LECTURE 2

What we have done so far

We have introduced two graph polynomials:

- the chromatic polynomial $\chi(G, X)$, and
- the characteristic polynomial $P(G, X)$.

We have looked at typical theorems about these graph polynomials.

Outline of Lecture 2

We introduce the matching polynomial in three versions.

- The **acyclic matching** polynomial $m(G; X)$;
- The **generating matching** polynomial $g(G; X)$;
- The **bivariate matching** polynomial $M(G; X)$.

We also get a glimpse the **Magic** of the **Tutte polynomial**.

Finally, we define when a graph invariant is
induced by a set of graph invariants.

Example 7 *The acyclic or matching defect polynomial, I*

We denote by $m_k(G)$ the number of k-matchings of a graph G, with $m_0(G) = 1$ by convention.

- The polynomial

$$m(G, X) = \sum_{k}^{\frac{n}{2}} (-1)^k m_k(G) X^{n-2k}$$

is called the **acyclic polynomial of G** and also the **reference polynomial** or **matching defect polynomial**.

The acyclic or matching defect polynomial, II

The acyclic polynomial has important applications in Chemistry (Hückel theory again) and and Molecular Physics of Ferromagnetisms. It was first studied in the 1970 (Heilman and Lieb, Kunz)

- L. Lovász and M.D. Plummer
  Matching Theory
  Annals of Discrete mathematics, vol. 29
  North-Holland 1986

- N. Trinajstić,
  Chemical Graph Theory
  CRC, 1992 (2nd edition)

- P.J. Garratt
  Aromaticity
  John Wiley and Sons, 19xx

Example 8 *The matching (generating) polynomial*

- The polynomial

$$g(G, X) = \sum_{k}^{n} m_k(G) X^k$$

is called the **matching polynomial of** G or the **matching generating polynomial of** G.

- It is easy to verify the identity

$$m(G, X) = X^n g(G, (-X^{-2}))$$

Example 9 *Multi-variate matching polynomial*

The two matching polynomials are special cases of a bivariate matching polynomial

$$M(G, X, Y) = \sum_{k}^{\frac{n}{2}} X^{n-2k} Y^k m_k(G) = \sum_{A} X^{|V(G)|-2|A|} Y^{|A|}$$

where A ranges over all subsets of $E(G)$ which are matchings.

Now we have

$$m(G; X) = M(G; X, -1)$$

and

$$g(G; X) = M(G; 1, X)$$

In other words, both $m(G; X)$ and $g(G; X)$ are substitution instances of $M(G; X, Y)$.

Interpretation: $\mid A \mid$ is the size of the matching A, and $\mid V(G) \mid -2 \mid A \mid$ is the number of vertices not incident with any edge in A.

Example 10 *Multi-variate Tutte polynomial*

Inspired by H. Whitney's work (1932) W.T. Tutte (1947, 1954) investigated generalizations of the chromatic polynomial to a polynomial in two variables, which he called the **dichromatic polynomial**, but now is called the **Tutte polynomial**, $T(G, X, Y)$.

The Tutte polynomial and its many generalizations became prominent, due to its many combinatorial interpretations in fields outside graph theory:

- Knot theory (via the Jones polynomial and its relatives)
- Statistical mechanics
- Quantum theory and quantum computing
- Chemistry

Example 11 *The Tutte polynomial*

Let $G = (V, E)$ be a graph,
and for $A \subseteq E$, let $G_A = (V, A)$ be a spanning subgraph.

The rank $r(G; A)$ is defined as $\mid V(G) \mid - k(G_A)$.

The **Tutte polynomial of** G is defined as

$$T(G; X, Y) = \sum_{A \subseteq E} (X-1)^{r(G;E)-r(G;A)} \cdot (Y-1)^{|A|-r(G;A)}$$

This looks confusing and innocent at the same time.

The fascination with the Tutte polynomial

The Tutte polynomial is like
a **magician's hat** with
rabbits, birds and other surprises coming out.

Easy manipulations produce various combinatorial counting functions. We have, at first glance surprisingly, the following

- $T(G, 1, 1)$ counts the number of spanning trees of G.
- $T(G, 2, 1)$ counts the number of forests of G.
- $T(G, 2, 0)$ counts the number of acyclic orientations of G.
- The chromatic polynomial is given by
$$\chi(G, X) = (-1)^{r(G;E)} X^{k(G)} T(G; 1 - X, 0)$$
- The reliability polynomial and the flow polynomial can also be derived with similar formulas.

Definition 12 *Complete graph invariants*

A graph invariant f is **graph-complete** if for any two graphs G_1, G_2 with $f(G_1) = f(G_2)$ we have also $G_1 \simeq G_2$.

The following is a graph-complete graph invariant.

- Let $X_{i,j}$ and Y be indeterminates.
  For a graph $\langle V, E \rangle$ with $V = [n]$ we put

$$Compl(G, Y, \bar{X}) = Y^{|V|} \cdot \left(\sum_{\sigma \in \mathfrak{S}_n} \prod_{(i,j) \in E} X_{\sigma(i),\sigma(j)} \right)$$

  Here $\mathfrak{S}_n$ is the permutation group of $[n]$.

Challenge: Find a polynomial in a constant finite number of indeterminates which is a graph-complete graph invariant.

An “unnatural” graph-complete invariant

Let $g : \mathcal{G} \rightarrow \mathbb{N}$ be a Gödel numbering for labeled graphs of the form $G = \langle [n], E, <_{nat} \rangle$.

We define a graph polynomial using g:

$$\Gamma(G, X) = \sum_{H \simeq G} X^{g(H)}$$

Clearly this is a graph invariant.

But it is “obviously unnatural” !

Can we make precise

what a **natural** graph polynomial should be?

Comparing graph invariants

In the literature we often find statements or questions of the form

- The Tutte polynomial is generalization of the chromatic polynomial.
- The Tutte polynomial does not determine the matching polynomial.
- Is there a natural most general graph polynomial?

We attempt to make this precise

Definition 13 *Induced graph invariants*

Let $\mathcal{H} \subseteq \mathcal{G}$ be a class of graphs closed under isomorphisms.
Let F be a set of graph invariants in a ring $\mathcal{R}$,
and let g be one more graph invariant.

We say that F **induces** g **on** $\mathcal{H}$,
or g **is a consequence of** F,
if for any two graphs $G_1, G_2 \in \mathcal{H}$ such that $f(G_1) = f(G_2)$ for all $f \in F$
we also have $g(G_1) = g(G_2)$.

We denote by $Ind_{\mathcal{R}}^{\mathcal{H}}(F)$ the set of graph invariants in $\mathcal{R}$ induced by F on $\mathcal{H}$.

We write also $F \models_{\mathcal{R}}^{\mathcal{H}} g$ for $g \in Ind_{\mathcal{R}}^{\mathcal{H}}(F)$.

How do we see if $F \models_{\mathcal{R}}^{\mathcal{H}} g$?

Example 14

Algebraically induced invariants

Let f, g be two graph invariants in $\mathcal{R}$.
Then the following are induced invariants of $F = \{f, g\}$:

- $f + g, f - g, f \times g$
- The formal derivative f'.
- Let $\phi : \mathcal{R}^2 \to \mathcal{R}$ be a function.
  Then $\phi(f, g)$ is induced by F.
- If $Q(G; \bar{X})$ is a graph polynomial and g is a substitution instance of Q, then g is induced by F.

Examples 15

Invariants induced by the characteristic polynomial

The characteristic polynomial $P(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of triangles of G.

We also have $P(K_{1,4}, X) = P(C_4 \sqcup E_1, X)$
but $K_{1,4}$ has no 2-matchings, whereas C_4 does.
Hence the $P(G, X)$ does not induce the number of connected components $k(G)$ nor $m(G, X)$.

Example 16

Invariants induced by the acyclic polynomial.

The acyclic polynomial $m(G, X)$ induces

- The number of vertices $\mid V \mid$.
- The number of edges $\mid E \mid$.
- The number of perfect matchings.
- the matching generating polynomial.

On the other side $g(E_n, X) = 1$ for all $n \in \mathbb{N}$,
whereas $m(E_n; X) = P(E_n, X) = X^n$.
Hence $g(G, X)$ does not induce the characteristic polynomial $P(G, X)$ nor the acyclic polynomial $m(G; X)$.

We shall discuss this further in Lecture 4.

Example 17

Invariants induced by the chromatic polynomial

The following are induced by $\chi(G, X) = \sum_{i=1}^{n} (-1)^{n-i} h_i X^i$:

- The cardinality of $V(G) = n$ is the degree of $\chi(G, X)$.
- The cardinality of $E(G) = m = h_{n-1}$.
- The chromatic number $\chi(G)$ is
  the smallest integer a such that $\chi(G, a) > 0$.
- The number of connected components $k(G)$
  is the multiplicity of zeros $X = 0$.
- The number of blocks $b(G)$ is the multiplicity of zeros $X = 1$.
- The girth $g = g(G)$ is given by the fact that
  for $0 \leq i \leq g - 2$ we have $h_{n-i} = \binom{E(G)}{i}$.

Example 18

The acyclic polynomial and the characteristic polynomial.

Theorem 19 (I. Gutman, 1977)

$P(G, X) = m(G, X)$ *iff* G *is a forest.*

For $\mathcal{H} = \mathcal{F}$ *the forests we have*

$$P(G, X) = m(G, X)$$

i.e., the acyclic (matching defect) polynomial and the characteristic polynomial coincide,
and we have

$$P(G, X) \models^{\mathcal{F}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{F}} P(G, X).$$

and

$$P(G, X) \models^{\mathcal{F}} g(G, X) \text{ and } g(G, X) \models^{\mathcal{F}} P(G, X).$$

In general, none induces the other.

I. Gutman, The acyclic polynomial of a graph, Publ. Inst. Math. (Beograd) (N.S.) 22 (36) (1977), pp. 63-69.

Example 20

The acyclic polynomial and the chromatic polynomial.

Definition 21
The complement graph of the simple graph $G = (V, E)$
is the graph $\bar{G} = (V, V^2 - D(V) - E)$.

For a graph polynomial $g = g(G, \bar{X})$ *the* **adjoint polynomial** $\hat{g}(G, \bar{X})$ *of* g *is defined by* $\hat{g}(G, \bar{X}) = g(\bar{G}, \bar{X})$.

Theorem 22 (E.J. Farrell and E.G. Whitehead Jr. 1992)
For $\mathcal{H} = \mathcal{TF}$, *the triangle free graphs, we have*

$$\hat{\chi}(G, X) \models^{\mathcal{TF}} m(G, X) \text{ and } m(G, X) \models^{\mathcal{TF}} \hat{\chi}(G, X).$$

i.e., the acyclic (matching defect) polynomial
and the adjoint chromatic polynomial mutually induce each other.

Note that $\chi(P_4) = \chi(K_{1,3})$, $P_4 \simeq \bar{P}_4$, *but* $m(P_4) \neq m(K_{1,3})$.
Hence, $\chi(G; X)$ *does not induce* $m(G; X)$.

Exercise: *Show that* $m(G; X)$ *does not induce* $\chi(G; X)$.

Graph polynomials on trees

Combining the Gutman Theorem with the Farrell-Whitehead Theorem we get

Corollary: On trees the chromatic polynomial $\chi(G; X)$,
the acyclic matching polynomial, $m(G; X)$
and the characteristic polynomial $P(G; X)$ induce each other.

Proof: Trees are triangle-free.

Note, this is not true for the generating matching polynomial $g(G; X)$.
Explain!

Project 1 (for this course)

- Prepare slides with a proof of the Farrell-Whitehead Theorem:
  E.J. Farrell and Earl Glen Whitehead, Jr.
  Connections between the matching and chromatic polynomials
  Internat. J. Math. & Math. Sci. VOL. 15 NO. 4 (1992) 757-766 pdf-file

- Find more Theorems similar to the Gutmann or Farrell-Whitehead Theorems.

Example 23

The chromatic polynomial and Tutte polynomial

(i) The chromatic polynomial $\chi(G, X)$ is not induced by the Tutte polynomial $T(G, X, Y)$.

(ii) On connected graphs $\mathcal{C}$ we have $T(G, X, Y) \models^{\mathcal{C}} \chi(G, X)$ for

(iii) Tutte polynomial $T(G, X, Y)$ is not induced by the the chromatic polynomial $\chi(G, X)$.

Proof:

(i) Let E_n be the graph with n vertices and no edges. We have $T(E_n, X, Y) = 1$ but $\chi(E_n, X) = X^n$.

(ii) (After W.T. Tutte, 1954) $\chi(G, X) = (-1)^{|V|-k(G)} X^{k(G)} T(G, 1-X, 0)$.

(iii) (After M. Noy, 2003) Let W_n be the wheel with n spokes. It is known that $T(G, X, Y) = T(W_n, X, Y)$ implies that $G \simeq W_n$ for all n.

But there is a $G \not\simeq W_5$ with $\chi(G, X, Y) = \chi(W_5, X, Y)$.

Example 24

The Tutte polynomial and the matching polynomials

- The matching polynomial is not induced by the Tutte polynomial, even on connected planar graphs.
- The Tutte polynomial is not induced by the matching polynomial, even on connected planar graphs.

Proof:

(i) For trees with n vertices t_n we have $T(t_n, X, Y) = X^{n-1}$. But it is easy to see that $K_{1,n-1}$ and P_n are both trees with n vertices and their matching polynomials differ, as $K_{1,n-1}$ has no 2-matching but P_n has for $n \geq 3$.

(ii) On the other hand $C_3 \sqcup_e C_5$ and $C_4 \sqcup_e C_4$ have the same matching polynomials (check by hand) but have different Tutte polynomials, as the Tutte polynomials counts cliques of given size.

What do we learn?
What do we ask?

- Polynomial graph invariants are still a mystery.

- Can we analyze the consequence relation for polynomial invariants?

- Can we identify "good invariants"?

- What are appropriate complexity classes for graph invariants?