LECTURE 3

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Last lecture

- We discussed the matching polynomials;
- We introduced a way of comparing graph invariants.
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Homework: Compute graph polynomials!

Lecture 3: Outline

- Variations around the chromatic polynomial;
- More graph polynomials
- Generating functions vs colorings
- Comparing graph invariants revisited.

The Chromatic Polynomial

and

Its Variations

The (vertex) chromatic polynomial (a reminder)

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-vertex-coloring** is a map

$$c : V(G) \to [\lambda]$$

such that $(u, v) \in E(G)$ implies that $c(u) \neq c(v)$.

We define $\chi(G, \lambda)$ to be the number of λ-vertex-colorings

Theorem 7 (G. Birkhoff, 1912) *$\chi(G, \lambda)$ is a polynomial in $\mathbb{Z}[\lambda]$.*

Proof:

(i) $\chi(E_n) = \lambda^n$ where E_n consists of n isolated vertices.

(ii) For any edge $e = E(G)$ we have $\chi(G - e, \lambda) = \chi(G, \lambda) + \chi(G/e, \lambda)$.

The edge-chromatic polynomial

Let $G = (V(G), E(G))$ be a graph, and $\lambda \in \mathbb{N}$.

A **λ-edge-coloring** is a map

$$c : E(G) \rightarrow [\lambda]$$

such that if $(e, f) \in E(G)$ have a common vertex then $c(e) \neq c(f)$.

We define $\chi_e(G, \lambda)$ to be the number of λ- edge-colorings

Fact: $\chi_e(G, \lambda)$ a polynomial in $\mathbb{Z}[\lambda]$.

Let $L(G)$ be the **line graph** of G.
$V(L(G)) = E(G)$ and $(e, f) \in E(L(G))$ iff e and f have a common vertex.

Observation: $\chi_e(G, \lambda) = \chi(L(G), \lambda)$, where $L(G)$ is the line graph of G.

Conclusion: $\chi_e(G, \lambda)$ **is a polynomial** in $\mathbb{Z}[\lambda]$.

Variations on coloring, I

We can count other coloring functions.

- Total colorings

  $f_V : V \to [\lambda_V]$, $f_E : E \to [\lambda_E]$ and $f = f_V \cup f_E$,
  with f_V a proper vertex coloring and f_E a proper edge coloring.

- Connected components

  $f_V : V \to [\lambda_V]$, If $(u, v) \in E$ then $f_V(u) = f_V(v)$.

- Pre-coloring extensions

  Given graph $G = (V, E)$ and an equivalence relation R on V and $f_V : V \to [\lambda_V]$, we require that if $(u, v) \in R$ they have the same color, and if $(u, v) \in E - R$ they have different colors.

Fact: The corresponding counting functions are polynomials in λ.

Variations on coloring, II

Encountered at CanaDam-07:

Let $f : V(G) \rightarrow [\lambda]$ be a function, such that Φ is one of the properties below and $\chi_\Phi(G, \lambda)$ denotes the number of such colorings with atmost λ colors.

* **convex:** Every monochromatic set induces a connected graph.

* **injective:** f is injectiv on the neighborhood of every vertex.

- **complete:** f is a proper coloring such that every pair of colors occurs along some edge.

* **harmonious:** f is a proper coloring such that every pair of colors occurs at most once along some edge.

- **equitable:** All color classes have (almost) the same size.

* **equitable, modified:** All non-empty color classes have the same size.

New fact: For (*), $\chi_\Phi(G, \lambda)$ is a polynomial in λ, for (-), it is not.

Variations on coloring, III

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* **path-rainbow:** Let $f : E \to [\lambda]$ be an edge-coloring. f is **path-rainbow** if between any two vertices $u, v \in V$ there as a path where all the edges have different colors.

New fact: $\chi_{rainbow}(G, \lambda)$, the number of path-rainbow colorings of G with λ colors, is a polynomial in λ

Rainbow colorings of various kinds arise in computational biology

* **-monochromatic components:** Let $f : V \to [\lambda]$ be an vertex-coloring and $t \in \mathbb{N}$. f is an mcc_t-coloring of G with λ colors, if all the connected components of a monochromatic set have size at most t.

New fact: For fixed $t \geq 1$ the function $\chi_{mcc_t}(G, \lambda)$, the number of mcc_t-colorings of G with λ colors, is a polynomial in λ. but not in t.

mcc_t colorings were first studied in:

N. Alon, G. Ding, B. Oporowski, and D. Vertigan. Partitioning into graphs with only small components. *Journal of Combinatorial Theory, Series B*, 87:231–243, 2003.

Variations on coloring, IV

Let $\mathcal{P}$ be any graphs property and let $n \in \mathbb{N}$.

We can define coloring functions $f : V \to [\lambda]$ by requiring that the union of any n color classes induces a graph in $\mathcal{P}$.

- For $n = 1$ and $\mathcal{P}$ the empty graphs $G = (V, \emptyset)$ we get the proper colorings.
- For $n = 1$ and $\mathcal{P}$ the connected graphs we get the convex colorings.
- For $n = 1$ and $\mathcal{P}$ the graphs which are disjoint unions of graphs of size at most t, we get the mcc_t-colorings.
- For $n = 2$ and $\mathcal{P}$ the acyclic graphs and $n = 2$ we get the **acyclic colorings**, studied in XXX-india.

Theorem: Let $\chi_{\mathcal{P},n}(G, \lambda)$ be the number of colorings of G with λ colors such that the union of any n color classes induces a graph in $\mathcal{P}$.

Then Let $\chi_{\mathcal{P},n}(G, \lambda)$ is a polynomial in λ.

Variations on colorings, V: coloring relations

Let $G = (V, E)$. Here we look at an example where the coloring is a relation $R \subseteq V \times [k]$ rather than a function $f : V \to [k]$.
We denote by C_v the set $\{c \in [k] : (v, c) \in R\}$.

Let $a, b \in \mathbb{N}$. An (a, b)-coloring relation with k colors is a relation $R \subseteq V \times [k]$ such that

- For each $v \in V$ there are at most a-many colors $c \in [k]$ such that $(v, c) \in R$.
- If $(u, v) \in E$ then $C_u \neq C_v$ and there are at most b-many distinct elements $c_1, \ldots, c_b$ in $C_u \cap C_v$.

Exercise:

- Compute the number of (a, b)-coloring relations of the complete graphs K_n for various $a, b, k \in \mathbb{N}$.
- Is the number (a, b)-coloring relations with k colors of a graph G a polynomial in a, b or k?
- Look at the corresponding definitions with "at most" replaced by "at least" or "exactly".

Variations on colorings, VI: Two kinds of colors.

Let $G = (V, E)$.
Here we look at two disjoint color sets $A = [k_1]$ and $B = [k_1 + k_2] - [k_1]$.
The colors in A are called proper colorings.
Our coloring is a function $f : V \rightarrow [k_1 + k_2] = [k]$ such that

- If $(u, v) \in E$ and $f(u) \in A$ and $f(v) \in A$ then $f(u) \neq f(v)$.
- We count the number of colorings with $k = k_1 + k_2$ colors such that k_1 colors are in k_1 (proper).

Theorem 8 (K. Dohmen, A. Pönitz and P. Tittman, 2003)
This gives us a polynomial $P(G, k_1, k)$ in k_1 and k.

Hypergraph colorings, I

Given hypergraph $H = (V, E)$ with $E \subset \wp(V)$, and a set of λ colors $[\lambda]$. Let $f : V \to [\lambda]$.

- f is a **weak hypergraph coloring**, if for each $e \in E$ with at least two vertices, there are $u \neq v$ and $u, v \in e$ with $f(u) \neq f(v)$.

- f is a **strong hypergraph coloring**, if for each $e \in E$ and for all $u, v \in e$ with $u \neq v$ we have $f(u) \neq f(v)$.

- f is a **conflict free hypergraph coloring**, if f is a weak hypergraph coloring and for each $e \in E$ there is $u \in e$ such that for all $v \in e, v \neq u$ we have $f(v) \neq f(u)$.

  This was introduced in Even, G., Lotker, Z., Ron, D., and Smorodinsky, S. (2003), K. Aardal, S. van Hoesel, A. Koster, C. Mannino, and A. Sassano (2003), cf. also J. Pach, E. Tardos, (2009)

Hypergraph colorings, II

Let $\chi_{h-weak}(H, \lambda)$, $\chi_{h-strong}(H, \lambda)$ and $\chi_{h-cf}(H, \lambda)$, denote the number of weak, strong and conflict free hypergraph colorings of G with λ colors, respectively.

Theorem: The counting functions

(i) $\chi_{h-weak}(H, \lambda)$,

(ii) $\chi_{h-strong}(H, \lambda)$ and

(iii) $\chi_{h-cf}(H, \lambda)$,

are hypergraph polynomials in λ.

Proof: For (i) and (ii) one can mimick Birkhoff's proof for graphs.

We shall give a **uniform proof** of these statements later on.

Hypergraph colorings, III

Given hypergraph $H = (V, D, E)$ with two kinds of hyperedges, $D, E \subset \wp(V)$, and a set of λ colors $[\lambda]$. Let $f : V \to [\lambda]$.

f is a **strong/weak mixed hypergraph coloring**, if

- for (V, E) the function f is a strong/weak hypergraph coloring, and
- for every $d \in D$ and for every $u, v \in d$ we have $f(u) = f(v)$.

Theorem: The number $\chi_{mixed}(H, \lambda)$ of mixed hypergraph colorings with λ colors is a polynomial in λ.

Proof: This was shown in

Vitaly I. Voloshin
Coloring Mixed Hypergraphs: Theory, Algorithms and Applications, AMS 2002

Our **uniform proof** applies also to this case.

Digression:

More typical theorems about the chromatic polynomial

Main reference:

[DKT] F.M. Dong, K.M. Koh and K.L. Teo
Chromatic polynomials and the chromaticity of graphs
World scientific, 2005

Expressive power of $\chi(G, \lambda)$, I

We denote by $k(G)$ the number of connected components, $b(G)$ the number of blocks (2-connected components), and $\chi(G)$ the chromatic number of G, respectively.

Theorem 9 (DKT, 3.2.1.)
Let G_1, G_2 be two graphs with $\chi(G_1, \lambda) = \chi(G_2, \lambda)$. Then

(i) $V(G_1) = V(G_2)$ and $E(G_1) = E(G_2)$.

(ii) $\chi(G_1) = \chi(G_2)$.

(iii) $k(G_1) = k(G_2)$, in particular G_1 is connected iff G_2 is connected.

(iv) $b(G_1) = b(G_2)$, in particular G_1 is 2-connected iff G_2 is 2-connected.

(v) G_1 is bipartite iff G_2 is bipartite.

Expressive power of $\chi(G, \lambda)$, II

We denote by $n_H(G)$ and $i_H(G)$ the number of subgraphs and induced subgraphs of G, respectively, which are isomorphic to H.
K_k is the clique (complete graph) on k vertices. C_k is the cycle on k vertices.
The girth $g(G)$ is the smallest k such that $n_{C_k}(G) \neq 0$.

Theorem 10 (DKT, 3.2.1.)
Let G_1, G_2 be two graphs with $\chi(G_1, \lambda) = \chi(G_2, \lambda)$. Then

(i) $n_{C_3}(G_1) = n_{C_3}(G_2)$

(ii) $g(G_1) = g(G_2)$.

(iii) $i_{C_4}(G_1) - 2n_{K_4}(G_1) = i_{C_4}(G_2) - 2n_{K_4}(G_2)$

(iv) $n_{C_k}(G_1) = n_{C_k}(G_2)$*, provided* $g = g(G_1) = g(G_2) \leq k \leq \lceil \frac{3g}{2} \rceil - 2$

Normal forms of $\chi(G,\lambda)$, I: Power form

As $\chi(G,\lambda)$ is a polynomial, we can write it as

$$\chi(G,\lambda) = \sum_{i}^{|V(G)|} a_i(G)\lambda^i$$

in power form.

For the disjoint union we note that

Proposition 11

$$\chi(G_1 \sqcup G_2,\lambda) = \chi(G_1,\lambda)\cdot\chi(G_2,\lambda).$$

Question: Is there a combinatorial interpretation of the $a_i(G)$?

Normal forms of $\chi(G,\lambda)$, II: Factorial form

We define $\lambda_{(i)} = \lambda \cdot (\lambda - 1) \cdot \ldots \cdot (\lambda - i + 1)$.

We write $\chi(G,\lambda)$

$$\chi(G,\lambda) = \sum_{i}^{|V(G)|} b_i(G)\lambda_{(i)}$$

There is a combinatorial interpretation of $b_i(G)$:

Theorem 12 *$b_i(G)$ is the number of partitions of V into i non-empty independent sets.*

Normal forms of $\chi(G,\lambda)$, II: Factorial form (continued)

We define a an operation $\circ$ on the $\lambda_{(i)}$ by $\lambda_{(i)} \circ \lambda_{(j)} = \lambda_{(i+j)}$ and extend it naturally to polynomials in $\lambda_{(i)}$.

The join of two graphs G_1, G_2, $G_1 \bowtie G_2$,
is obtained by taking the disjoint union and
adding all the edges between $V(G_1)$ and $V(G_2)$.

Theorem 13

$$\chi(G_1 + G_2, \lambda) = \left(\sum_i^{|V(G)|} c_i(G_1)\lambda_{(i)} \circ \sum_i^{|V(G)|} c_i(G_2)\lambda_{(i)} \right)$$

Normal forms of $\chi(G,\lambda)$, III: Binomial Form

We note that $\frac{\lambda_{(i)}}{i!} = \binom{\lambda}{i}$.

We write $\chi(G,\lambda)$

$$\chi(G,\lambda) = \sum_{i}^{|V(G)|} c_i(G)\binom{\lambda}{i}$$

There is a combinatorial interpretation of $c_i(G)$:

Theorem 14 *$c_i(G)$ is the number of proper colorings of G with exactly i colors.*

Corollary 15 $\frac{b_i(G)}{i!} = c_i(G)$

Exercise: Give a direct proof of the corollary!

The complexity of the chromatic polynomial, I

Let us look at the chromatic polynomial $\chi(G, \lambda)$.

- $\chi(G, \lambda)$ has integer coefficients,
  and for $\lambda \geq 0$ non-negative values,
  hence evaluating it at $\lambda = a, a \in \mathbb{N}$ is in $\sharp\mathbf{P}$.
- For $a = 0, 1, 2$ evaluating $\chi(G, \lambda)$ is in $\mathbf{P}$.
- For integer $a \geq 3$ evaluating $\chi(G, \lambda)$ is $\sharp\mathbf{P}$-complete.
- What about evaluating $\chi(G, \lambda)$ for $\lambda = b$ with
  $b \in \mathbb{Z}$, $b \leq 0$?
  $b \in \mathbb{R}$ or $b \in \mathbb{C}$?

Given evaluations of $\chi(G, \lambda)$ at $| V(G) | +1$ many points, we can compute the coefficients of $\chi(G, \lambda)$ efficiently.

The complexity of the chromatic polynomial, II

Theorem 16 *(N. Linial, 1986)*

For any two points $a, b \in \mathbb{C} - \mathbb{N}$
there is a **polynomial time algebraic reduction**
from the evaluation of $\chi(G, a)$ *to the evaluation of* $\chi(G, b)$.

Hence they are all **equally difficult**.

Proof: We note that

$$\chi(G \bowtie K_n, \lambda + n) = n_\lambda \cdot \chi(G, \lambda).$$

We use this to compute sufficently many points of $\chi(G, \lambda)$, and then use Lagrange interpolation. Q.E.D.

End of digression on typical theorems about the chromatic polynomial