Catalogue of graph polynomials

J.A. Makowsky

November 30, 2010

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Chapter 1

Graph polynomials

A graph is a relational structure $G = \langle V, E; R \rangle$, where V is a set of vertices, E is a set of edges, and R is the a ternary relation $R \subseteq V \times E \times V$, where R(u, e, v) holds if and only the edge has u as its source and v its target. An edge labeling of a graph is a function $c_E : E \to \mathbb{N}$, A vertex labeling of a graph is a function $c_V : V \to \mathbb{N}$. A labeled graph is a tuple of the form $\langle V, E, c_V, c_E \rangle$. c_V or c_E could be the empty map. Let \mathcal{G} denote the class of finite graphs, \mathcal{G}_{lab} denote the class of finite labeled graphs.

1.1 Comparing graph polynomials

We want to compare the combinatorial expressive power of graph polynomials. Several approaches have been used explicitly or implicitly in the literature. We shall state our definitions for PNGIs, rather than restricting ourselves to graph polynomials, whenever it makes sense.

1.1.1 Distinctive power

Let f, g be two NPGIs for a class of labeled graphs \mathcal{G} .

Definition 1. f is less distinctive than g, and we write $f <_{dist} g$, if whenever $G_1, G_2 \in \mathcal{G}$ and $g(G_1) = g(G_2)$ then also $f(G_1) = f(G_2)$.

- 1.1.2 Substitution instances
- 1.1.3 Uniform algebraic reductions
- 1.1.4 Substitution instances
- 1.1.5 Substitution instances
- 1.1.6 Substitution instances
- 1.2 Definability of graph polynomials
- 1.2.1 Static definitions
- 1.2.2 Dynamic definitions
- 1.2.3 SOL-definable polynomials
- 1.2.4 Generalized chromatic polynomials

Chapter 2

A catalogue of graph polynomials

2.1 Polynomials from the zoo

2.1.1 Chromatic polynomial

Author: Introduced by G. Birkhoff in 1912, [Bir12].

Motivation: To prove the four-color conjecture.

Definition: $\chi(G, k)$ is the number of proper k-vertex colorings of G. It is a polynomial by Birkhoff's argument via the recurrence relations. It is also a polynomial by the Makowsky-Zilber argument.

Reducibility Instance of the Tutte polynomial for connected graphs.

$$\chi(G,k) = (-1)^{|V(G)| - k(G)} X^{k(G)} \cdot T(G, 1 - k, 0)$$

Recurrence relations Deletion-Contraction rules.

- **Complexity** \sharp **P**-hard almost everywhere. On the other hand, there exists an $O(n \cdot log^2 n)$ algorithm to compute all coefficients of the chromatic polynomial of an n vertex graph of bounded treewidth, [Für10].
- Difficult point property: Proven in [Lin86].

Definability: $MSOL_2$ over ordered graphs, order invariant. (via the Tutte polynomial).

- **Combinatorial interpretations:** For k = -1 it counts the number of acyclic orientations, [Sta73].
- **Roots:** In [BT69] first found zeros of the chromatic polynomials. In [Bro98], it was proved that the chromatic polynomial of a connected graph with n vertices and m edges has a root with modulus at least (m-1)/(n-2); this bound is best possible for trees and 2-trees (only). It as also proved that the chromatic polynomial of a graph with few triangles that is not a forest has a nonreal root and that there is a graph with n vertices whose chromatic polynomial has a root with imaginary part greater than $\sqrt{(\frac{n}{4})}$. In [Roy07], an infite family of 3-connected non-bipartite graphs with chromatic roots in the interval (1; 2) was inverstigated. See also [BKW80].

References A survey monograph is [DKT05]. See also [MW82], [AT97], [NW99], [DPT03], [Sar00]

2.1.2 Chromatic symmetric function

Authors: Introduced by Stanley, [Sta95].

Motivation: Certain conjectures regarding immanants of Jacobi-Trudi matrices

Static definition: $Y_G(\mathbf{x}, t)$ is given by

$$Y_G(\mathbf{x},t) = \sum_{\chi} x_{\chi(v_1)} x_{\chi(v_2)} \cdots x_{\chi(v_n)} (1+t)^{b(\chi)},$$

the sum ranges over all proper colorings χ , $b(\chi)$ is the number of monochromatic edges.

Reducibility: Related to the Universal polynomial, cf. [MN09]. For any $G, Y_G(\mathbf{x}, t)$ is given by

$$Y_G(\mathbf{x},t) = t^{|V|} U_G(x_j = p_j(\mathbf{x})/t, y = t+1),$$

 $p_j(\mathbf{x}) = \sum_{i=1}^{\infty} x_i^j.$

References: [Sta98], [MN09].

2.1.3 Adjoint polynomials

Authors: Introduced by Liu, [Liu87].

- **Motivation:** To relate the chromatic uniqueness of a graph to the structure of its complement and employ the classical theory of polynomial algebra. A graph G is chromatically unique if $P(H, \lambda) = P(G, \lambda)$ imlies that H is isomorphic to G.
- **Static definition:** If G_0 is a spanning subgraph of G and every component of G_0 is complete, then G_0 is called an ideal subgraph of G. For a given graph G, let N(G, k) be the number of ideal sungraphs with k components. The polynomial

$$h(G, x) = \sum_{i=1}^{p} N(G, i) x^{i}$$

is called the adjoint polynomial of the graph G, where p = |V(G)|.

Recursive definition: If $uv \in E(G)$, and if uv does not belong to any triangle in G, then

$$h(G, x) = h(G - uv, x) + x \cdot h(G - u, v, x)$$

Reducibility: Related to chromatic polynomial [Liu97]. Let $\alpha(G, i)$ denote the number of *i*-independent partitions of G and $\sigma(G, x) = \sum_{i=1}^{p} \alpha(G, i) x^{i-\chi(G)}$. Then

$$h(G, x) = x^{\chi(G)} \cdot \sigma(\bar{G}, x)$$

Roots: In [ZL09], the nimimal roots $\beta(G)$, such that $-(2 + \sqrt{5}) \leq \beta(G) \leq -4$ characterized for connected graphs G.

References: [Wa] and [Za].

2.1.4 The Tutte polynomial

Authors: Introduced by W.T. Tutte [Tut54], building on H. Whitney [Whi32].

Motivation: To be checked.

Static definition: To be checked.

Recurrence relations: To be checked.

Recursive definition: To be checked.

Reducibility: To be checked

Related polynomials: Chromatic polynomial, flow polynomial, and more (to be added).

Complexity: #P-hard for most points.

Difficult point property: Proved in [JVW90].

Definability: To be checked.

Combinatorial interpretations: To be added.

Roots: To be checked.

References: Monographs dealing with the Tutte polynomial are [Bol99, GR01].

2.1.5 Strong Tutte symmetric function

Authors: Introduced by C. Merino and S. Noble, [MN09].

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Motivation: To extend Stanley's Tutte symmetric function by replacing the t variable by countably infinitely many commuting variables.

Static definition: $\bar{Y}_G(\mathbf{x}, \mathbf{t})$ is given by

$$\bar{Y}_G(\mathbf{x}, \mathbf{t}) = \sum_{\chi} (\prod_{i=1}^n x_{\chi(v_i)}) (\prod_{i=1}^\infty (1+t_i)^{b_i(\chi)})$$

the sum ranges over all proper colorings χ , $b_i(\chi)$ is the number of monochromatic edges for which both endpoints have colour *i*.

Reducibility: Related to the strong *U*-polynomial, cf. [MN09]. The strong *U*-polynomial and the strong Tutte symmetric function are aquivalent:

$$Y_G(\mathbf{x}, \mathbf{t}) = U_G(z_{i,j} = \bar{p}_{i,i+j-1}(\mathbf{x}, \mathbf{t})),$$

 $\bar{p}_{r,s}(\mathbf{x}, \mathbf{t}) = \sum_{i=1}^{\infty} x_i^r t_i^s.$

References: See [Goo08].

2.1.6 Tutte-Grothendieck invariants

Authors: Introduced by A.Goodall, [Goo08].

- Motivation: Investigate the Tutte polynomial of a grapf as a Hamming weight enimerator of its tensions (or flows). Consider a family polynomials containing the graph polynomials. The question is: whether other Tutte-Grothendieck invariants can be obtained in a similar way.
- **Definition of Tutte-Grothendieck invariants :** A function F from (isomorphism classes of) graps to $C[\alpha, \beta, \gamma, x, y]$ is a Tutte-Grothendieck invariant if it satisfies, for each graph G = (V, E) and any edge $e \in E$,

$$F(G) = \begin{cases} \gamma^{|V|} & E = \emptyset\\ xF(G/e) & e \text{ a bridge}\\ yF(G\backslash e) & e \text{ a loop}\\ \alpha F(G/e) + \beta F(G\backslash e) & e \text{ not a bridge or loop} \end{cases}$$

Reducibility: If F is a Tutte-Grothendieck invariant then

$$F(G) = \gamma^{k(G)} \alpha^{r(G)} \beta^{n(G)} T(G; \frac{x}{\alpha}, \frac{y}{\beta}).$$

Fact: Let Q be a set of size q and $\mathbf{w} = (w_{a,b})$ a tuple of complex numbers indexed by $(a, b) \subset Q \times q$. Assume that the edges u, v of G = (V, E) have been given an arbitrary, fixed orientation (u, v). Denote by \vec{E} the resulting set of derected edges. Consider the partition function for a vertex Q-colouring model that assigns a weight wa, b to a directed edge (u, v) coloured (a, b):

$$F(G; \mathbf{w}) = \sum_{\mathbf{c} \in Q^V} \prod_{(u,v) \in \vec{E}} w_{c_u, c_v} = \sum_{\mathbf{c} \in Q^V} \prod_{(a,b) \in Q \times Q} w_{a,b}^{\sharp\{(u,v) \in \vec{E}: (c_u, c_v) = (a,b)\}}$$

The graphical invariant $F(G; \mathbf{w})$ is a Tutte-Grothendieck invariant if and only if there are constants y, w such that

$$w_{a,b} = \begin{cases} w & a \neq b \\ y & a = b \end{cases}$$

proven in [MN09].

2.1.7 A weighted graph polynomial

Authors: Introduced by S. Noble and D. Welsh in [NW99].

- Motivation: To extend the treatment of the relationship between chord diagrams and Vassiliev invariants of knots [CDL94] to Tutte-Grothendieck invariants.
- Static definition: The weighted graph polynomial $U(G, \bar{x}, y)$ for a graph G = (V, E) is defined as

$$U(G, \bar{x}, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{s(i,A)} y^{|A| - r(A)}$$

where s(i, A) denotes the number of connected components of size *i* in the spanning subgraph (V, A), and r(A) = |V| - k(A) is the rank of (V, A).

- Reducibility and Related polynomials: The weighted graph polynomial contains many other graph invariants as specialization, for instance the 2-polymatroid rank generating function of Exley and Whittle [OW93], and as a consequence the matching polynomial [HL72] and the symmetric function generalization of the chromatic polynomial [Sta95]. In [Sar00], it was shown that the coefficients of U and the polychromate determine one another. Related to the weighted integral gain graph of [FZ07].
- **Complexity:** For graphs of tree-width atmost k, computable in polynomial time, with exponent depending on k, [Nob09]. Not known to be in FPT.
- Definability: Not even SOL-definable. Proof sketched in [AGM07].
- **References:** The dichromatic polynomial of [FZ07] on a graph whose edges are labelled invertibly from a group and vertices weighted from a semigroup, includes in particular the Noble and Welsh's for graphs with positive interger weights. To be checked.

2.1.8 Chain polynomial

Authors: Introduced by R.C. Read and E.G. Whitehead Jr. [RJ99].

Motivation: Chromatic polynomials for homeomorphisms classes of graphs.

Static definition: r(S) denotes the rank of the spanning subgraph induced by S.

$$Ch(G,\omega,\bar{q}) = \sum_{S \subseteq E} (1-\omega)^{|S|-r(S)} \prod_{e \in E-S} q_e$$

Recursive definition: • $Ch(E_1) = 1;$

• For G_1 and G_2 with at most one vertex in common we have

$$Ch(G_1 \cup G_2) = Ch(G_1) \cdot Ch(G_2)$$

• If $e \in E$ is a loop, we have

$$Ch(G) = (q_e - \omega)Ch(G - e)$$

• If $e \in E$ is not a loop, we have

$$Ch(G) = (q_e - 1)Ch(G - e) + Ch(G/e)$$

Reducibility: Related to the labeled Tutte polynomial. Suppose U is an edge-labeled multigraph and M is doubly weighted matroid on E = E(G) obtrained from the circuit matroid of G by letting the weights of an edge labeled a be p(a) = 1 and q(a) = a - 1. Then the chain polynomial Ch(G) is obtained from the Tutte polynomial t(M) by evaluating y = 2 - w and x = 2, cf. [Tra02]. In [Ja10], a formula for computing the Tutte polynomial of the signed graph formed from a weighed graph by edge replacement in terms of the chain polynomial of the weighed graph presented.

Complexity: #P-hard for many instances.

Definability: Order invariant $MSOL_2$ -definable with auxiliary order on edges.

Roots: If a poset is a semi-order (i.e. 1+3 free and 2+2 free) then the roots of its chain polynomial are real. It is conjectured that if a poset is a distributive lattice, then again, the roots are real. In [BR05], it was proven that the number c_i of chains of length i in the proper part of a distributed lattice L of length l+2 satisfy the inequalities:

$$c_0 \leq \ldots \leq c_{|l/2|}$$
 and $c_{|3l/2|} \leq \ldots \leq c_l$

References: [RJ99, RJ03, Tra02, BR05, Ja10, AGM08, AGM10].

2.1.9 Characteristic polynomial

- **Motivation:** In linear algebra, one associates a polynomial to every square matrix: its characteristic polynomial. This polynomial encodes several important properties of the matrix, most notably its eigenvalues, its determinant and its trace. The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix.
- **Static definition:** We start with a field K (such as the real or complex numbers) and an $n \times n$ matrix A over K. The characteristic polynomial of A, denoted by pA(t), is the polynomial defined by

$$pA(t) = det(t \cdot \mathbf{I} - A)$$

where I denotes the n-by-n identity matrix and the determinant is being taken in K(t), the field of rational functions in t.

- **Recursive definition:** Let's call the collection of the characteristic polynomials of the vertexdeleted subgraphs the polynomial *deck*. Tutte [Tut79] proved that the characteristic polynomial of a graph is reconstructible from its deck. More results may be found in particular in [CL98, Sch79, Sim92, Hag00, Sci02]. In the molecular context, it was given in [Hei53] and developped in [RG96].
- **Reducibility and Related polynomials:** On forest it equals the matching polynomial.
- **Complexity** Polynomial time computable (Determinant).
- **Definability** Substitution instance of MSOL-polynomial for I(G).
- Combinatorial interpretations: Coefficient is related to Expanders.
- **Roots:** Roots of the characteristic polynomial are known as the spectrum (or eigenvalues) of the matrix (or of the graph); differing graphs with the same characteristic polynomial (and so the same roots) are known as *Isospectral Graphs*.
- **References** Fan Rong K. Chung Spectral Graph Theory, CBMS Conference on Recent Advances in Spectral Graph Theory held at California State University at Fresno, June 6-10, 1994"–T.p. verso. Published 1997 American Mathematical Society ISBN 0821803158

Chemical Graphs: Looking Back and Glimpsing Ahead AT Balaban - Journal of Chemical Information and Computer Sciences, 1995

2.1.10 Matching polynomial

Authors: Introduced by C.J. Heilmann and E.H. Lieb in 1972 [HL72]. See alse [Far79].

Motivation: Physics and chemistry. See [Tri92, Bal93a, Bal95].

Static definition: A matching of a graph G is a set of paiwise disjoint edges of G. The matching polynomial $\mu(G, x)$ of G is given by

$$\mu(G, x) = \sum_{k < 0} (-1)^k \cdot p(G, k) \cdot x^{n-2k},$$

where p(G, k) is the number of matchings with k edges in G. It is possible to recover the order, degree, girth and number of minimal cycles for a regular graph from its matching polynomial, [BF95].

Recurrence relations: In the chemeistry contex, for polygraph, given in [BGMP86] and developped in [BX93]. See also [ABBK03].

Recursive definition: See [AGM07].

- Related polynomials: Characteristic polynomial [Gut80], the matching polynomial of a graph coincides with the chracteristic polynomial if and only if the graph is a forest [GG81], acyclic polynomial [Far80], Hermite and Laguerre polynomials [GG81, Las04], rook polynomial [Far88], independent set polynomial, clique polynomial, vertex-cover polynomial.
- **Complexity:** On arbitrary graphs, or even planar graphs, computing the matching polynomial is #P-complete, [Jer87]. However, it can be computed more efficiently when additional structure about the graph is known. In particular, computing the matching polynomial on *n*-vertex graphs of tree-width k is fixed-parameter tractable: there exists an algorithm whose running time, for any fixed constant k, is a polynomial in n with an exponent that does not depend on k, [CMR01]. The matching polynomial of a graph with n vertices and clique-width k may be computed in time $n^{O(k)}$, [MRAG06]. For the case of Gallai-Edmonds decomposition, see [Gee00]. For the case of (edge-colouring) bipartite graphs, see [Sch03].
- **Difficult point property:** Proved in I. Averbouch's thesis.

Definability: *MSOL*₂-definable.

- **Roots:** It is well known that the roots of the matching polynomial are real, [GG81]. It is also well known that the multiplicity of a root changes by at most one upon deleting a vertex from G. In the context of [HL72], the polynomial has its zeros on the imaginary axis when the dimer activities are non-negative. The following was proven in [God93]:
 - (i) The maximal multiplicity of a root of the matching polynomial $\mu(G, x)$ is at most the minimal number of vertex disjoint paths needed to cover the vertex set of G.
 - (ii) If G is a Hamiltonial path, then all roots of its matching polynomial are simple.

The necessary and sufficient condition for the maximum multiplicity of a root of the matching polynomial of a tree to be equal to the minimum number of vertex disjoint paths needed to cover it was proven in [KW09]. The detailed investigation of the spectra of matching polynomials of graphs relevant to chemical physisc may be found in [Bal93b].

References: Generalized in different ways in [GH83, ZX85, AEMR05, AM06]. The matchingequivalence of graphs investigated in [Pra99].

2.1.11 The independent set polynomial

Authors: Introduced by I. Gutman and F. Harary in [GH83].

Motivation: Generalizations of the matching polynomial.

Static definition: Let in(G, r) denote the number of independent sets of size r, which are induced subgraphs of G.

$$In(G, x) = \sum_{r=0}^{|V|} in(G, r) \cdot x^r$$

The following table summarizes closed forms for the independence polynomials of some common classes of graphs. For more graphs, see [Wei].

${\rm Graph}\;G$	In(G,x)
Complete K_n	1 - nx
Complete bipartite graph $K_{n,n}$	$2(x+1)^n - 1$
Star graph S_n	$x + (1+x)^{n-1}$
Wheel graph W_n	$\frac{-(1-t)^n - (1-t)^n t + (t-1)(t+1)^n + 2^{n-1}x^2}{2^{n+2}x}, \ t = \sqrt{1+4x}$

Recurrence relations: In [HL94] and [LM05] proven:

Let G = (E, V) be a graph and $U \subset V$ such that G(U) is a complete sungraph of G. Then the following holds:

$$In(G, x) = In(G - w; x) + x \cdot In(G - N[w]; x);$$
$$In(G, x) = In(G - e; x) - x^2 \cdot In(G - N[u] \cup N(v); x);$$
$$In(G, x) = In(G - U; x) - x \cdot \Sigma_{v \in U} In(G - N[v]; x);$$

The following table summarizes the recurrence relation of some common classes of graphs. For more graphs, see [Wei].

${\rm Graph}\;G$	Recurrence
Complete bipartite graph $K_{n,n}$	$In_n(x) = (x+2)In_{n-1}(x) - (x+1)In_{n-2}(x)$
Star graph S_n	$In_n(x) = (x+2)In_{n-1}(x) - (x+1)In_{n-2}(x)$
Wheel graph W_n	$In_n(x) = 2In_{n-1}(x) + (x-1)In_{n-2}(x) - xIn_{n-3}(x)$
Cyrcle graph C_n	$In_n(x) = In_{n-1}(x) - xIn_{n-2}(x)$
Path graph C_n	$In_n(x) = In_{n-1}(x) - xIn_{n-2}(x)$

Further results see in [WZ11].

Reducibility: In [Aro84], shown that

$$In(P_n; x) = F_{n+1}(x)$$
 and $In(C_n; x) = F_{n-1} + 2xF_{n-2}(x)$,

where $F_n(x), n \leq 0$, are so-called *Fibonacci polynomials*, i.e., the polynomials defined recurcively by

$$F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x).$$

Related polynomials: Matching polynomial, vertex-cover polynomial.

- **Complexity:** The independent set polynomials are almost everywhere #P-hard to evaluate, it is even hard to approximate at every point except at 1 and 0, see [BH08].
- Difficult point property: See I. Averbouch's thesis.

Definability: $MSOL_1$ -definable.

Roots: A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. In [CS07] proven that the roots of independence polynomial of a claw-free graph are real. Moreover, the average independence polynomial has all real, simple roots. On the other hand, with probability tending to 1, the independence polynomial of a graph has a nonreal root. See [BN01] for these results. More results are summarized in [LM05, LM08]. Development of the results see in [WZ11].

References: Surveys are [HL94, LM05]. See also [GP06].

2.1.12 The clique polynomial

Authors: Introduced by I. Gutman and F. Harary in [GH83] and [Far89].

- Motivation: Generalizations of the matching polynomial.
- Static definition: Let cl(G, r) denote the number of cliques of size r, which are induced subgraphs of G.

$$Cl(G, x) = \sum_{r=0}^{|V|} cl(G, r) \cdot x^r$$

Recurrence relations: The recurrence relation for the weighted version of the clique graph was obtained in [Far89]. In the original notation, let G' be the graph obtained from G by deleting of an edge e, G" be the graph obtained from G by removing the nodes at the ends of e and G^* be the graph obtained from G by incorporating e. Then

$$Cl(G) = Cl(G') + Cl(G") + Cl(G")$$

- **Related polynomials:** Clique polynomials are related to \mathcal{H} -polinomials of [BIK05] as well as to trace monoids, see [GS98]. For the relation to the Hibert series of an algebra whose generators correspond to vertices and whose ideal of relations has generators that are graded commutators corresponding to edges, see [BG10].
- **Roots:** In [GS98], it was shown that clique polynomials can not have a negative real root, moreover, every clique polynomial of dergee 2 has real roots. In [GS00], it was shown that clique polynomials have a unique root of smallest modulus.

2.1.13 The vertex-cover polynomial

Authors: Introduced by F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little in [DHTL02].

Motivation: Biological systematics. Chemistry.

Static definition: Let cv(G, r) denote the number of vertex-covers of G of size r.

$$\Psi(G, x) = \sum_{r=0}^{|V|} cv(G, r)x^r$$

Recursive definition: Denote by $N_G(v)$ the neighborhood of v, i.e., $N_G(v) = \{u \in V(G) : u \neq v, (u, v) \in E\}$. If G has no loops and $|V(G)| \ge 2$ we have

$$\Psi(G(x) = x \cdot \Psi(G - v, x) + x^d \cdot \Psi(G - v - N_G(v), x)$$

Furthermore, Ψ is multiplicative.

Algorithms: An algorithm to evaluate Ψ is given in [Cas07].

Complexity: In 1972, Karp [Kar72] introduced a list of twenty-one NP-complete problems, one of which was the problem of finding a minimum vertex cover in a graph. Some progress made in [Dha]. The corresponding counting problem is $\sharp P$ -complete [PB83].

Difficult point property: See the dichotomy theorem of [BK09].

Definability: $MSOL_1$ -definable.

Related polynomials: Independent set polynomial, clique polynomial, matching polynomial.

Let w(G, r) denote the number of independet sets of size r. For n = |V(G)| it is known that w(G, n - k) = cv(G, k).

References: [AGM10].

2.1.14 The edge-cover polynomial

Authors:

Motivation:

Static definition: To be checked.

Recurrence relations: To be checked.

Recursive definition: To be checked.

Reducibility: To be checked

Related polynomials: To be checked

Complexity: To be checked

Difficult point property:

Definability: To be checked.

Roots: For every graph G with no isolated vertex, all the roots of E(G, x) are in the ball

$$\{z \in \mathbf{C} | |z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}\}$$

If every block of the graph G is K_2 or cycle, then all real roots of E(G, x) are in the interval (-4, 0]. For every tree T of order n, we have

$$\xi_{\mathbf{R}}(K_{1,n-1}) \le \xi_{\mathbf{R}}(T) \le \xi_{\mathbf{R}}(P_n),$$

where $-\xi_{\mathbf{R}}(T)$ is the smallest real root of E(T, x), and $P_n, K_{1,n-1}$ are the path and the star of order *n*, respectively, [CO10].

References: To be checked.

2.1.15 The Martin polynomial

Authors: Introduced by P. Martin in his thesis [Mar77]. Further developed by M. Las Vergnas [Ver83].

Motivation: Information about Eulerian paths.

Static definition: In a graph, a circuit is a sequence $v_1e_1v_2e_2\cdots v_{k1}e_{k1}v_ke_k$, where the v_i are vertices and the e_i are distinct edges or loops such that the endvertices of e_i are v_i and v_{i+1} , with $v_{k+1} \equiv v_1$. In a digraph, the circuit are oriented: e_i must be oriented from v_i to v_{i+1} . A digraph **G** is said to be Eulerian if for every vertex v, the outdegree is the same as the indegree. Also, a graph **G** is Eulerian if every vertex has even degree counting, as usual, two for every loop at the vertex. Thus a graph or digraph is Eulerian if and only if it is the edge-disjoint union of some circuits (or cycles) and isolated vertices. Given an Eulerian digraph **G** and an integer $k \geq 0$, we write $r_k(\mathbf{G})$ for the number of partitions of **G** into k edge-disjoint circuits and isolated vertices. The *circuit partition polynomial* of **G** is

$$r_{\mathbf{G}}(x) = r(\mathbf{G}; x) = \Sigma_k r_k(\mathbf{G}) x^k.$$

The circuit partition polynomial rG(x) of an Eulerian graph G is defined similarly

$$r_G(x) = r(G; x) = \Sigma_k r_k(G) x^k.$$

The Martin polynomial of a graph G is

$$m_G(x) = m(G; x) = \frac{1}{x - 1} r_G(x1),$$

and that of a digraph ${\bf G}$ is

$$m_{\mathbf{G}}(x) = m(\mathbf{G}; x) = \frac{1}{x - 2} r_{\mathbf{G}}(x^2).$$

Reducibility: The circuit partition polynomials are simple transformations of the Martin polynomials, [EM04].

Related polynomials: Tutte polynomial, [EM04].

- **Complexity:** Let G be a directed graph, and let k be a positive integer. In [MR10], q(G;k) is defined as follows. At each vertex v, we place a k-dimensional complex vector x_v . We take the product, over all edges (u, v), of the inner product $\langle x_u, x_v \rangle$. Finally, q(G;k) is the expectation of this product, where the x_v are chosen uniformly and independently from all vectors of norm 1 (or, alternately, from the Gaussian distribution). [MR10] shows that q(G;k) is proportional to G's circuit partition polynomial, and therefore that it is $\sharp P$ -complete for any k > 1. There exists a polynomial-time algorithms for link-like graphs tree width [MM03].
- **Roots:** If maximal $deg(G) > 2^k$ then $2 2^k$ is a root of m(G; x), [EM98].
- **References:** J. A. Ellis-Monaghan in [EM98] used Hopf algebra techniques to prove some combinatorial interpretations of the Martin polynomial for unoriented graphs. In [Bol02], B. Bollobás gives proofs of similar interpretations for a considerably wider class of values.

2.1.16 Interlace polynomial

Authors: Introduced by R. Arratia and B. Bolloás and G.B. Sorkin in [ABS00a, ABS00b].

- Motivation: Originally motivated by a problem relating to DNA sequencing by hybridization. From the mathematical point of view: by circle graphs, and the enumeration of Euler circuits.
- **Recursive definition:** Given an Euler circuit C of a 2-in, 2-out directed graph D, two vertices a and b of D are interlaced if C visits them in the sequence $\ldots, a, \ldots, b, \ldots, a, \ldots, b, \ldots$. The interlace graph H = H(C) corresponding to C has the same vertex set as D, with an edge ab in H if a and b are interlaced in C.

Let G be any undirected graph, and ab a pair of distinct vertices in G. Construct the pivot G^{ab} as follows. Partition the vertices other than a and b into four classes:

- (i) vertices adjacent to a alone;
- (ii) vertices adjacent to b alone;
- (iii) vertices adjacent to both a and b;
- (iv) vertices adjacent to neither a nor b.

Begin by setting $G^{ab} = G$. For any vertex pair xy where x is in one of the classes (i-iii) and y is in a different class (i-iii), toggle the pair xy: if it was an edge, make it a non-edge, and if it was a non-edge, make it an edge.

There exists a function q_1 , from the set of interlace graphs to the integers, such that for any 2-in, 2-out digraph D with Euler circuit C, the number of Euler circuits of D is equal to $q_1(H(C))$. Moreover, q_1 is uniquely defined by the following recursion:

$$q_1(H) = \begin{cases} q_1(H-a) + q_1(H^{ab} - b) & \text{if } ab \text{ is an edge of H} \\ 1 & \text{if } H \text{ has no edges} \end{cases}$$

The $q_1(H)$ can be generalized to a one-variable graph polynomial q(G; x), which is defined for all graphs G, not just interlace graphs, and which for interlace graphs G satisfies $q(G; 1) = q_1(G)$. The q(G; x) is called the interlace polynomial of G.

There is a unique map $q: \mathcal{G} \to \mathbf{Z}[x], G \to q(G)$, such that the following two conditions hold:

(i) If G contains an edge ab then

$$q(G) = q(G-a) + q(G^{ab} - b)$$

(ii) On E_n , the edgeless graph with *n* vertices, $q(E_n) = x^n$.

The interlace polynomials of edgeless graphs, complete graphs, stars, and complete bipartite graphs are given by

- (i) $q(E_n) = x^n$ for $n \ge 0$
- (ii) $q(K_n) = 2^{n-1}x$ for $n \ge 1$
- (iii) $q(K_{1n}) = 2x + x^2 + x^3 + \dots + x^n$ for $n \ge 2$

(iv) $q(K_{mn}) = (1 + \dots + x^{m-1})(1 + \dots + x^{n-1}) + x^m + x^n - 1$ for $m, n \ge 1$

For $n \ge 2$, the interlace polynomial of the path P_n with n+1 vertices and n edges satisfies

$$q(P_n) = q(P_{n-1}) + xq(P_{n-2})$$

Static definition: For $n \ge 2$, the interlace polynomial of the path P_n with n + 1 vertices and n edges satisfies:

$$q(P_n) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n-r}{r} + \binom{n-r-1}{r} \right\} x^{r+1}$$

For the path P_n , $q(P_n)(1) = F_{n+2}$, the (n+2)'nd Fibonacci number (with $F_0 = 0, F_1 = 1$. The interlace polynomials of the cycles C_n are $q(C_3) = 4x$, for $n \ge 4$ and with $y = \sqrt{1+4x}$,

$$q(C_n)(x) = \left(\frac{1-y}{2}\right)^n + \left(\frac{1+y}{2}\right)^n + \left(\frac{y^4 - 10y^2 - 7}{16}\right) \text{ for } n \text{ even}$$
$$q(C_n)(x) = \left(\frac{1-y}{2}\right)^n + \left(\frac{1+y}{2}\right)^n + \left(\frac{y^2 - 7}{4}\right) \text{ for } n \text{ odd}$$

For the static definition on trees and forests, see [ACRT10].

- **Related polynomials:** The interlace graph polynomial may be viewed as a special case of the Martin polynomial of an isotropic system, which underlies its connections with the circuit partition polynomial and the Kauffman brackets of a link diagram. The graph polynomial also has a two-variable generalization that is unknown for the Martin polynomial.
- **Complexity:** \sharp **P**-hard almost everywhere, [BH08]. In [Cou08], an algorithm to evaluate the multivariate interlace polynomial of a graph with *n* vertices given a tree decomposition of the graph of width *k* with running time f(k)n given, where f(k) is doubly exponential in *k*. Analyzing the GF(2)-rank of adjacency matrices in the context of tree decompositions, [BH10] give a faster algorithm that uses $2^{3k^2+O(k)}n$ arithmetic operations and can be efficiently implemented in parallel.

Difficult point property: Proved in [BH07].

Definability: Order independent *CMSOL*₂-definable show by Courcelle.

References: define two-variable interlace polynomials. Further developed in [ABS04a, ABS04b, AvdH04]. In [Cou08], the multivariate interlace polynomial defined and investigated. In [RP05], spectral interpretations of the interlace polynomial, including the characterisation of the values of the power spectra for quadratics, concidered in application to cryptography. The weighted version introduced in [Tra10].

2.1.17 The cover polynomial

Authors: Introduced by F.R.K. Chung and R.L. Graham in [CG95].

Motivation: To find a Tutte-like polynomial for directed graphs.

Static definition: To be checked.

Recurrence relations: To be checked.

Recursive definition: To be checked.

Reducibility: To be checked

Related polynomials: To be checked

Complexity: To be checked

Difficult point property: Proved in [BD07].

Definability: To be checked.

Roots: To be checked.

References: To be checked.

2.1.18 Go polynomial

Authors: Introduced by G.E. Farr in [Far03a].

- **Motivation:** Some analogy to the notion of "captured territory" in the game of Go. The questionto unswer is: What is the probability that the resulting partial uniform random assignment has the following property: for every colour class, each component of the induced subgraph has a vertex that is adjacent to an uncoloured vertex?
- Static definition: Let $(r, p_1, \ldots, p_{\lambda})$ be a probability distribution. For each $v \in V(G)$, assign it colour *i* with probability p_i (and no colour with probability $r = 1 \sum_{i=1}^{\lambda}$), with the choice of the vertices being independent and identically deistributed. The colour class C_i and the uncolored set U are now set-valued random variables. Let f be the partial λ -assignment generated randomly in this way. Let's consider Pr(f is free in G). If $\lambda = 2$ and $p_1 = p_2 \leq \frac{1}{2}$ then this probability is a polynomial in p. The polynomial is the *Go polynomial* of G. The number of free partial λ -assignments in G (for general λ) is also a polynomial in λ and denoted by $Go^{\sharp}(G; \lambda)$. If $r = p_1 = \cdots = p_{\lambda} = \frac{1}{(\lambda+1)}$ and n = |V(G)| then

 $Go^{\sharp}(G;\lambda)(\lambda+1)^n = Pr(f \text{ is free in } G)$

Recurrence relations: To be checked.

Recursive definition: Pr(f is free in G) satisfies recursive relation for *intermediate graphs* $\mathcal{G}(\mathcal{L})$, i.e. graph with exactly one label from $\{J, L, A, a\}$ on each edge and zero, one or two labels from $\{C, S\}$. The intuition is as follows: *joining* vertices of the same colour together, we help to form a group, and being a *lifeline* by making a coloured vertex adjancent to an uncoloured one, we help to preserve the former's group. If an adge is labelled A then we forbid its endpoints from receiving the same colour, but allow them to receive two different colours. In addition, one or both endpoints may remain uncoloured. Label a is similar, exept that at most one of the edges's endpoints can be uncoloured. Vertex label C means that a vertex so labelled cannot be left uncoloured, and a vertex labelled S guarantees the freedom of any group of which it is a part, regardless of whether the group has a lifeline to an uncoulored vertex. Let Z be a label. If $e \in E(G)$ then $G[e \leftarrow Z]$ denotes the intermediate graph obtained by replacing the label of e by Z. If $v \in V(G)$ then $N_G^{(Z)}(v)$ denotes the set of those neighbours w of v such that vw has label Z.

If $G \in \mathcal{G}(J, L)$, f is a random partial λ -assignment of G and $e_J \in E_J(G)$ and $e_L \in E_L(G)$ are parallel edges, then

 $Pr(f \text{ is free in } G) + Pr(f \text{ is free in } G - e_J - e_L) =$

 $Pr(f \text{ is free in } G - e_J) + Pr(f \text{ is free in } G - e_L)$

It leads to: If G is a stage 1 graph and $e_J \in E_J(G)$ and $e_L \in E_L(G)$ are parallel edges, then

$$Go(G; p) + Go(G - e_J - e_L; p) = Go(G - e_J; p) + Go(G - e_L; p)$$
$$Go^{\sharp}(G; \lambda) + Go^{\sharp}(G - e_J - e_L; \lambda) = Go^{\sharp}(G - e_J; \lambda) + Go^{\sharp}(G - e_L; \lambda)$$

Let's $G \in \mathcal{G}(J, L, C, A)$ and $e = uv \in E_J(G)$. Let f be a random partial λ -assignment whose distribution satisfies $p_1 = \cdots = p_{\lambda} = p \leq \frac{1}{\lambda}$. Then

$$Pr(f \text{ is free in } G) = p \cdot Pr(f \text{ is free in } (G/e)[u \leftarrow C]) + Pr(f \text{ is free in } G[e \leftarrow A])$$

It leads to: If $G \in \mathcal{G}(J, L, C, A)$ and $e = uv \in E_J(G)$ then

$$Go(G;p) = p \cdot Go((G/e)[u \leftarrow C];p) + Go(G[e \leftarrow A];p)$$
$$Go^{\sharp}(G;\lambda) = Go^{\sharp}((G/e)[u \leftarrow C];\lambda) + Go(G^{\sharp}[e \leftarrow A];\lambda)$$

If $G \in \mathcal{G}(J, L, a)$ and $e \in e_J(G)$ then

$$Go^{\sharp}(G;\lambda) = Go^{\sharp}((G/e);\lambda) + Go(G^{\sharp}[e \leftarrow a];\lambda)$$

If $G \in \mathcal{G}(L, A, a, C, S)$ and $v \in V \setminus V_C$ then

 $Pr(f \text{ is free in } G) = r \cdot Pr(f \text{ is free in } (G-v)[N_G^{(L)}(v) \leftarrow S]) + Pr(f \text{ is free in } G[v \leftarrow C])$ If $G \in \mathcal{G}(L, A, a, C, S)$ and $v \in V_C \cap V_S$ then

$$Pr(f \text{ is free in } G) = Pr(f \text{ is free in } G - E_L(\{v\}, N_G^{(L)}(v))))$$

The above leads to: If $G \in \mathcal{G}(L, A, a, C, S)$ and

• $v \in V \setminus V_C$ then

$$Go(G;p) = (1 - 2 \cdot p)Go((G - v)[N_G^{(L)}(v) \leftarrow S];p) + Go(G[v \leftarrow C];p)$$
$$Go^{\sharp}(G;\lambda) = Go^{\sharp}((G - v)[N_G^{(L)}(v) \leftarrow S];\lambda) + Go^{\sharp}(G[v \leftarrow C];\lambda)$$

• $v \in V_C \cap V_S$ then

$$Go(G; p) = Go(G - E_L(\{v\}, N_G^{(L)}(v)); p)$$
$$Go^{\sharp}(G; \lambda) = Go^{\sharp}(G - E_L(\{v\}, N_G^{(L)}(v)); \lambda)$$

Reducibility and Related polynomials: If we have an intermediate $\{A, C, S\}$ -graph with all vertices labelled both C and S then

$$Go(G; p) = p^n \cdot P(G; 2)$$
 and $Go^{\sharp}(G; \lambda) = P(G, \lambda)$

Complexity: #P-hard to compute in general.

Definability: Not found.

Combinatorial interpretations: Not found.

Roots: Not found.

References: In [TF05], the exact values for the number of legal positions on square boards of size upto 16×16 were obtained, as well as the limiting behaviour of the number of legal positions on large rectangular boards of fixed width upto seven was deternined. They also find estimates for the number of legal *games* of Go. In [FS07], bounds for the number of legal Go positions for various numbers of players on some planar lattice graphs investigated.

2.1.19 Stability polynomial

Authors: Introduced by Farr, [Far93].

Static definition: $A_{G(p)}$ is given by

$$A_{G(p)} = \sum_{U \in S(G)} p^{|U|} (1-p)^{|V(G)U|}$$

S(G) denotes the set of all stable sets of G, i.e. all sets in a graph G, for which G has no edge with both endpoints in S.

Recursive definition: To be completeed

Reducibility: Related to the Universal polynomial, cf. [MN09]. If G is loopless then $A_{G(p)}$ is given by

$$A_{G(p)} = U_G(x_i = 1; x_j = -(-p)^j for \ j \ge 2, y = 0)$$

References: See [Nob09] and [MN09]. Results of [Far93] are extended in [McD10].

2.1.20 Strong U-polynomial

Authors: Introduced by C. Merino and S. Noble, [MN09].

Motivation: To extend universal polynomial by countably many commuting variables $z_{i,j}$.

Static definition:

$$\bar{U}_G(z_{i,j}) = \sum_{A \subseteq E} z_{c_1, e_1 - c_1 + 1} z_{c_2, e_2 - c_2 + 1} \cdots z_{c_{k(G|A)}, e_{k(G|A)} - c_{k(G|A)} + 1}$$

where c_i and e_i are, respectively, the number of vertices and edges in the *i*th connected compotent of G|A.

Recursive definition: There is a recurcive relation fro the strong U-polynomial, [MN09]. Let (G, ω) desbribe graph G with a strictly positive integer weight $\omega(v)$ attached at each vertex v. Then

$$\bar{W}_{(G,w)}(\mathbf{z}) = \sum_{A \subseteq E} z_{w_1, e_1 - c_1 + 1} z_{w_2, e_2 - c_2 + 1} \cdots z_{w_{k(G|A)}, e_{k(G|A)} - c_{k(G|A)} + 1}$$

If $\omega(v) = 1$ for all v, then $\overline{U}_G(\mathbf{z}) = \overline{W}_{(G,w)}(\mathbf{z})$. On the other hand, [MN09]:

$$W_{(G,w)} = W_{(G,w)-e} + W_{(G,w)e}$$

Reducibility: Related to the strong Tutte symmetric function, cf. [MN09]. The strong *U*-polynomial and the strong Tutte symmetric function are aquivalent:

$$Y_G(\mathbf{x}, \mathbf{t}) = U_G(z_{i,j} = \bar{p}_{i,i+j-1}(\mathbf{x}, \mathbf{t})),$$

 $\bar{p}_{r,s}(\mathbf{x}, \mathbf{t}) = \sum_{i=1}^{\infty} x_i^r t_i^s.$

2.1.21 Two polymatroid rank generating function

Authors: Introduced by Oxley and Whittle, [OW93].

Motivation: To develop a theory of Tutte invariants for 2-polymatroids.

Static definition: Let f(A) denote the number of vertices of G that are endpoint of an adge in A.

$$S_G(u, v) = \sum_{A \in E(G)} u^{|V(G)| - f(A)} v^{2|A| - f(A)}$$

Reducibility: Related to the Universal polynomial, cf. [MN09]. If G is loopless graph with no isolated vertices, then

$$S_G(u,v) = U_G(x_1 = u; v_2 = 1; x_j = v^{j-2} for \ j \ge 2, y = v^2 + 1)$$

Recursive definition: Via the Universal polynomial.

References: See [Far03b], [LM03], [Kun06], [Nob06], [MN09].

2.1.22 Rook polynomial

Authors: Introduced by Stanley, [Sta95].

Motivation: To be checked.

Static definition:

Recursive definition: To be completeed

Reducibility:

Complexity: To be completed

Difficult point property: To be completed

Definability: To be completed

Roots:

References:

2.1.23 Acyclic polynomial

Authors: Introduced by Stanley, [Sta95].

Motivation: To be checked.

Static definition:

Recursive definition: To be completeed

Reducibility:

Complexity: To be completed

Difficult point property: To be completed

Definability: To be completed

Roots:

References:

2.1.24 Pott's polynomial

Authors: Introduced by Stanley, [Sta95].

Motivation: To be checked.

Static definition:

Recursive definition: To be completeed

Reducibility:

Complexity: To be completed

Difficult point property: To be completed

Definability: To be completed

Roots:

References:

2.2 New graph polynomials

2.2.1 mcc-polynomial

Author Introduced in [MZ08] based on the corresponding graph property discussed in [LMST07].

Definition $mcc_t(G, k)$ is the number of colorings of G with k colors such that all the connected components of a monochromatic set have size at most t.

For fixed $t \in \mathbb{N}$ this is a polynomial by the Makowsky-Zilber argument. [MZ06, MZ08]. For variable t the function mcc(G, k, t) is not a polynomial, as it vanishes for large enough t.

Motivation: To be checked.

Static definition: To be checked.

Reducibility $\chi(G,k) = mcc_1(G,k)$

Recurrence relations To be checked.

Recursive definition: To be checked.

Complexity To be checked.

Difficult point property: To be checked.

Definability To be checked.

Combinatorial interpretations: To be checked.

Related polynomials: Chromatic polynomial.

Roots: To be checked.

References To be checked.

2.2.2 hc-polynomial

Author Introduced in [MZ08] based on the corresponding graph property discussed in [HK83].

Definition hc(G, k) is the number of harmonious proper colorings of G with at most k colors. This is a polynomial by the Makowsky-Zilber argument, [MZ06, MZ08].

hc(G) is the smallest k such that $hc(G, k) \neq 0$.

Recurrence relations To be added.

Complexity hc(G) is NP-hard already for trees, [EM95].

Definability If $NP \neq P$, it is not *MSOL*-definable in any expansion of *G*. Otherwise it would be polynomial time computable on graphs of tree-width at most *k*, in particular on trees, [Mak04].

References For hc(G) only a survey is given in [Edw97].

2.2.3 The cc-polynomial

Author JAM

Definition cc(G, k) is the number of colorings of G with k colors such that each monochromatic set is connected.

Reducibility

Recurrence relations

Complexity Conjecture: $\sharp P$ -hard for k = 2.

The conjecture implies the difficult point property.

Definability

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