-1}(s) = \{G : cP(G) = s\}$.

Now assume $P(G, X) \preceq_{d.p.} Q(G, X)$.

If $Q^{-1}(s) \neq \emptyset$, then for every $G_1, G_2 \in Q^{-1}(s)$ we have $cQ(G_1) = cQ(G_2)$, and therefore $cP(G_1) = cP(G_2)$.

Hence $P[Q^{-1}(s)] = \{t_s\}$ for some $t_s \in \mathbb{Z}^{<\omega}$.

Now we define

$$F_{P,Q}(s) = \begin{cases} t_s & Q^{-1}(s) \neq \emptyset \\ s & \text{else} \end{cases}$$

Q.E.D.

Example, I: The two matching polynomials

$dm(G,x) = \sum_r (-1)^r m_r(G) x^{n-2r}$

$gm(G,x) = \sum_r m_r(G) x^r$

We have $dm(G;x) = x^n gm(G;(-x)^{-2})$ where $n =\mid V \mid$.

- The degree of dm is n
- If $m_r(G) \neq 0$ the $n - 2r > 0$.
- Hence
$$\frac{dm(G;x)}{X^n}$$
is a polynomial, and we can compute the coefficients of gm from the coefficents of dm.
- We cannot compute the coefficients of dm from gm without knowing the value of $\mid V \mid= n$.

Example II: The Tutte polynomial and the chromatic polynomial

The Tutte polynomial and the chromatic polynomial are related by the formula

$$\chi(G,X) = (-1)^{r(G)} \cdot X^{k(G)} \cdot T(G;1-X,0)$$

- To compute the coefficients of $\chi(G;X)$ from $T(G;X,Y)$ we have to know the parity of $r(G)$ and the number of connected components of G.
- For connected graphs $k(G) = 1$ and $r(G) = \mid V \mid -1$.

Introducing auxiliary parameters $\mathcal{S}$

Let $\mathcal{S} = \{S_1(G), \ldots, S_t(G)\}$ be graph parameters (polynomials), and $\mathcal{C}$ a graph property.
Let $P(G, \bar{X})$ and Let $Q(G, \bar{Y})$ be two graph polynomials.

Definition 13
We say that Q determines P relative to $\mathcal{S}$ over $\mathcal{C}$, or
Q is at least as distinctive than P relative to $\mathcal{S}$ over $\mathcal{C}$, and write $P \preceq^{\mathcal{S},\mathcal{C}}_{r.d.p.} Q$
if for all graphs $G_1, G_2 \in \mathcal{C}$ with $S_i(G_1) = S_i(G_2) : i \leq t$ we have

$$Q(G_1) = Q(G_2) \text{ implies that } P(G_1) = Q(P_2).$$

Definition 14
We say that Q determines coefficient-wise P relative to $\mathcal{S}$ over (C)
and write $P \preceq^{\mathcal{S},(C)}_{relcoeff} Q$
if there is a function $F : (\mathbb{Z}^{<\omega})^{t+1} \rightarrow \mathbb{Z}^{<\omega}$ such that for all graphs $G \in \mathcal{P}$

$$F(cS_1(G), \ldots, cS_t(G), cQ(G)) = cP(G)$$

The equivalence relations $P \sim^{\mathcal{S},(C)}_{r.d.p.} Q$ and $P \sim^{\mathcal{S},(C)}_{relcoeff} Q$, are defined as usual.

Theorem 15 $P \preceq^{\mathcal{S}}_{relcoeff} Q$ *iff* $P \preceq^{\mathcal{S}}_{r.d.p.} Q$

The proof is left as an exercise!

Conclusion of Lecture 11

We have established a framework for comparing graph polynomials.

What remains to do?

- In the seminar 238901 next semester
  - Comparing uniform sequences of polynomials.
  - Introducing complexity.
- In Lecture 12
  - Introducing Logic
  - Linear recurrences for graph polynomials