Catalogue of graph polynomials

J.A. Makowsky

April 6, 2011

Contents

Chapter 1

Graph polynomials

A graph is a relational structure $G =< V, E; R >$, where V is a set of vertices, E is a set of edges, and R is the a ternary relation $R \subseteq V \times E \times V$, where $R(u, e, v)$ holds if and only the edge has u as its source and v its target. An edge labeling of a graph is a function $c_E : E \rightarrow \mathbb{N}$, A vertex labeling of a graph is a function $c_V : V \rightarrow \mathbb{N}$. A labeled graph is a tuple of the form $< V, E, c_V, c_E >$. c_V or c_E could be the empty map. Let $\mathcal{G}$ denote the class of finite graphs, $\mathcal{G}_{lab}$ denote the class of finite labeled graphs.

1.1 Comparing graph polynomials

We want to compare the combinatorial expressive power of graph polynomials. Several approaches have been used explicitly or implicitly in the literature. We shall state our definitions for PNGIs, rather than rstricting ourselves to graph polynomials, whenever it makes sense.

1.1.1 Distinctive power

Let f, g be two NPGIs for a class of labeled graphs $\mathcal{G}$.

Definition 1. *f is* less distinctive *than g, and we write $f <_{dist} g$, if whenever $G_1, G_2 \in \mathcal{G}$ and $g(G_1) = g(G_2)$ then also $f(G_1) = f(G_2)$.*

1.1.2 Substitution instances

1.1.3 Uniform algebraic reductions

1.1.4 Substitution instances

1.1.5 Substitution instances

1.1.6 Substitution instances

1.2 Definability of graph polynomials

1.2.1 Static definitions

1.2.2 Dynamic definitions

1.2.3 SOL-definable polynomials

1.2.4 Generalized chromatic polynomials

Chapter 2

A catalogue of graph polynomials

2.1 Polynomials from the zoo

2.1.1 Chromatic polynomial

Author: Introduced by G. Birkhoff in 1912, [Bir12].

Motivation: To prove the four-color conjecture.

Definition: $\chi(G, k)$ is the number of proper k-vertex colorings of G. It is a polynomial by Birkhoff's argument via the recurrence relations. It is also a polynomial by the Makowsky-Zilber argument.

Reducibility Instance of the Tutte polynomial for connected graphs.

$$\chi(G, k) = (-1)^{|V(G)|-k(G)} X^{k(G)} \cdot T(G, 1-k, 0)$$

Recurrence relations Deletion-Contraction rules.

Complexity $\sharp$**P**-hard almost everywhere. On the other hand, there exists an $O(n \cdot log^2 n)$ algorithm to compute all coefficients of the chromatic polynomial of an n vertex graph of bounded tree-width, [Für10].

Difficult point property: Proven in [Lin86].

Definability: $MSOL_2$ over ordered graphs, order invariant (via the Tutte polynomial).

Combinatorial interpretations: For $k = -1$ it counts the number of acyclic orientations, [Sta73].

Roots: In [BT69] first found zeros of the chromatic polynomials. In [Bro98], it was proved that the chromatic polynomial of a connected graph with n vertices and m edges has a root with modulus at least $(m-1)/(n-2)$; this bound is best possible for trees and 2-trees (only). It as also proved that the chromatic polynomial of a graph with few triangles that is not a forest has a nonreal root and that there is a graph with n vertices whose chromatic polynomial has a root with imaginary part greater than $\sqrt{(\frac{n}{4})}$. Moreover, the real closure of its roots is $\{0\} \bigcup \{1\} \bigcup [\frac{32}{27}, \infty)$, and the complex closure of its roots is $\mathcal{C}$, [Hos07]. In [Roy07], an infite family of 3-connected non-bipartite graphs with chromatic roots in the interval $(1;2)$ was inverstigated. See also [BKW80].

References A survey monograph is [DKT05]. See also [MW82], [AT97], [NW99], [DPT03], [Sar00] .

2.1.2 Chromatic symmetric function

Authors: Introduced by Stanley, [Sta95].

Motivation: Certain conjectures regarding immanants of Jacobi-Trudi matrices

Static definition: $Y_G(\mathbf{x}, t)$ is given by

$$Y_G(\mathbf{x}, t) = \sum_{\chi} x_{\chi(v_1)} x_{\chi(v_2)} \cdots x_{\chi(v_n)} (1+t)^{b(\chi)},$$

the sum ranges over all proper colorings χ, $b(\chi)$ is the number of monochromatic edges.

Reducibility: Related to the Universal polynomial, cf. [MN09]. For any G, $Y_G(\mathbf{x}, t)$ is given by

$$Y_G(\mathbf{x}, t) = t^{|V|} U_G(x_j = p_j(\mathbf{x})/t, y = t+1),$$

$p_j(\mathbf{x}) = \sum_{i=1}^{\infty} x_i^j$.

References: [Sta98], [MN09].

2.1.3 Adjoint polynomials

Authors: Introduced by Liu, [Liu87].

Motivation: To relate the chromatic uniqueness of a graph to the structure of its complement and employ the classical theory of polynomial algebra. A graph G is chromatically unique if $P(H, \lambda) = P(G, \lambda)$ imlies that H is isomorphic to G.

Static definition: If G_0 is a spanning subgraph of G and every component of G_0 is complete, then G_0 is called an ideal subgraph of G. For a given graph G, let $N(G, k)$ be the number of ideal sungraphs with k components. The polynomial

$$h(G, x) = \sum_{i=1}^{p} N(G, i) x^i$$

is called the adjoint polynomial of the graph G, where $p = |V(G)|$.

Recursive definition: If $uv \in E(G)$, and if uv does not belong to any triangle in G, then

$$h(G, x) = h(G - uv, x) + x \cdot h(G - u, v, x)$$

Reducibility: Related to chromatic polynomial [Liu97]. Let $\alpha(G, i)$ denote the number of i-independent partitions of G and $\sigma(G, x) = \sum_{i=1}^{p} \alpha(G, i) x^{i - \chi(G)}$. Then

$$h(G, x) = x^{\chi(\bar{G})} \cdot \sigma(\bar{G}, x)$$

Roots: In [ZL09], the nimimal roots $\beta(G)$, such that $-(2+\sqrt{5}) \leq \beta(G) \leq -4$ characterized for connected graphs G.

References: [Wa] and [Za04].

2.1.4 The Tutte polynomial

Authors: Introduced by W.T. Tutte [Tut54], building on H. Whitney [Whi32].

Motivation: To be checked.

Static definition: To be checked.

Recurrence relations: To be checked.

Recursive definition: To be checked.

Reducibility: To be checked

Related polynomials: Chromatic polynomial, flow polynomial, and more (to be added).

Complexity: $\sharp\mathbf{P}$-hard for most points.

Difficult point property: Proved in [JVW90].

Definability: To be checked.

Combinatorial interpretations: To be added.

Roots: To be checked.

References: Monographs dealing with the Tutte polynomial are [Bol99, GR01].

2.1.5 Strong Tutte symmetric function

Authors: Introduced by C. Merino and S. Noble, [MN09].

Motivation: To extend Stanley's Tutte symmetric function by replacing the t varible by countably infinitely many commuting varialbles.

Static definition: $\bar{Y}_G(\mathbf{x}, \mathbf{t})$ is given by

$$\bar{Y}_G(\mathbf{x}, \mathbf{t}) = \sum_{\chi} (\prod_{i=1}^{n} x_{\chi(v_i)})(\prod_{i=1}^{\infty} (1 + t_i)^{b_i(\chi)})$$

the sum ranges over all proper colorings χ, $b_i(\chi)$ is the number of monochromatic edges for wich both endpoints have colour i.

Reducibility: Related to the strong U-polynomial, cf. [MN09]. The strong U-polynomial and the strong Tutte symmetric function are aquivalent:

$$\bar{Y}_G(\mathbf{x}, \mathbf{t}) = \bar{U}_G(z_{i,j} = \bar{p}_{i,i+j-1}(\mathbf{x}, \mathbf{t})),$$

$\bar{p}_{r,s}(\mathbf{x}, \mathbf{t}) = \sum_{i=1}^{\infty} x_i^r t_i^s$.

References: See [Goo08].

2.1.6 Tutte-Grothendieck invariants

Authors: Introduced by A.Goodall, [Goo08].

Motivation: Investigate the Tutte polynomial of a grapf as a Hamming weight enimerator of its tensions (or flows). Consider a family polynomials containing the graph polynomials. The question is: whether other Tutte-Grothendieck invariants can be obtained in a similar way.

Definition of Tutte-Grothendieck invariants : A function F from (isomorphism classes of) graps to $C[\alpha, \beta, \gamma, x, y]$ is a Tutte-Grothendieck invariant if it satisfies, for each graph $G = (V, E)$ and any edge $e \in E$,

$$F(G) = \begin{cases} \gamma^{|V|} & E = \emptyset \\ xF(G/e) & e \text{ a bridge} \\ yF(G\backslash e) & e \text{ a loop} \\ \alpha F(G/e) + \beta F(G\backslash e) & e \text{ not a bridge or loop} \end{cases}$$

Reducibility: If F is a Tutte-Grothendieck invariant then

$$F(G) = \gamma^{k(G)}\alpha^{r(G)}\beta^{n(G)}T(G; \frac{x}{\alpha}, \frac{y}{\beta}).$$

Fact: Let Q be a set of size q and $\mathbf{w} = (w_{a,b})$ a tuple of complex numbers indexed by $(a, b) \subset Q \times q$. Assume that the edges u, v of $G = (V, E)$ have been given an arbitrary, fixed orientattion (u, v). Denote by $\vec{E}$ the resulting set of derected edges. Consider the partition function for a vertex Q-colouring model that assigns a weight wa, b to a directed edge (u, v) coloured (a, b):

$$F(G; \mathbf{w}) = \sum_{\mathbf{c} \in Q^V} \prod_{(u,v) \in \vec{E}} w_{c_u, c_v} = \sum_{\mathbf{c} \in Q^V} \prod_{(a,b) \in Q \times Q} w_{a,b}^{\sharp\{(u,v) \in \vec{E} : (c_u, c_v) = (a,b)\}}$$

The graphical invariant $F(G; \mathbf{w})$ is a Tutte-Grothendieck invariant if and only if there are constants y, w such that

$$w_{a,b} = \begin{cases} w & a \neq b \\ y & a = b \end{cases}$$

proven in [MN09].

2.1.7 A weighted graph polynomial

Authors: Introduced by S. Noble and D. Welsh in [NW99].

Motivation: To extend the treatment of the relationship between chord diagrams and Vassiliev invariants of knots [CDL94] to Tutte-Grothendieck invariants.

Static definition: The weighted graph polynomial $U(G, \bar{x}, y)$ for a graph $G = (V, E)$ is defined as

$$U(G, \bar{x}, y) = \sum_{A \subseteq E} \prod_{i=1}^{|V|} x_i^{s(i,A)} y^{|A| - r(A)}$$

where $s(i, A)$ denotes the number of connected components of size i in the spanning subgraph (V, A), and $r(A) = |V| - k(A)$ is the rank of (V, A).

Reducibility and Related polynomials: The weighted graph polynomial contains many other graph invariants as specialization, for instance the 2-polymatroid rank generating function of Exley and Whittle [OW93], and as a consequence the matching polynomial [HL72] and the symmetric function generalization of the chromatic polynomial [Sta95]. In [Sar00], it was shown that the coefficients of U and the polychromate determine one another. Related to the weighted integral gain graph of [FZ07].

Complexity: For graphs of tree-width atmost k, computable in polynomial time, with exponent depending on k, [Nob09]. Not known to be in FPT.

Definability: Not even SOL-definable. Proof sketched in [AGM07].

References: The dichromatic polynomial of [FZ07] on a graph whose edges are labelled invertibly from a group and vertices weighted from a semigroup, includes in particular the Noble and Welsh's for graphs with positive interger weights.

2.1.8 Chain polynomial

Authors: Introduced by R.C. Read and E.G. Whitehead Jr. [RJ99].

Motivation: Chromatic polynomials for homeomorphisms classes of graphs.

Static definition: $r(S)$ denotes the rank of the spanning subgraph induced by S.

$$Ch(G, \omega, \bar{q}) = \sum_{S \subseteq E} (1 - \omega)^{|S| - r(S)} \prod_{e \in E - S} q_e$$

Recursive definition:

- $Ch(E_1) = 1$;
- For G_1 and G_2 with at most one vertex in common we have

$$Ch(G_1 \cup G_2) = Ch(G_1) \cdot Ch(G_2)$$

- If $e \in E$ is a loop, we have

$$Ch(G) = (q_e - \omega) Ch(G - e)$$

- If $e \in E$ is not a loop, we have

$$Ch(G) = (q_e - 1) Ch(G - e) + Ch(G/e)$$

Reducibility: Related to the labeled Tutte polynomial. Suppose U is an edge-labeled multigraph and M is doubly weighted matroid on $E = E(G)$ obtrtained from the circuit matroid of G by letting the weights of an edge labeled a be $p(a) = 1$ and $q(a) = a - 1$. Then the chain polynomial $Ch(G)$ is obtained from the Tutte polynomial $t(M)$ by evaluating $y = 2 - w$ and $x = 2$, cf. [Tra02]. In [Ja10], a formula for computing the Tutte polynomial of the signed graph formed from a weighed graph by edge replacement in terms of the chain polynomial of the weighed graph presented.

Complexity: $\sharp\mathbf{P}$-hard for many instances.

Definability: Order invariant $MSOL_2$-definable with auxiliary order on edges.

Roots: If a poset is a semi-order (i.e. 1+3 free and 2+2 free) then the roots of its chain polynomial are real. It is conjectured that if a poset is a distributive lattice, then again, the roots are real. In [BR05], it was proven that the number c_i of chains of length i in the proper part of a distributed lattice L of lenght $l+2$ satisfy the inequalities:

$$c_0 \leq \ldots \leq c_{\lfloor l/2 \rfloor} \text{ and } c_{\lfloor 3l/2 \rfloor} \leq \ldots \leq c_l$$

References: [RJ99, RJ03, Tra02, BR05, Ja10, AGM08, AGM10].

2.1.9 Characteristic polynomial

Motivation: In linear algebra, one associates a polynomial to every square matrix: its characteristic polynomial. This polynomial encodes several important properties of the matrix, most notably its eigenvalues, its determinant and its trace. The characteristic polynomial of a graph is the characteristic polynomial of its adjacency matrix.

Static definition: We start with a field K (such as the real or complex numbers) and an $n \times n$ matrix A over K. The characteristic polynomial of A, denoted by $pA(t)$, is the polynomial defined by

$$pA(t) = det(t \cdot \mathbf{I} - A)$$

where $\mathbf{I}$ denotes the n-by-n identity matrix and the determinant is being taken in $K(t)$, the field of rational functions in t.

Recursive definition: Let's call the collection of the characteristic polynomials of the vertex-deleted subgraphs the polynomial *deck*. Tutte [Tut79] proved that the characteristic polynomial of a graph is reconstructible from its deck. More results may be found in particular in [CL98, Sch79, Sim92, Hag00b, Sci02]. In the molecular context, it was given in [Hei53] and developped in [RG96].

Reducibility and Related polynomials: On forest it equals the matching polynomial.

Complexity Polynomial time computable (Determinant).

Definability Substitution instance of MSOL-polynomial for $I(G)$.

Combinatorial interpretations: Coefficient is related to Expanders.

Roots: Roots of the characteristic polynomial are known as the spectrum (or eigenvalues) of the matrix (or of the graph); differing graphs with the same characteristic polynomial (and so the same roots) are known as *Isospectral Graphs*.

References Fan Rong K. Chung Spectral Graph Theory, CBMS Conference on Recent Advances in Spectral Graph Theory held at California State University at Fresno, June 6-10, 1994"–T.p. verso. Published 1997 American Mathematical Society ISBN 0821803158

Chemical Graphs: Looking Back and Glimpsing Ahead AT Balaban - Journal of Chemical Information and Computer Sciences, 1995

2.1.10 Matching polynomial

Authors: Introduced by C.J. Heilmann and E.H. Lieb in 1972 [HL72]. See alse [Far79].

Motivation: Physics and chemistry. See [Tri92, Bal93a, Bal95].

Static definition: A *matching* of a graph G is a set of paiwise disjoint edges of G. The *matching polynomial* $\mu(G, x)$ of G is given by

$$\mu(G, x) = \Sigma_{k \leq 0}(-1)^k \cdot p(G, k) \cdot x^{n-2k},$$

where $p(G, k)$ is the number of matchings with k edges in G. It is possible to recover the order, degree, girth and number of minimal cycles for a regular graph from its matching polynomial, [BF95].

Recurrence relations: In the chemeistry contex, for polygraph, given in [BGMP86] and developped in [BX93]. See also [ABBK03].

Recursive definition: See [AGM07].

Related polynomials: Characteristic polynomial [Gut80], the matching polynomial of a graph coincides with the chracteristic polynomial if and only if the graph is a forest [GG81], acyclic polynomial [Far80], Hermite and Laguerre polynomials [GG81, Las04], rook polynomial [Far88], independent set polynomial, clique polynomial, vertex-cover polynomial.

Complexity: On arbitrary graphs, or even planar graphs, computing the matching polynomial is #P-complete, [Jer87]. However, it can be computed more efficiently when additional structure about the graph is known. In particular, computing the matching polynomial on n-vertex graphs of tree-width k is fixed-parameter tractable: there exists an algorithm whose running time, for any fixed constant k, is a polynomial in n with an exponent that does not depend on k, [CMR01]. The matching polynomial of a graph with n vertices and clique-width k may be computed in time $n^{O(k)}$, [MRAG06]. For the case of Gallai-Edmonds decomposition, see [Gee00]. For the case of (edge-colouring) bipartite graphs, see [Sch03].

Difficult point property: Proved in I. Averbouch's thesis, [Ave11].

Definability: $MSOL_2$-definable.

Roots: It is well known that the roots of the matching polynomial are real, [GG81]. Moreover, the real and complex closure of its roots is $[0, \infty)$, [Hos07]. It is also well known that the multiplicity of a root changes by at most one upon deleting a vertex from G. In the context of [HL72], the polynomial has its zeros on the imaginary axis when the dimer activities are non-negative. The following was proven in [God93]:

(i) The maximal multiplicity of a root of the matching polinomial $\mu(G, x)$ is at most the minimal number of vertex disjoint paths needed to cover the vertex set of G.

(ii) If G is a Hamiltonial path, then all roots of its matching polynomial are simple.

The necessary and sufficient condition for the maximum multiplicity of a root of the matching polinomial of a tree to be equal to the minimum number of vertex disjoint paths needed to cover it was proven in [KW09]. The detailed investigation of the spectra of matching polynomials of graphs relevant to chemical physisc may be found in [Bal93b].

References: Generalized in different ways in [GH83, ZX85, AEMR05, AM06]. The matching-equivalence of graphs investigated in [Pra99].

2.1.11 The independent set polynomial

Authors: Introduced by I. Gutman and F. Harary in [GH83].

Motivation: Generalizations of the matching polynomial.

Static definition: Let $in(G,r)$ denote the number of independent sets of size r, which are induced subgraphs of G.

$$In(G,x) = \sum_{r=0}^{|V|} in(G,r) \cdot x^r$$

The following table summarizes closed forms for the independence polynomials of some common classes of graphs. For more graphs, see [Wei].

Graph G	$In(G,x)$
Complete K_n	$1 - nx$
Complete bipartite graph $K_{n,n}$	$2(x+1)^n - 1$
Star graph S_n	$x + (1+x)^{n-1}$
Wheel graph W_n	$\frac{-(1-t)^n-(1-t)^n t+(t-1)(t+1)^n+2^{n-1}x^2}{2^{n+2}x}$, $t = \sqrt{1+4x}$

Recurrence relations: In [HL94] and [LM05] proven:
Let $G = (E,V)$ be a graph and $U \subset V$ such that $G(U)$ is a complete sungraph of G. Then the following holds:

$$In(G,x) = In(G-w;x) + x \cdot In(G-N[w];x);$$

$$In(G,x) = In(G-e;x) - x^2 \cdot In(G-N[u] \cup N(v);x);$$

$$In(G,x) = In(G-U;x) - x \cdot \Sigma_{v \in U} In(G-N[v];x);$$

The following table summarizes the recurrence relation of some common classes of graphs. For more graphs, see [Wei].

Graph G	Recurrence
Complete bipartite graph $K_{n,n}$	$In_n(x) = (x+2)In_{n-1}(x) - (x+1)In_{n-2}(x)$
Star graph S_n	$In_n(x) = (x+2)In_{n-1}(x) - (x+1)In_{n-2}(x)$
Wheel graph W_n	$In_n(x) = 2In_{n-1}(x) + (x-1)In_{n-2}(x) - xIn_{n-3}(x)$
Cyrcle graph C_n	$In_n(x) = In_{n-1}(x) - xIn_{n-2}(x)$
Path graph C_n	$In_n(x) = In_{n-1}(x) - xIn_{n-2}(x)$

Further results see in [WZ11].

Reducibility: In [Aro84], shown that

$$In(P_n; x) = F_{n+1}(x) \text{ and } In(C_n; x) = F_{n-1} + 2xF_{n-2}(x),$$

where $F_n(x), n \leq 0$, are so-called *Fibonacci polynomials*, i.e., the polynomials defined recurcively by

$$F_0(x) = 1, F_1(x) = 1, F_n(x) = F_{n-1}(x) + xF_{n-2}(x).$$

Related polynomials: Matching polynomial, vertex-cover polynomial.

Complexity: The independent set polynomials are almost everywhere $\#P$-hard to evaluate, it is even hard to approximate at every point except at 1 and 0, see [BH08].

Difficult point property: See I. Averbouch's thesis, [Ave11].

Definability: $MSOL_1$-definable.

Roots: A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. In [CS07] proven that the roots of independence polynomial of a claw-free graph are real. Moreover, the real closure of its roots is $(-\infty, 0]$, and the complex closure of its roots is $\mathcal{C}$, [Hos07]. Moreover, the average independence polynomial has all real, simple roots. On the other hand, with probability tending to 1, the independence polynomial of a graph has a nonreal root. See [BN01] for these results. More results are summarized in [LM05, LM08]. Development of the results see in [WZ11].

References: Surveys are [HL94, LM05]. See also [GP06].

2.1.12 The clique polynomial

Authors: Introduced by I. Gutman and F. Harary in [GH83] and [Far89].

Motivation: Generalizations of the matching polynomial.

Static definition: Let $cl(G, r)$ denote the number of cliques of size r, which are induced subgraphs of G.

$$Cl(G, x) = \sum_{r=0}^{|V|} cl(G, r) \cdot x^r$$

Recurrence relations: The recurrence relation for the weighted version of the clique graph was obtained in [Far89]. In the original notation, let G' be the graph obtained from G by deleting of an edge e, G" be the graph obtained from G by removing the nodes at the ends of e and G^* be the graph obtained from G by incorporating e. Then

$$Cl(G) = Cl(G') + Cl(G") + Cl(G^*)$$

Related polynomials: Clique polynomials are related to $\mathcal{H}$-polinomials of [BIK05] as well as to trace monoids, see [GS98]. For the relation to the Hibert series of an algebra whose generators correspond to vertices and whose ideal of relations has generators that are graded commutators corresponding to edges, see [BG10].

Roots: In [GS98], it was shown that clique polynomials can not have a negative real root, moreover, every clique polynomial of dergee 2 has real roots. In [GS00], it was shown that clique polynomials have a unique root of smallest modulus.

2.1.13 The vertex-cover polynomial

Authors: Introduced by F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little in [DHTL02].

Motivation: Biological systematics. Chemistry.

Static definition: Let $cv(G, r)$ denote the number of vertex-covers of G of size r.

$$\Psi(G, x) = \sum_{r=0}^{|V|} cv(G, r)x^r$$

Recursive definition: Denote by $N_G(v)$ the neighborhood of v, i.e., $N_G(v) = \{u \in V(G) : u \neq v, (u, v) \in E\}$. If G has no loops and $|V(G)| \geq 2$ we have

$$\Psi(G(x) = x \cdot \Psi(G - v, x) + x^d \cdot \Psi(G - v - N_G(v), x)$$

Furthermore, Ψ is multiplicative.

Algorithms: An algorithm to evaluate Ψ is given in [Cas07].

Complexity: In 1972, Karp [Kar72] introduced a list of twenty-one NP-complete problems, one of which was the problem of finding a minimum vertex cover in a graph. Some progress made in [Dha06]. The corresponding counting problem is $\sharp P$-complete [PB83]. In [BK09], it was proven that vertex cover polynomials are VNP-complete.

Difficult point property: See the dichotomy theorem of [BK09].

Definability: $MSOL_1$-definable.

Related polynomials: Independent set polynomial, clique polynomial, matching polynomial.

Let $w(G, r)$ denote the number of independet sets of size r. For $n = |V(G)|$ it is known that $w(G, n-k) = cv(G, k)$.

References: [AGM10].

2.1.14 The edge-cover polynomial

Authors: The order and the size of a graph G denote the number of vertices and the number of edges of G, respectively. Let G be a simple graph of order n and size m. An edge covering of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. The concept of the edge cover polynomial was introduced by S. Akbari and M.R. Oboudi in [AO].

Motivation: To explore the notion of the vertex-cover polynomials.

Static definition: Let G be a graph without isolated vertex. A minimum edge covering is an edge covering of the smallest possible size. The edge covering number $\rho(G)$ is the size of a minimal edge covering of G. Let $\epsilon(G, i)$ denote the family of edge covering sets of G with cardinality i. Let $e(G, i) = |\epsilon(G, i)|$. The edge cover polynomial $E(G, x)$ of G is defined as follows:

$$E(G, i) = \Sigma_{i=\rho(G)}^{m} e(G, i) x^i.$$

Recursive definition: Let G be a graph, $u, v \in V(G)$ and uv be an edge of G. Then, [AO]:

$$E(G, x) = (x + 1)E(G \setminus uv, x) + x(E(G \setminus u, x) + E(G \setminus v, x) + E(G \setminus \{u, v\}, x)).$$

A pendant vertex is a vertex with degree 1. Let G be a graph and u be a pendant vertex of G. Suppose that v is the neighbor of u. Then, [AO]:

$$E(G, x) = xE(G \setminus u, x) + xE(G \setminus \{u, v\}, x).$$

Let P_n be a path of size n. For every natural number $n \geq 3$, [AO]:

$$E(P_n, x) = xE(P_{n-1}, x) + xE(P_{n-2}, x).$$

Roots: For every natural number $n \geq 3$, all roots of $E(P_n, x)$ are real, [AO]. Moreover, the real roots of the edge cover polynomials are dense in interval $[-4, 0]$, [AO]. $E(G, x)$ has exactly one distinct root if and only if every connected component of G is star, [AO]. For every graph G with no isolated vertex, all the roots of $E(G,x)$ are in the ball

$$\{z \in \mathbf{C}|\ |z| < \frac{(2+\sqrt{3})^2}{1+\sqrt{3}}\}$$

If every block of the graph G is K_2 or cycle, then all real roots of $E(G,x)$ are in the interval $(-4,0]$. For every tree T of order n, we have

$$\xi_{\mathbf{R}}(K_{1,n-1}) \le \xi_{\mathbf{R}}(T) \le \xi_{\mathbf{R}}(P_n),$$

where $-\xi_{\mathbf{R}}(T)$ is the smalest real root of $E(T,x)$, and $P_n, K_{1,n-1}$ are the path and the star of order n, respectively, [CO10].

2.1.15 The Martin polynomial

Authors: Introduced by P. Martin in his thesis [Mar77]. Further developed by M. Las Vergnas [Ver83].

Motivation: Information about Eulerian paths.

Static definition: In a graph, a circuit is a sequence $v_1e_1v_2e_2\cdots v_{k1}e_{k1}v_ke_k$, where the v_i are vertices and the e_i are distinct edges or loops such that the endvertices of e_i are v_i and v_{i+1}, with $v_{k+1} \equiv v_1$. In a digraph, the circuit are oriented: e_i must be oriented from v_i to v_{i+1}. A digraph $\mathbf{G}$ is said to be Eulerian if for every vertex v, the outdegree is the same as the indegree. Also, a graph $\mathbf{G}$ is Eulerian if every vertex has even degree counting, as usual, two for every loop at the vertex. Thus a graph or digraph is Eulerian if and only if it is the edge-disjoint union of some circuits (or cycles) and isolated vertices. Given an Eulerian digraph $\mathbf{G}$ and an integer $k \ge 0$, we write $r_k(\mathbf{G})$ for the number of partitions of $\mathbf{G}$ into k edge-disjoint circuits and isolated vertices. The *circuit partition polynomial* of $\mathbf{G}$ is

$$r_{\mathbf{G}}(x) = r(\mathbf{G};x) = \Sigma_k r_k(\mathbf{G})x^k.$$

The circuit partition polynomial rG(x) of an Eulerian graph G is defined similarly

$$r_G(x) = r(G;x) = \Sigma_k r_k(G)x^k.$$

The *Martin polynomial* of a graph G is

$$m_G(x) = m(G;x) = \frac{1}{x-1} r_G(x1),$$

and that of a digraph $\mathbf{G}$ is

$$m_{\mathbf{G}}(x) = m(\mathbf{G};x) = \frac{1}{x-2} r_{\mathbf{G}}(x2).$$

Reducibility: The circuit partition polynomials are simple transformations of the Martin polynomials, [EM04].

Related polynomials: Tutte polynomial, [EM04].

Complexity: Let G be a directed graph, and let k be a positive integer. In [MR10], $q(G;k)$ is defined as follows. At each vertex v, we place a k-dimensional complex vector x_v . We take the product, over all edges (u,v), of the inner product $< x_u, x_v >$. Finally, $q(G;k)$ is the expectation of this product, where the x_v are chosen uniformly and independently from all vectors of norm 1 (or, alternately, from the Gaussian distribution). [MR10] shows that $q(G;k)$ is proportional to G's circuit partition polynomial, and therefore that it is $\sharp P$-complete for any $k > 1$. There exists a polynomial-time algorithms for link-like graphs tree width [MM03].

Roots: If maximal $deg(G) > 2^k$ then $2 - 2^k$ is a root of $m(G;x)$, [EM98].

References: J. A. Ellis-Monaghan in [EM98] used Hopf algebra techniques to prove some combinatorial interpretations of the Martin polynomial for unoriented graphs. In [Bol02], B. Bollobás gives proofs of similar interpretations for a considerably wider class of values.

2.1.16 Interlace polynomial

Authors: Introduced by R. Arratia and B. Bolloás and G.B. Sorkin in [ABS00a, ABS00b].

Motivation: Originally motivated by a problem relating to DNA sequencing by hybridization. From the mathematical point of view: by circle graphs, and the enumeration of Euler circuits.

Recursive definition: Given an Euler circuit C of a 2-in, 2-out directed graph D, two vertices a and b of D are interlaced if C visits them in the sequence $\ldots, a, \ldots, b, \ldots, a, \ldots, b, \ldots$. The interlace graph $H = H(C)$ corresponding to C has the same vertex set as D, with an edge ab in H if a and b are interlaced in C.

Let G be any undirected graph, and ab a pair of distinct vertices in G. Construct the pivot G^{ab} as follows. Partition the vertices other than a and b into four classes:

(i) vertices adjacent to a alone;

(ii) vertices adjacent to b alone;

(iii) vertices adjacent to both a and b;

(iv) vertices adjacent to neither a nor b.

Begin by setting $G^{ab} = G$. For any vertex pair xy where x is in one of the classes (i-iii) and y is in a different class (i–iii), *toggle* the pair xy: if it was an edge, make it a non-edge, and if it was a non-edge, make it an edge.

There exists a function q_1, from the set of interlace graphs to the integers, such that for any 2-in, 2-out digraph D with Euler circuit C, the number of Euler circuits of D is equal to $q_1(H(C))$. Moreover, q_1 is uniquely defined by the following recursion:

$$q_1(H) = \begin{cases} q_1(H-a) + q_1(H^{ab} - b) & \text{if } ab \text{ is an edge of H} \\ 1 & \text{if H has no edges} \end{cases}$$

The $q_1(H)$ can be generalized to a one-variable graph polynomial $q(G;x)$, which is defined for all graphs G, not just interlace graphs, and which for interlace graphs G satisfies $q(G;1) = q_1(G)$. The $q(G;x)$ is called *the interlace polynomial of* G.

There is a unique map $q : \mathcal{G} \to \mathbf{Z}[x]$, $G \to q(G)$, such that the following two conditions hold:

(i) If G contains an edge ab then

$$q(G) = q(G - a) + q(G^{ab} - b)$$

(ii) On E_n, the edgeless graph with n vertices, $q(E_n) = x^n$.

The interlace polynomials of edgeless graphs, complete graphs, stars, and complete bipartite graphs are given by

(i) $q(E_n) = x^n$ for $n \geq 0$

(ii) $q(K_n) = 2^{n-1}x$ for $n \geq 1$

(iii) $q(K_{1n}) = 2x + x^2 + x^3 + \cdots + x^n$ for $n \geq 2$

(iv) $q(K_{mn}) = (1 + \cdots + x^{m-1})(1 + \cdots + x^{n-1}) + x^m + x^n - 1$ for $m, n \geq 1$

For $n \geq 2$, the interlace polynomial of the path P_n with $n + 1$ vertices and n edges satisfies

$$q(P_n) = q(P_{n-1}) + xq(P_{n-2})$$

Static definition: For $n \geq 2$, the interlace polynomial of the path P_n with $n + 1$ vertices and n edges satisfies:

$$q(P_n) = \sum_{r=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n-r}{r} + \binom{n-r-1}{r} \right\} x^{r+1}$$

For the path P_n, $q(P_n)(1) = F_{n+2}$, the $(n + 2)$'nd Fibonacci number (with $F_0 = 0, F_1 = 1$.

The interlace polynomials of the cycles C_n are $q(C_3) = 4x$, for $n \geq 4$ and with $y = \sqrt{1 + 4x}$,

$$q(C_n)(x) = \left(\frac{1-y}{2}\right)^n + \left(\frac{1+y}{2}\right)^n + \left(\frac{y^4 - 10y^2 - 7}{16}\right) \text{ for } n \text{ even}$$

$$q(C_n)(x) = \left(\frac{1-y}{2}\right)^n + \left(\frac{1+y}{2}\right)^n + \left(\frac{y^2 - 7}{4}\right) \text{ for } n \text{ odd}$$

For the static definition on trees and forests, see [ACRT10].

Related polynomials: The interlace graph polynomial may be viewed as a special case of the Martin polynomial of an isotropic system, which underlies its connections with the circuit partition polynomial and the Kauffman brackets of a link diagram. The graph polynomial also has a two-variable generalization that is unknown for the Martin polynomial.

Complexity: $\sharp\mathbf{P}$-hard almost everywhere, [BH08]. In [Cou08], an algorithm to evaluate the multivariate interlace polynomial of a graph with n vertices given a tree decomposition of the graph of width k with running time $f(k)n$ given, where $f(k)$ is doubly exponential in k. Analyzing the $GF(2)$-rank of adjacency matrices in the context of tree decompositions, [BH10] give a faster algorithm that uses $2^{3k2+O(k)}n$ arithmetic operations and can be efficiently implemented in parallel. In [BK09], it was proven that the interlace polynomials are VNP-complete.

Difficult point property: Proved in [BH07].

Definability: Order independent $CMSOL_2$-definable show by Courcelle.

References: define two-variable interlace polynomials. Further developed in [ABS04a, ABS04b, AvdH04]. In [Cou08], the multivariate interlace polynomial defined and investigated. In [RP05], spectral interpretations of the interlace polynomial, including the characterisation of the values of the power spectra for quadratics, concidered in application to cryptography. The weighted version introduced in [Tra10].

2.1.17 The cover polynomial

Authors: Introduced by F.R.K. Chung and R.L. Graham in [CG95].

Motivation: To find a Tutte-like polynomial for directed graphs.

Recursive definition: Let $D = (V; E)$ be a directed graph, or digraph, for short. That is, V is some (finite) set of vertices of D, and $E \subseteq V \times V$ is the set of edges of D. For an edge $e = uv \in E$, $u \neq v$, denote by $D \setminus e$ the digraph with vertex set V , and edge set $E \setminus \{e\}$; this is called a deletion of D. Similarly, the contraction D/e of the digraph with vertex set is obtained by replacing two vertices u and v in V by a single new vertex w, and with edge set formed from E by removing exactly those edges of the form ux or yv.
$C(D) = C(D; x, y)$ of D is defined recursively as follows:

(i) For the digraph with n independent vertices and no edges,

$$C(I_n) = x^{\underline{n}} := x(x-1)\dots(x-n+1);$$

For the special case of $n = 0$, the corresponding digraph D_Φ having no vertices or edges has cover polynomial

$$C(D_\Phi) = 1;$$

(ii) If e is an edge of D which is not a loop then

$$C(D) := C(D \setminus e) + C(D/e);$$

(iii) If e is a loop of D then

$$C(D) := C(D \setminus e) + yC(D/e);$$

Static definition:

$$C(D; x, y) := \Sigma_{i,j} c_D(i, j) x^{\underline{i}} y^j,$$

where $x^{\underline{i}} := x(x-1)\dots(x-i+1)$ and $x^{\underline{0}} := 1$.

Facts: Note that

(i) $c_D(i, j)$ is the number of ways of (disjointly) covering all the vertices of D with i directed paths and j directed cycles.

(ii) $C(D; \lambda, \mu)$ is the number of (λ, μ)-colorings of D.

(iii) $C(D)$ is multiplicative.

(iv) $C(D)$ is a polynomial iff D is acyclic.

(v) $C(D; x, 1) = x^n$,

(vi) $C(D; 1, 0)$ is the number of Hamiltonial paths in D,

(vii) $C(D; 0, 1)$ is the number of ways of covering D with disjoint cycles,

(viii) $c_D(0, 1)$ is the number of Hamiltonial cycles in D,

(ix) $c_D(1, 0) + c_D(2, 0)$ is the number of covering D with at most two disjoint paths,

(x) $\Sigma_j c_D(0, 2j)$ is the number of covering D with an even number of cycles.

Reducibility: Let board B be an n by n array of cells on wich certain cells have been designated as forbidden. An arrangement of i non-attacking rooks on B corresponds to a placement of i (chess) rooks on non-forbidden cells so that no rook attacks any other rook. Let $r_B(i)$ denote the number of possible such arrangements, [Rio58].

$$c_D(n - i) = r_B(i).$$

Given a digraph $D = (V, E)$ with $|V| = n$, let's denote by $Sym(D)$ the set of all $n!$ bijections $\pi : V \longrightarrow V$. if $u\pi(u) \in E$ then π has a drop at u. $\delta_D(k$ denotes the number of $\pi \in Sym(D)$ having k drops. For all j, [Sta86]:

$$\Sigma_k \delta_D(k) \binom{n}{r} = r_B(j)(n - j)!$$

The binomial drop polynomila for D is defined as

$$\Delta(D; x) := \Sigma_j \Sigma_k \delta_D(k) \binom{k}{j} \binom{x}{n-j} = C(D; x, 1)$$

To each D can be associated a poset $P = (V, \prec)$: $x \prec y$ iff $xy \in E$. Let $inc(P)$ denote the incomparability graph assiciated to P, so that G has vertex set V, and $\{x, y\}$ is an edge of G iff x and y incomparable in P. For any poset P with n elements:

$$\Delta(P; x) = \Sigma_k \delta_P(k) \binom{x+k}{n} = \chi(inc(P); x),$$

where χ is a chromatic polynomial.

Related polynomials: Rook's polynomial, binomial drop polynomil, chromatic polynomial, sf. [CG95].

Complexity: The problem of evaluating the cover polynomial is $\sharp P$-hard under polynomial-time Turing reductions, while only three points are easy, [BD07]. In [BDF08], the complexity of approximately evaluating the geometric cover polynomial was investigated. It was shown that under the reasonable complexity assumptions $RP \neq NP$ and $RFP \neq \sharp P$, there exists a succinct characterization of a large class of points at which approximating the geometric cover polynomial within any polynomial factor is not possible.

Difficult point property: Proved in [BD07].

Definability: $MSOL$-definable [Mak08].

References: In [Cho96], Chow generalized the cover polynomial to a symmetric function. In [DM00], *cyclepath indicator polynomial* of D introduced in order to classify the cycle–path covers of D by types. This polynomial can be obtained by a deletion-contraction recurrence relation. Then some specializations of the cycle-path indicator polynomial, such as *the geometric cover polynomial* (the geometric version of the cover polynomial), *the cycle cover polynomial*, and *the path cover polynomial* studied. As an application, chromatic arrangements of nonattacking rooks on a board B, where the principal sub-boards determined by the rooks of the same color are disjoint, considered. The associated polynomial is symmetric and generalizes the usual rook polynomial.

2.1.18 Go polynomial

Authors: Introduced by G.E. Farr in [Far03a].

Motivation: Some analogy to the notion of "captured territory" in the game of Go. The questionto unswer is: What is the probability that the resulting partial uniform random assignment has the following property: for every colour class, each component of the induced subgraph has a vertex that is adjacent to an uncoloured vertex?

Static definition: Let $(r, p_1, \dots, p_\lambda)$ be a probability distribution. For each $v \in V(G)$, assign it colour i with probability p_i (and no colour with probability $r = 1 - \Sigma_{i=1}^{\lambda}$), with the choice of the vertices being independent and identically deistributed. The colour class C_i and the uncolored set U are now set-valued random variables. Let f be the partial λ-assignment generated randomly in this way. Let's consider $Pr(f \text{ is free in } G)$. If $\lambda = 2$ and $p_1 = p_2 \leq \frac{1}{2}$ then this probability is a polynomial in p. The polynomial is the *Go polynomial* of G. The number of free partial λ-assignments in G (for general λ) is also a polynomial in λ and denoted by $Go^\sharp(G; \lambda)$. If $r = p_1 = \dots = p_\lambda = \frac{1}{(\lambda+1)}$ and $n = |V(G)|$ then

$$Go^\sharp(G; \lambda)(\lambda + 1)^n = Pr(f \text{ is free in } G)$$

Recursive definition: $Pr(f \text{ is free in } G)$ satisfies recursive relation for *intermediate graphs* $\mathcal{G}(\mathcal{L})$, i.e. graph with exactly one label from $\{J, L, A, a\}$ on each edge and zero, one or two labels from $\{C, S\}$. The intuition is as follows: *joining* vertices of the same colour together, we help to form a group, and being a $lifeline$ by making a coloured vertex adjancent to an uncoloured one, we help to preserve the former's group. If an adge is labelled A then we forbid its endpoints from receiving the same colour, but allow them to receive two different colours. In addition, one or both endpoints may remain uncoloured. Label a is similiar, exept that at most one of the edges's endpoints can be uncoloured. Vertex label C means that a vertex so labelled cannot be left uncoloured, and a vertex labelled S guarantees the freedom of any group of which it is a part, regardless of whether the group has a lifeline to an uncoulored vertex. Let Z be a label. If $e \in E(G)$ then $G[e \leftarrow Z]$ denotes the intermediate graph obtained by replacing the label of e by Z. If $v \in V(G)$ then $N_G^{(Z)}(v)$ denotes the set of those neighbours w of v such that vw has label Z.

If $G \in \mathcal{G}(J, L)$, f is a random partial λ-assignment of G and $e_J \in E_J(G)$ and $e_L \in E_L(G)$ are parallel edges, then

$$Pr(f \text{ is free in } G) + Pr(f \text{ is free in } G - e_J - e_L) =$$

$$Pr(f \text{ is free in } G - e_J) + Pr(f \text{ is free in } G - e_L)$$

It leads to: If G is a stage 1 graph and $e_J \in E_J(G)$ and $e_L \in E_L(G)$ are parallel edges, then

$$Go(G;p) + Go(G - e_J - e_L;p) = Go(G - eJ;p) + Go(G - e_L;p)$$

$$Go^\sharp(G;\lambda) + Go^\sharp(G - e_J - e_L;\lambda) = Go^\sharp(G - eJ;\lambda) + Go^\sharp(G - e_L;\lambda)$$

Let's $G \in \mathcal{G}(J,L,C,A)$ and $e = uv \in E_J(G)$. Let f be a random partial λ-assignmwent whose distribution satisfies $p_1 = \cdots = p_\lambda = p \leq \frac{1}{\lambda}$. Then

$$Pr(f \text{ is free in } G) = p \cdot Pr(f \text{ is free in } (G/e)[u \leftarrow C]) + Pr(f \text{ is free in } G[e \leftarrow A])$$

It leads to: If $G \in \mathcal{G}(J,L,C,A)$ and $e = uv \in E_J(G)$ then

$$Go(G;p) = p \cdot Go((G/e)[u \leftarrow C];p) + Go(G[e \leftarrow A];p)$$

$$Go^\sharp(G;\lambda) = Go^\sharp((G/e)[u \leftarrow C];\lambda) + Go(G^\sharp[e \leftarrow A];\lambda)$$

If $G \in \mathcal{G}(J,L,a)$ and $e \in e_J(G)$ then

$$Go^\sharp(G;\lambda) = Go^\sharp((G/e);\lambda) + Go(G^\sharp[e \leftarrow a];\lambda)$$

If $G \in \mathcal{G}(L,A,a,C,S)$ and $v \in V \setminus V_C$ then

$$Pr(f \text{ is free in } G) = r \cdot Pr(f \text{ is free in } (G - v)[N_G^{(L)}(v) \leftarrow S]) + Pr(f \text{ is free in } G[v \leftarrow C])$$

If $G \in \mathcal{G}(L,A,a,C,S)$ and $v \in V_C \cap V_S$ then

$$Pr(f \text{ is free in } G) = Pr(f \text{ is free in } G - E_L(\{v\}, N_G^{(L)}(v))))$$

The above leads to: If $G \in \mathcal{G}(L,A,a,C,S)$ and

- $v \in V \setminus V_C$ then

$$Go(G;p) = (1 - 2 \cdot p)Go((G - v)[N_G^{(L)}(v) \leftarrow S];p) + Go(G[v \leftarrow C];p)$$

$$Go^\sharp(G;\lambda) = Go^\sharp((G - v)[N_G^{(L)}(v) \leftarrow S];\lambda) + Go^\sharp(G[v \leftarrow C];\lambda)$$

- $v \in V_C \cap V_S$ then

$$Go(G;p) = Go(G - E_L(\{v\}, N_G^{(L)}(v));p)$$

$$Go^\sharp(G;\lambda) = Go^\sharp(G - E_L(\{v\}, N_G^{(L)}(v));\lambda)$$

Reducibility and Related polynomials: If we have an intermediate $\{A,C,S\}$-graph with all vertices labelled both C and S then

$$Go(G;p) = p^n \cdot P(G;2) \text{ and } Go^\sharp(G;\lambda) = P(G,\lambda)$$

Complexity: $\#P$-hard to compute in general.

References: In [TF05], the exact values for the number of legal positions on square boards of size upto 16×16 were obtained, as well as the limiting behaviour of the number of legal positions on large rectangular boards of fixed width upto seven was deternined. They also find estimates for the number of legal *games* of Go. In [FS07], bounds for the number of legal Go positions for various numbers of players on some planar lattice graphs investigated.

2.1.19 Stability polynomial

Authors: Introduced by Farr, [Far93].

Static definition: $A_{G(p)}$ is given by

$$A_{G(p)} = \sum_{U \in S(G)} p^{|U|}(1-p)^{|V(G)U|}$$

$S(G)$ denotes the set of all stable sets of G, i.e. all sets in a graph G, for which G has no edge with both endpoints in S.

Recursive definition: To be completeed

Reducibility: Related to the Universal polynomial, cf. [MN09]. If G is loopless then $A_{G(p)}$ is given by

$$A_{G(p)} = U_G(x_i = 1; x_j = -(-p)^j for\ j \geq 2, y = 0)$$

References: See [Nob09] and [MN09]. Results of [Far93] are extented in [McD10].

2.1.20 Strong U-polynomial

Authors: Introduced by C. Merino and S. Noble, [MN09].

Motivation: To extend universal polynomial by countably many commuting variables $z_{i,j}$.

Static definition:

$$\bar{U}_G(z_{i,j}) = \sum_{A \subseteq E} z_{c_1, e_1 - c_1 + 1} z_{c_2, e_2 - c_2 + 1} \cdots z_{c_{k(G|A)}, e_{k(G|A)} - c_{k(G|A)} + 1}$$

where c_i and e_i are, respectively, the number of vertices and edegs in the ith connected compotent of $G|A$.

Recursive definition: There is a recurcive relation fro the strong U-polynomial, [MN09]. Let (G, ω) desbribe graph G with a strictly positive integer weight $\omega(v)$ attached at each vertex v. Then

$$\bar{W}_{(G,w)}(\mathbf{z}) = \sum_{A \subseteq E} z_{w_1, e_1 - c_1 + 1} z_{w_2, e_2 - c_2 + 1} \cdots z_{w_{k(G|A)}, e_{k(G|A)} - c_{k(G|A)} + 1}$$

If $\omega(v) = 1$ for all v, then $\bar{U}_G(\mathbf{z}) = \bar{W}_{(G,w)}(\mathbf{z})$. On the other hand, [MN09]:

$$\bar{W}_{(G,w)} = \bar{W}_{(G,w)-e} + \bar{W}_{(G,w)e}$$

Reducibility: Related to the strong Tutte symmetric function, cf. [MN09]. The strong U-polynomial and the strong Tutte symmetric function are aquivalent:

$$\bar{Y}_G(\mathbf{x}, \mathbf{t}) = \bar{U}_G(z_{i,j} = \bar{p}_{i,i+j-1}(\mathbf{x}, \mathbf{t})),$$

$\bar{p}_{r,s}(\mathbf{x}, \mathbf{t}) = \sum_{i=1}^{\infty} x_i^r t_i^s$.

2.1.21 Two polymatroid rank generating function

Authors: Introduced by Oxley and Whittle, [OW93].

Motivation: To develop a theory of Tutte invariants for 2-polymatroids.

Static definition: Let $f(A)$ denote the number of vertices of G that are endpoint of an adge in A.

$$S_G(u,v) = \sum_{A \in E(G)} u^{|V(G)|-f(A)} v^{2|A|-f(A)}$$

Reducibility: Related to the Universal polynomial, cf. [MN09]. If G is loopless graph with no isolated vertices, then

$$S_G(u,v) = U_G(x_1 = u; v_2 = 1; x_j = v^{j-2} for\ j \geq 2, y = v^2 + 1)$$

Recursive definition: Via the Universal polynomial.

References: See [Far03b], [LM03], [Kun06], [Nob06], [MN09].

2.1.22 Rook polynomial

Authors: The study of the rook polynomials was begun in 1946 by Kaplansky and Riordan [KR46] with applications to cardmatching problems. Riordans 1958 book [Rio58] is considered the first systematic analysis, and remains a classic treatment of the subject. Since then, various researchers have applied rook polynomials to make important connections to Fibonacci theory [Fie04], group theory [MV00], hypergeometric series summation [Hag96], and the computation of the permanents of various matrices [CHS03, Hag00a, HOW99]. A series of papers by Goldman et al. [GJW75, GJRW76, GJW78, GJW77, GJW76] in the 1970s expanded the field by applying more advanced combinatorial methods.

Motivation: The classification of permutations by ascending runs was first studied by MacMahon [Mac13]. Applications of his results to significance tests in statistics have been made by Moore and Wallis [MW43] and Mann [Man45]. The numbers arising in this study have occurred several times in other contexts. Rather extensive references to older literature (going back to Laplace) are given by v. Schrutka in [Sch41]. The numbers have also appeared in papers by Dwyer [Dwy38, Dwy40], who calls them "cumulative numbers", and in papers by Toscano on summation of series, [Tos36]. These results may be unified and generalized by the study of a chessboard recreation: in how many ways can a given number of rooks be placed on a chessboard so that no two attack each other?

Static definition: A general board B is any sunset of $m \times n$ rectagular chessboard. Let B be a general board and $r_k(B)$ the number of non-taking placements of k rooks on B. The rook polynomial $\mu_B(t)$ for the board B is given by

$$\mu_B(t) = r_0(B) + r_1(B)t + r_2(B)t^2 + \cdots = \Sigma_{k \geq 0} r_k(B) t^k.$$

Note: $r_k = 0$ if $k > min(m,n)$.

Recursive definition Let B be a board, B' be the board obtained by removing the row and column corresponding to a cell from B, and B'' be the board obtained by restrinting the same cell on B. Then, [Rio58]:

$$\mu_B(t) = t \cdot \mu_{B'}(t) + \mu_{B''}(t).$$

Let B be a board containing a row i with n cells. Let $B - \{i\}$ be the board obtained from B by removing row i. Let B_j^n be the board obtained from B by removing row i and j – one of the n columns containing cells in row i. Then, [Rio58]:

$$\mu_B(t) = \mu_{B-\{i\}}(t) + t \cdot \Sigma_{j=1}^{n} \mu_{B_j^n}(t).$$

Related polynomials: Rook polynomials are equal to the matching polynomial of the associated board G_B. The *hit polynomial* $T(z; B)$ of B is

$$T(z; B) = \Sigma_{k=0}^{n} k! r_{n-k}(B)(z-1)^{n-k}.$$

If B is a board, then its coefficints of z^k is called the k^{th} *hit number*, the number of ways of placing n non-attacking rooks on B where exactly k rooks lie on non-zero.

In mathematics, the *Laguerre polynomials*, named after *Edmond Laguerre* (1834 - 1886), are the canonical solutions of Laguerre's equation:

$$x\,y'' + (1-x)\,y' + n\,y = 0$$

which is a second-order linear differential equation. This equation has nonsingular solutions only if n is a non-negative integer. The Laguerre polynomials are also used for Gaussian quadrature to numerically compute integrals of the form

$$\int_0^\infty f(x) e^{-x}\, dx.$$

These polynomials, usually denoted $L_0, L_1, \ldots$, are a polynomial sequence which may be defined by the Rodrigues formula

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^n\right).$$

They are orthogonal to each other with respect to the inner product given by

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) e^{-x}\, dx.$$

The sequence of Laguerre polynomials is a Sheffer sequence.

The Laguerre polynomials arise in quantum mechanics, in the radial part of the solution of the Schrödinger equation for a one-electron atom. The rook polynomial of a square chessboard is closely related to the generalized Laguerre polynomial $L_n^\alpha(x)$ by the identity:

$$\mu_{m,n}(x) = n! x^n L_n^{(m-n)}(-x^{-1}).$$

Facts:

- If the highest power of t in $\mu_B(t)$ is r, then all the terms in t^m, where $0 \le m \le r$, occur in $\mu_B(t)$ with non-zero coefficients.
- Let
$$\mu_{p,q}(t) = \Sigma_k r_k t^k$$
for a board b with p rows and q cells. Then
$$r_0 = 1, r_1 = q \text{ and } r_2 = \binom{q}{2} - \Sigma_{i=1}^{q} \binom{d_i}{2},$$
where d_i is the number of cells in row i.
- The coefficient of t^k in $\mu_B(t)$ is the number of sets of k cells with pairwise disjoint row and column indices.
- On rectangular board
$$\mu_{m,n}(x) = \Sigma_k \frac{(m)_k (n)_k}{k!} x^k.$$

Complexity: The rook polynomials are $\sharp P$-hard [MM03].

Definability: The rook polynomials are $MSOL$-definable [MM03].

Roots: Nijenhuis [Nij76] proved that if the elements of the board matrix are non-negative real numbers, then all the roots of the corresponding rook polynomial are non-negative real numbers. Moreover, the closure of its roots is $(-\infty, 0]$, [Hos07]. Let A be a board with the property that if $a_{ij} = 1$, then $a_{st} = 1$ whenever $i \le s \le n$ and $j \le n$. Then A is called *a Ferrers board*. In [HOW98], it was proven that all the roots of the hit polynomial of a Ferrers board are real.

References: In [Ges89], analogs of the theory of rook polynomials that are related to general Laguerre and Charlier polynomials in the same way that ordinary rook polynomials are related to simple Laguerre polynomials developped. The generalizations of the definition and properties of rook polynomials to "boards" in three and higher dimensions was given in [KA09].

2.1.23 Acyclic polynomial

Authors: Introduced by I .Gutman, [Gut77a]. Different consequences of this formula can be found in an article by the author and H. Hosoya [GH78].

Motivation: Inspired by the works of [Hos71], [Aih76], [GMT77].

Static definition: Let G be a graph with n vertices and m edges. The acyclic polynomial of G is defined as
$$\alpha(G) = \alpha(G, \lambda) = \Sigma_{j=0}^{[n/2]} (-1)^j p(G,k) \lambda^{n-2j},$$
where $p(G,k)$ is the number of ways in which one can select k independent edges in G. In addition, $p(G,0) = 1$. Explicit expressions are obtained for the acyclic polynomials of certain special graphs (e.g., for the cycle, the wheel, the complete graph, the complete bipartite graphs, etc.).

Recursive definition: Let e be an edge of G incident to the vertices v and w. Then the acyclic polynomial satisfies the recursion formula

$$\alpha(G) = \alpha(G-e) - \alpha(G-v-w).$$

The acyclic polynomial of a graph can be expressed as a linear combination of characteristics polynomials of its subforests. Let the characteristic polynomial of G be denoted by Φ, C_n and P_n be the cycle and the path, respectively, with n vertices. Then, [Gut77a],

$$\alpha(C_n) = \Phi(P_n) - \Phi(P_{n-2}).$$

Reducibility: $\alpha(G)$ coincides with the characteristic polynomial of G if and only if G is a forest. Note that there exist nonisomorphic graphs having cycles with equal acyclic polynomials. The acyclic polynomial of a graph is a special case of its matching polynomial with appropriately chosen weights: $\alpha(G) = M(G; \lambda, -1)$, see [Far80],

Complexity: An interesting and simple way (i.e., without use of computers) to obtain acyclic polynomial for graphs is based on a theorem, due to Godsil [God82],, which claims that the acyclic polynomial is a factor in the characteristic polynomial of a path-tree P-T derived for a cyclic graph G. Descriptions of algorithms, optimizations, and the efficient computer program in the chemistry context are presented in [MT82, HRZ88], etc.

Difficult point property: Considered in [Ave11] for matching polynomial.

Definability: Definable in MSOL, [Mak04b].

Roots: I.Gutman conjectured that the zeros of $\alpha(G)$ are real, the proof see, for example, in [GG81]. Moreover, it can be shown [Gut77b] that $\Phi(P_n) = \frac{sin(n+1)t}{sin\ t}$ that leads to the roots of $\alpha(C_n)$: $2cos\frac{(2j+1)\pi}{2n}, j = 1, \ldots, n$.

References: Chemestry, see, for example, [TKR86].

2.1.24 Potts polynomial

Author: The classical Potts model was introduced by Potts in 1952 [Pot52] and in it is a generalization of the Ising model [Isi25] to more-than-two components. A four-component version of the model was first studied in [AT43]. Consider a finite lattice L_n of N sites or general graph G of N vertices and suppose that each site (=vertex) can have associated with it a spin, which can have one of Q values. The energy between two interacting spins is taken to be zero if the spins are the same and equal to a constant if they are different.

Motivation: In statistical mechanics, the Potts model, a generalization of the Ising model, is a model of interacting spins on a crystalline lattice. By studying the Potts model, one may gain insight into the behaviour of ferromagnets and certain other phenomena of solid state physics. For the complete survey see in [Wu82].

Static definition: In the simplest description of the Potts model with Q states $1, 2, ..., Q$, the Hamiltonian H is given by

$$H = J\Sigma_{i\sim j}(1 - \delta(\sigma_i, \sigma_j)),$$

where the sum is over all nearest-neighbor pairs of sites i, j and σ_i is the spin at site i. Here J is the (constant) interaction. The model is ferromagnetic when $J > 0$ and antiferromagnetic if $J < 0$. The probability of finding the system in state σ is then given by

$$Pr[\sigma] = e^{-\beta H(\sigma)}/Z,$$

where Z, the normalizing constant, is the partition function and $\beta = 1/kT$, where k is Boltzmanns constant and T is the temperature. Thus the partition function is

$$Z(G; Q, K) = \Sigma_\sigma exp(-K\Sigma_{i\sim j}(1 - \delta(\sigma_i, \sigma_j))),$$

where $K = J/kT$, the summation in the exponential is over all near-neighbor pairs (i, j), and the first summation is over all possible spin configurations. The Ising model with zero external field is just the special case when $Q = 2$ and then the spins are usually taken to be ± 1.

Recursive definition: To be completeed

Reducibility: For the relation of Potts model to Tutte (and chromatic) polynomial, see [WM00].

$$Z_{Potts}(G; Q, k) = Q(e^k - 1)^{|V|-1}e^{-K|E|}T(G; \frac{e^K + Q - 1}{e^K - 1}, e^K).$$

When G has k connected components then there is an extra factor of Q^{k-1} on the right-hand side of the above.

Complexity: To be completed

Difficult point property: To be completed

Definability: To be completed

Roots:

References:

2.2 New graph polynomials

2.2.1 mcc-polynomial

Author Introduced in [MZ08] based on the corresponding graph property discussed in [LMST07].

Definition $mcc_t(G, k)$ is the number of colorings of G with k colors such that all the connected components of a monochromatic set have size at most t.

For fixed $t \in \mathbb{N}$ this is a polynomial by the Makowsky-Zilber argument. [MZ06, MZ08]. For variable t the function $mcc(G, k, t)$ is not a polynomial, as it vanishes for large enough t .

Motivation: To be checked.

Static definition: To be checked.

Reducibility $\chi(G,k) = mcc_1(G,k)$

Recurrence relations To be checked.

Recursive definition: To be checked.

Complexity To be checked.

Difficult point property: To be checked.

Definability To be checked.

Combinatorial interpretations: To be checked.

Related polynomials: Chromatic polynomial.

Roots: To be checked.

References To be checked.

2.2.2 hc-polynomial

Author Introduced in [MZ08] based on the corresponding graph property discussed in [HK83].

Definition $hc(G,k)$ is the number of harmonious proper colorings of G with at most k colors. This is a polynomial by the Makowsky-Zilber argument, [MZ06, MZ08].

$hc(G)$ is the smallest k such that $hc(G,k) \neq 0$.

Recurrence relations To be added.

Complexity $hc(G)$ is NP-hard already for trees, [EM95].

Definability If $NP \neq P$, it is not $MSOL$-definable in any expansion of G. Otherwise it would be polynomial time computable on graphs of tree-width at most k, in particular on trees, [Mak04a].

References For $hc(G)$ only a survey is given in [Edw97].

2.2.3 The cc-polynomial

Author JAM

Definition $cc(G,k)$ is the number of colorings of G with k colors such that each monochromatic set is connected.

Reducibility

Recurrence relations

Complexity Conjecture: $\sharp P$-hard for $k = 2$.

The conjecture implies the difficult point property.

Definability

References

Bibliography

[ABBK03] N. Anzalone, J. Baldwin, I. Bronshtein, and T. Kyle. A reciprocity theorem for Monomer-Dimer covering. *Discrete Mathematics and Theoretical Computer Science*, AB:179–194, 2003.

[ABS00a] R. Arratia, B. Bollobas, and G.B. Sorkin. The interlace polynomial: A new graph polynomial. IBM Research Report, RC 21813 (98165), 2000.

[ABS00b] R. Arratia, B. Bollobás, and G.B. Sorkin. The interlace polynomial: a new graph polynomial. In *Proceedings of the 11th Annual ACM-SIAM Symposium on Discrete Mathematics*, pages 237–245, 2000.

[ABS04a] R. Arratia, B. Bollobás, and G.B. Sorkin. The interlace polynomial of a graph. *Journal of Combinatorial Theory, Series B*, 92:199–233, 2004.

[ABS04b] R. Arratia, B. Bollobás, and G.B. Sorkin. A two-variable interlace polynomial. *Combinatorica*, 24.4:567–584, 2004.

[ACRT10] C. Anderson, J. Cutler, A.J. Radcliffe, and L. Traldi. On the interlace polynomials of forests. *Discrete Mathematics*, 310, 2010.

[AEMR05] O. Araujo, M. Estrada, D. Morales, and J. Rada. The higher-order matching polynomial of a graph. *International journal of Mathematics and mathematical Sciences*, 2005:1565–1576, 2005.

[AGM07] I. Averbouch, B. Godlin, and J.A. Makowsky. The most general edge elimination polynomial. submitted, 2007.

[AGM08] I. Averbouch, B. Godlin, and J.A. Makowsky. A most general edge elimination polynomial. In *WG'08*, volume 5344 of *Lecture Notes in Computer Science*, pages 31–42, 2008.

[AGM10] I. Averbouch, B. Godlin, and J.A. Makowsky. An extension of the bivariate chromatic polynomial. *European Journal of Combinatorics*, 31.1:1–17, 2010.

[Aih76] J. Aihara. A new definition of Dewar-type resonance energy. *J. Amer. Chem. Soc.*, 98:2750–2758, 1976.

[AM06] I. Averbouch and J.A. Makowsky. The most general matching polynomial. In preparation, 2006.

[AO] S. Akbari and M.R. Oboudi. On the edge cover polynomial of a graph. Preprint.

[Aro84] J.L. Arocha. Propriedades del polinomio independente de un grafo. *Revista Ciencias Matematicas*, V:103–110, 1984.

[AT43] J. Ashkin and E. Teller. Statistics of two-dimensional lattices with four components. *Physical Review*, 64:178–184, 1943.

[AT97] N. Alon and M. Tarski. A note on graph colorings and graph polynomials. *Journal of Combinatorial Theory*, 70:197–201, 1997.

[AvdH04] M. Aigner and H. van der Holst. Interlace polynomials. *Linear Algebra and Applications*, 377:11–30, 2004.

[Ave11] I. Averbouch. *Completeness and Universality Properties of Graph Invariants and Graph Polynomials.* PhD thesis, Technion - Israel Institute of Technology, Haifa, Israel, 2011.

[Bal93a] A. T. Balaban. Solved and unsolved problems in chemical graph theory. quo vadis, graph theory? *Ann. Discrete Math.*, 35:109–126, 1993.

[Bal93b] K. Balasubramanian. Spectra of matching polynomials. *Chemecal Physics Letters*, 208:219–224, 1993.

[Bal95] A. T. Balaban. Chemical graphs: Looking back and glimpsing ahead. *Journal of Chemical Information and Computer Science*, 35:339–350, 1995.

[BD07] M. Bläser and H. Dell. Complexity of the cover polynomial. In L. Arge, C. Cachin, T. Jurdziński, and A. Tarlecki, editors, *Automata, Languages and Programming, ICALP 2007*, volume 4596 of *Lecture Notes in Computer Science*, pages 801–812. Springer, 2007.

[BDF08] M. Bläzer, H. Dell, and M. Fouz. Compexity and approximability of the cover polynomial. http://www2.informatik.hu-berlin.de/ dell/papers/bdf08coverpoly.pdf, 2008.

[BF95] R.A. Beezer and E.J. Farrell. The matching polynomial of a regular graph. *Discrete Mathematics*, 137:7–18, 1995.

[BG10] P. Bubenik and L.H. Gold. Graph products of spheres, associative graded algebras and hilbert series. *Mathematische Zeitschrift*, 3, 2010.

[BGMP86] D Babic, A Gravovac, B Mohar, and T Pisanski. The matching polynomial of a polygraph. *Discrete Appl. Math.*, 15(1):11–24, 1986.

[BH07] M. Bläser and C. Hoffmann. On the complexity of the interlace polynomial. arXive 0707.4565, 2007.

[BH08] Markus Bläser and Christian Hoffmann. On the complexity of the interlace polynomial. In *STACS*, volume 08001 of *Dagstuhl Seminar Proceedings*, pages 97–108. Internationales Begegnungs- und Forschungszentrum fuer Informatik (IBFI), Schloss Dagstuhl, Germany, 2008.

[BH10] Markus Bläser and Christian Hoffmann. Fast evaluation of interlace polynomials on graphs of bounded treewidth. *Alogorithmica*, pages 1–33, 2010.

[BIK05] B. Brešar, W. Imrich, and S. Klavžar. Reconstracting subgraph-counting graph polynomials of increasing families of graphs. Submitted Elsevier Science, 2005.

[Bir12] G.D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Annals of Mathematics*, 14:42–46, 1912.

[BK09] I. Briquel and P. Koiran. A dichotomy theorem for polynomial evaluation. In *Mathematical Foundations of Computer Science (MFCS'09)*, volume 5734 of *Lecture Notes in Computer Science*, pages 187–198. Springer Verlag, 2009.

[BKW80] S. Beraha, J. Kahane, and N.J. Weiss. Limits of chromatic zeros of some families of graphs. *Journal of Combinatorial Theory*, 28:52–65, 1980.

[BN01] J.I. Brown and R.J. Nowakowsky. Average independence polynomials. *Journal of Combinatorial Theory*, 58:113–120, 2001.

[Bol99] B. Bollobás. *Modern Graph Theory.* Springer, 1999.

[Bol02] B. Bollobás. Evaluation of the circuit partition polynomial[1]. *Journal of Combinatorial Theory*, 85:261–268, 2002.

[BR05] A. Björner and J.D. Rafley. Chain polynomials of distributive lattices are 75 % unimodal. *The Electronic Journal of Combinatorics*, 12, 2005.

[Bro98] J.I. Brown. On the roots of chromatic polynomials. *Journal of Combinatorial Theory*, 72:251–256, 1998.

[BT69] G. Berman and W.T. Tutte. The golden root of a chromatic polynomial. *Journal of Combinatorial Theory*, 6:301–302, 1969.

[BX93] H.J. Broersma and L. Xueliang. On T̈he matching polynomial of a polygraph ·: *Discrete Appl. Math.*, 46:79–86, 1993.

[Cas07] G.G. Cash. An algorithm for calculating the independence and vertex-cover polynomials of a graph. *Applied Mathematics and Computation*, 190:1487–1491, 2007.

[CDL94] S.V. Chmutov, S.V. Duzhin, and S.K. Lando. Vassiliev knot invariants III. forest algebra and weighed graphs. *Advances Soviet Math.*, 21:135–145, 1994.

[CG95] F.R.K. Chung and R.L. Graham. On the cover polynomial of a digraph. *Journal of Combinatorial Theory, Ser. B*, 65(2):273–290, 1995.

[Cho96] T.Y. Chow. The path-cycle symmetric function of a digraph. *Advances in Mathematics*, 118:71–98, 1996.

[CHS03] G-S. Cheon, S-G. Hwang, and S-Z. Song. Rook polynomials to and from permanents. *Discrete Mathematics*, 264:25–36, 2003.

[CL98] D. Cvetković and M. Lepovic. Seeking counterexamples to the reconstruction conjecture for characteristic polynomials of graphs and a positive result. *Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math.*, 116(23):91–100, 1998.

[CMR01] B. Courcelle, J.A. Makowsky, and U. Rotics. On the fixed parameter complexity of graph enumeration problems definable in monadic second order logic. *Discrete Applied Mathematics*, 108(1-2):23–52, 2001.

[CO10] P. Csikvári and M.R. Oboudi. On the roots of edge cover polynomials of graphs. Unpublished, personal communication, 2010.

[Cou08] B. Courcelle. A multivariate interlace polynomial and its computation for graphs of bounded clique-width. *The electronic journal of combinatorics*, 15, 2008.

[CS07] M. Chudnovsky and P. Seymour. The roots of the independence polynomial of a clwafree graph. *Journal of Combinatorial Theory*, 97:350–357, 2007.

[Dha06] A. Dharwadker. The vertex cover algorithm. http://www.dharwadker.org, 2006.

[DHTL02] F.M. Dong, M.D. Hendy, K.L. Teo, and C.H.C. Little. The vertex-cover polynomial of a graph. *Discrete Mathematics*, 250:71–78, 2002.

[DKT05] F.M. Dong, K.M. Koh, and K.L. Teo. *Chromatic Polynomials and Chromaticity of Graphs.* World Scientific, 2005.

[DM00] O.M. D'Antona and E. Munarini. The cycle-path indicator polynomila of a digraph. *Advances in Apllied Mathematics*, 25:41–56, 2000.

[DPT03] K. Dohmen, A. Pönitz, and P. Tittmann. A new two-variable generalization of the chromatic polynomial. *Discrete Mathematics and Theoretical Computer Science*, 6:69–90, 2003.

[Dwy38] P.S. Dwyer. The computation of moments with the use of cumulative totals. *Ann. Math. Statist*, 9:288–304, 1938.

[Dwy40] P.S. Dwyer. The cumulative numbers and their polynomials. *Ann. Math. Statist*, 11:66–71, 1940.

[Edw97] K. Edwards. The harmonious chromatic number and the achromatic number. In R. A. Bailey, editor, *Survey in Combinatorics*, volume 241 of *London Math. Soc. Lecture Note Ser.*, pages 13–47. Cambridge Univ. Press, 1997.

[EM95] K. Edwards and C. McDiarmid. The complexity of harmonious colouring for trees. *Discrete Appl. Math.*, 57(2-3):133–144, 1995.

[EM98] J. Ellis-Monaghan. New results for the Martin polynomial. *Journal of Combinatorial Theory, Series B*, 74:326–352, 1998.

[EM04] J. Ellis-Monaghan. Identities for circuit partition polynomials, with applications to the Tutte polynomial. *Advances in Applied Mathematics*, 32:188–197, 2004.

[Far79] E.J. Farrell. An introduction to matching polynomials. *Journal of Combinatorial Theory, Series B*, 27:75–86, 1979.

[Far80] E.J. Farrell. The matching polynomial and its relation to the acyclic polynomial of a graph. *Ars Combin.*, 9:221–228, 1980.

[Far88] E.J. Farrell. A graph-theoretic approach to rook theory. *Caribbean J. Math.*, 7(1-2):1–47, 1988.

[Far89] E.J. Farrell. On a class of polynomials associated with the cliques in a graph and its applications. *Iternat. J. Math & Math.Sci*, 12(1):77–84, 1989.

[Far93] G.E. Farr. A correlation inequality involving stable sets and chromatic polynomials. *Journal of Combinatorial Theory, Series B*, 58:14–21, 1993.

[Far03a] G.E. Farr. The go polynomials of a graph. *Theoretical Computer Science*, 306:1–18, 2003.

[Far03b] G.E. Farr. The Go polynomials of a graph. *Theoretical Computer Science*, 306:1–18, 2003.

[Fie04] D. Fielder. Some thoughts on rook polynomials on square chessboards. In *Applications of Fibonacci Numbers*, volume 9, pages 101–108. Kluwer, Dordrecht, 2004.

[FS07] G. Farr and J. Schmidt. On the number of Go positions on latice graphs. *Information Processing Letters*, 105:124–130, 2007.

[Für10] M. Fürer. Almost linear time computation of the chromatic polynomial of a graph of bounded tree-width. *LNCS*, 6034:49–59, 2010.

[FZ07] D. Forge and T. Zaslavsky. Lattice point counts for the shi arrangement and other affinographic hyperplane arrangements. *Journal of Combinatorial Theory*, 114:97–109, 2007.

[Gee00] J.F. Geelen. An algebraic matching algorithm. *Combinatorica*, 20:61–70, 2000.

[Ges89] I. Gessel. Generalized rook polynomials and orthogonal polynomials. In *IMA Volumes in Mathematics and Its Applications*, volume 18, pages 159–176. Springer, 1989.

[GG81] C.D. Godsil and I. Gutman. On the theory of the matching polynomial. *Journal of Graph Theory*, 5:137–144, 1981.

[GH78] I. Gutman and H. Hosoya. On the calculation of the acyclic polynomial. In *Theoretical Chemistry Accounts: Theory, Computation, and Modeling (Theoretica Chimica Acta)*, pages 279–286. Springer, 1978.

[GH83] I. Gutman and F. Harary. Generalizations of the matching polynomial. *Utilitas Mathematicae*, 24:97–106, 1983.

[GJRW76] J.R. Goldman, J.T. Joichi, D.I. Reiner, and D.E. White. Rook theory. II. Boards of binomial types. *SIAM J. Appl. Math*, 31:618–633, 1976.

[GJW75] J.R. Goldman, J.T. Joichi, and D.E. White. Rook theory. I. Rook equivalence of Ferrers boards. *Proc. Amer. Math. Soc.*, 52:485–492, 1975.

[GJW76] J.R. Goldman, J.T. Joichi, and D.E. White. Rook theory. V. Rook polynomials, Möbius inversion and the umbral calculus. *J. Combin. Theory*, 21:230–239, 1976.

[GJW78] J.R. Goldman, J.T. Joichi, and D.E. White. Rook theory. III. Rook polynomials and the chromatic structure of graphs. *J. Combin. Theory*, 25:135–142, 1978.

[GJW77] J.R. Goldman, J.T. Joichi, and D.E. White. Rook theory. IV. Orthogonal sequences of rook polynomials. *Studies in Appl. Math.*, 56:267–272, 1976/77.

[GMT77] I. Gutman, M. Milun, and Trinajstič. Graph theory and molecular orbitals. XIX. Non-parametric resonance energies of arbitrary conjugated systems. *J. Amer. Chem. Soc.*, 99:2332–2339, 1977.

[God82] G.D. Godsil. Some graphs with characteristic polynomilas which are not solvable by radicals. *Journal of Graph Theory*, 6:211–214, 1982.

[God93] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, 1993.

[Goo08] A. Goodall. Graph polynomials and Tutte-Grothendieck invariants: an application of elementary finite Fourier analysis. 18 pages, based on talks given at the Patejdlovka Enumeration Workshop (19-23 November 2007), 2008.

[GP06] R. Glantz and M. Pelillo. Graph polynomials from pricipal pivoting. *Discrete Mathematics*, 306:3253–3266, 2006.

[GR01] C. Godsil and G. Royle. *Algebraic Graph Theory.* Graduate Texts in Mathematics. Springer, 2001.

[GS98] M. Goldwurm and M. Santini. Clique polynomials and trace monoids. Rapporto Interno n.222-98, Departmento di Scienze dell'Informazione, Università degli Studi di Milano, Milano, Italy, 1998.

[GS00] M. Goldwurm and M. Santini. Clique polynomials have a unique root of smallest modulus. Rapporto Interno n.247-00, Departmento di Scienze dell'Informazione, Università degli Studi di Milano, Milano, Italy, 2000.

[Gut77a] I. Gutman. The acyclic polynomial of a graph. *Publications de línstitut math*, 22(36):63–69, 1977.

[Gut77b] I. Gutman. Generalization of a recurrence relation for the characteristic polynomials of trees. *Publications de línstitut math*, 21(35):75–80, 1977.

[Gut80] I. Gutman. Characteristic and matching polynomials of some compound graphs. *Publications de línstitut math*, 27(41):61–66, 1980.

[Hag96] J. Haglund. Rook theory and hypergeometric series. *Advances in Applied Mathematics*, 17:408–459, 1996.

[Hag00a] J. Haglund. Further investigations involving rook polynomials with only real zeros. *Europian Journal of Combinatorics*, 21:1017–1037, 2000.

[Hag00b] E.M. Hagos. The characteristic polynomial of a graph is reconstructible from the characteristic polynomials of its vertex-deleted subgraphs and their complements. *The electronic journal of combinatorics*, 7, 2000.

[Hei53] E. Heilbronner. Das komposition-prinzip: Eine anschauliche methode zur elektron-theoretischen behandlung nicht oder niedrig symmetrischer molekeln im rahmen der mo-theorie. *Helv. Chim. Acta*, 21:135–145, 1953.

[HK83] J.E. Hopcroft and M.S. Krishnamoorthy. On the harmonious coloring of graphs. *SIAM J. Algebraic Discrete Methods*, 4:306–311, 1983.

[HL72] C.J. Heilmann and E.H. Lieb. Theory of monomer-dimer systems. *Comm. Math. Phys*, 25:190–232, 1972.

[HL94] C. Hoede and X. Li. Clique polynomials and independent set polynomials of graphs. *Discrete Mathematics*, 125:219–228, 1994.

[Hos71] H. Hosoya. Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons. *Bull. Chem. Soc. Japan*, 44:2332–2339, 1971.

[Hos07] R. Hoshino. *Independence Polynomials of Circulant Graphs*. PhD thesis, Dalhousie University, Halifax, Nova Scotia, 2007.

[HOW98] J. Haglund, K. Ono, and D.G. Wagner. Theorems and conjectures involving rook polynomials with only real zeros. ttp://www.math.wisc.edu/ ono/reprints/041.pdf, 1998.

[HOW99] J. Haglund, K. Ono, and D.G. Wagner. Theorems and conjectures involving rook polynomials with only real zeros. In *Topics in Number Theory, Mathematics and its Applications*, volume 467, pages 207–221. Kluwer, Dordrecht, 1999.

[HRZ88] W.C. Herndon, T.P. Radhakrishnan, and T.P. Zivkovic. Characteristic and matching polynomials of chemical graphs. *Chemical Physics Letters*, 152(2,3):233–238, 1988.

[Isi25] E. Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*, 31:253–258, 1925.

[Ja10] X. Jin and all. Zeros of the Jones polynomial are dense in the complex plane. *The Electronic Journal of Combinatorics*, 17, 2010.

[Jer87] M. Jerrum. Two–dimensional monomer–dimer systems are computationally intractable. *Journal Stat. Phys.*, 48:121–134, 1987. Erratum in vol. 59 (1990), pages 1087-1088.

[JVW90] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Camb. Phil. Soc.*, 108:35–53, 1990.

[KA09] N. Krzywonos and F. Alayont. Rook polynomials in higher dimensions. *Student Summer Scholars*, Paper 29, 2009.

[Kar72] Richard M. Karp. Reducibility among combinatorial problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85 – 104. Plenum Press, 1972.

[KR46] I. Kaplansky and J. Riordan. The problem of rooks and its application. *Duke Math. Journal*, 13:259–268, 1946.

[Kun06] Joseph P. S. Kung. A tutte decomposition for matrices and bimatroids. *J. Comb. Theory Ser. B*, 96(1):50–66, 2006.

[KW09] Ch.Y. Ku and K.B. Wong. Maximal multiplicity of a root of the matching polynomial of a tree and minimal path cover. *The Electronic Journal of Combinatorics*, 16, 2009.

[Las04] D. Lass. Matching polynomials and duality. *Combinatorica*, 24:427–440, 2004.

[Lin86] M. Linial. Hard enumeration problems in geometry and combinatorics. *SIAM Journal of Algebraic and Discrete Methods*, 7:331–335, 1986.

[Liu87] R.Y. Liu. A new method to find chromatic polynomial of graph and its applications. *Kexue Tongbao*, 32:1508–1509, 1987.

[Liu97] R.Y. Liu. Adjoint polynomilas and chromatically unique graphs. *Discrete Mathematics*, 172:85–92, 1997.

[LM03] Lemos and Manoel. Uniqueness of the decomposition of the rank function of a 2-polymatroid. *Discrete Mathematics*, 269(1-3):161–179, 2003.

[LM05] V.E. Levit and E. Mandrescu. The independence polynomial of a graph - a survey. In S. Bozapalidis, A. Kalampakas, and G. Rahonis, editors, *Proceedings of the 1st International Conference on Algebraic Informatics*, pages 233–254. Aristotle University of Thessaloniki, Department of Mathematics, Thessaloniki, 2005.

[LM08] V.E. Levit and E. Mandrescu. On the roots of the independence polynomials of almost all very well-covered graphs. *Discrete Mathematics*, 156:478–491, 2008.

[LMST07] N. Linial, J. Matousek, O. Sheffet, and G. Tardos. Graph coloring with no large monochromatic components. arXiv:math/0703362, 2007.

[Mac13] P.A. MacMahon. The indices of permutations and derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects. *Amer. J. Math*, 35:281–322, 1913.

[Mak04a] J.A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. *Annals of Pure and Applied Logic*, 126.1-3:159–213, 2004.

[Mak04b] J.A. Makowsky. algoritmic uses of the Feferman-Vaught Theorm. *Annals of Pure and Apllied Logic*, 16:159–213, 2004.

[Mak08] J.A. Makowsky. Uniform algebraic redicibility between parametrized invariants. *Logic and Teory of Algorithms*, 5028:403–406, 2008.

[Man45] H.B. Mann. Nonparametric test against trend. *Econometrica*, 13:245–259, 1945.

[Mar77] P. Martin. *Enumérations Euleriennes dans les multigraphes et invariantes de Tutte-Grothendieck*. PhD thesis, Grenoble, France, 1977.

[McD10] C. McDiamid. On a correlation inequality of Farr. Draft manuscript, 2010.

[MM03] J.A. Makowsky and J.P. Mariño. Farrell polynomials on graphs of bounded treewidth. *Advances in Applied Mathematics*, 30:160–176, 2003.

[MN09] C. Merino and S. Noble. The equivalence of two graph polynomials and symmetric function. *Combinatorics, Probability, and Computing*, 18:601–615, 2009.

[MR10] Ch. Moore and A. Russell. Circuit partitions and #P-complete products of inner products. http://www.cs.unm.edu/ treport/tr/10-01/paper-2010-04.pdf, 2010.

[MRAG06] J.A. Makowsky, U. Rotics, I. Averbouch, and B. Godlin. Computing graph polynomials on graphs of bounded clique-width. In F. V. Fomin, editor, *Graph-Theoretic Concepts in Computer Science, 32nd International Workshop, WG 2006, Bergen, Norway, June 22-23, 2006, Revised Papers*, volume 4271 of *Lecture Notes in Computer Science*, pages 191–204. Springer, 2006.

[MT82] B. Mohar and N. Trinajstić. On computation of the topological resonance energy. *Journal of Computational Chemistry*, 3(1):28–36, 1982.

[MV00] T. Mansour and A. Vainshtein. Avoiding maximal parabolic subgroups of S_k. *Discrete Mathematics and Theoretical Computer Science*, pages 67–75, 2000.

[MW43] G.H. Moore and W.A. Wallis. Time series significance tests based on signs of differences. *Journal of the American Statistical Association*, 38:153, 1943.

[MW82] A.J. Mansfield and D.J.A. Welsh. Some colouring problems and their complexity. *Annals of Discrete mathematics*, 13:159–170, 1982.

[MZ06] J.A. Makowsky and B. Zilber. Polynomial invariants of graphs and totally categorical theories. MODNET Preprint No. 21, `http://www.logique.jussieu.fr/modnet/Publications/Preprint%20 server`, 2006.

[MZ08] J.A. Makowsky and B. Zilber. Generalized chromatic polynomials. in prepararion, 2008.

[Nij76] A. Nijenhuis. On permanents and the zeros of rook polynomials. *Journal of Combinatorial Theory*, 21:240–244, 1976.

[Nob06] S. D. Noble. Evaluating the rank generating function of a graphic 2-polymatroid. *Comb. Probab. Comput.*, 15(3):449–461, 2006.

[Nob09] S. Noble. Evaluating a weighted graph polynomial for graphs of bounded tree-width. *The Electronic Journal of Combinatorics*, 16, 2009.

[NW99] S.D. Noble and D.J.A. Welsh. A weighted graph polynomial from chromatic invariants of knots. *Ann. Inst. Fourier, Grenoble*, 49:1057–1087, 1999.

[OW93] J.G. Oxley and G.P. Whittle. A characterization of Tutte invariants of 2-polymatroids. *Journal of Combinatorial Theory Series B*, 59:210–244, 1993.

[PB83] J.S. Provan and M.O. Ball. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J. Comp.*, 12(4):777–788, 1983.

[Pot52] R.B. Potts. Some generalized order-disorder transformations. In *Proceedings of the Cambrige Philosophical Society*, volume 48, pages 106–109, 1952.

[Pra99] C.R. Pranesachar. A class of 'matching-equivalent' bipartite graphs. *Discrete Mathematics*, 203:207–213, 1999.

[RG96] V. Rosenfeld and I. Guttman. A new recursion relation for the characteristic polynomial of a molecular graph. *J. Chem. Inf. Compt. Sci.*, 36:527–530, 1996.

[Rio58] J. Riordan. *An Introduction to Combinatorial Analysis.* Wiley, 1958.

[RJ99] R.C. Read and E.G. Whitehead Jr. Chromatic polynomials of homeomorphism classes of graphs. *Discrete Mathematics*, 204:337–356, 1999.

[RJ03] R.C. Read and E.G. Whitehead Jr. Chain polynomials of graphs. *Discrete Mathematics*, 265.1-3:213–235, 2003.

[Roy07] G.F. Royle. Graphs with chromatic roots in the interval $(1; 2)$. *The Electronic Journal of Combinatorics*, 14, 2007.

[RP05] C. Riera and M.G. Parker. Spectral interpretations of the interlace polynomial. In *Proceedings of the Workshop on Coding and Cryptography*, LNCS, Bergen, 2005.

[Sar00] I. Sarmiento. The polychromate and a chord diagram polynomial. *Annals of Combinatorics*, 4:227–236, 2000.

[Sch41] L.v. Schrutka. Eine neue einteilung der permutationen. *Mathematische Annalen*, 118:246–250, 1941.

[Sch79] A.J. Schwenk. Spectral reconstruction problems. *Ann. N.Y. Acad. Sci.*, 328:183–189, 1979.

[Sch03] A. Schrijver. Matching, edge-colouring, dimers. In H.L.Bodlaender, editor, *Proceedings of the 29th International Workshop on Graph-Theoretic Concepts in Computer Science, WG03*, volume 2880 of *LNCS*, pages 13–22, Berlin, 2003. Springer.

[Sci02] I. Sciriha. Polynomial reconstruction and terminal vertices. *Linear Algebra and its Applications*, 356, 2002.

[Sim92] S.K. Simić. A note on reconstructing the characteristic polynomial of a graph. In J. Nešetřil and M. Fiedler, editors, *Fourth Czechoslovakian Symposium on Combinatorics, Graphs and Complexity*, pages 315–319. Elsevier Science Publishers, 1992.

[Sta73] R. P. Stanley. Acyclic orientations of graphs. *Discrete Mathematics*, 5:171–178, 1973.

[Sta86] R.P. Stanley. *Enumerative Combinatorics*, volume 1. Wadsworth and Brooks, 1986.

[Sta95] R.P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics*, 111:166–194, 1995.

[Sta98] R. P. Stanley. Graph coloring and related symmetric functions: Ideas and applications. a description of results, interesting applications, notable open problems. *Discrete Mathematics*, 193:267–286, 1998.

[TF05] J. Tromp and G. Farneback. Combinatorics of Go. *prerint*, 2005.

[TKR86] N. Trinajstic, D.J. Klein, and M. Randic. On some solved and unsolved problems of chemical graph theory. *International Journal of Quantum Chemistry: Quantum Chemistry Symposium*, 20:699–742, 1986.

[Tos36] L. Toscano. Successioni ricorrenti e polinomi di bernoulli e di eulero. *Accad. Sci. Fis. e Mat. Napoli, Rend., IV.S*, 6:55–61, 1936.

[Tra02] L. Traldi. Chain polynomials and Tutte polynomials. *Discrete Mathematics*, 248:279–282, 2002.

[Tra10] L. Traldi. Weighted interlace polynomials. *Combinatorics, Probability and Computing*, 19:133–157, 2010.

[Tri92] N. Trinajstić. *Chemical Graph Theory.* CRC Press, 2nd edition, 1992.

[Tut54] W.T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian Journal of Mathematics*, 6:80–91, 1954.

[Tut79] W.T. Tutte. All the king's horses. In J.A. Bondy and U.S.R. Murty, editors, *Graph Theory and Related Topics*, pages 15–33. Academic Press, New York, 1979.

[Ver83] M. Las Vergnas. Le polynôme de Martin d'un graphe eulerien. *Annals of Discrete Mathematics*, 17:397–411, 1983.

[Wa] J. Wang and all. Trees characterized by the second smallest real adjoint root - I. Manuscript.

[Wei] E.W. Weisstein. Independence polynomial. http://mathworld.wolfram.com/IndependencePolynomial.html.

[Whi32] H. Whitney. The coloring of graphs. *Ann. Math.*, 33:26–40, 1932.

[WM00] D.J.A. Welsh and C. Merino. The Potts model and the Tutte polynomial. *Journal of Mathematical Physics*, 41(3):1127–1152, 2000.

[Wu82] F-Y. Wu. The Potts model. *Reviews of modern physics*, 54:235–268, 1982.

[WZ11] Y. Wang and B.X. Zhu. On the unimodality of independance polynomials of some graphs. *European Journal of Combinatorics*, 32:10–20, 2011.

[Za04] H. Zhao and all. On properties of adjoint polynomial of graphs and its applications. *AUSTRALASIAN JOURNAL OF COMBINATORICS*, 30:291306, 2004.

[ZL09] H. Zhao and R.Y. Liu. On the nimimum real roots of the adjoint polynomial of a graph. *Discrete Mathematics*, 309:4635–4641, 2009.

[ZX85] F.J. Zhang and L.Y. Xun. Two classes of general polynomials for graphs (Chinese). *Acta Math. Sinica*, 28(1):122–130, 1985.