Connections Between the Matching and Chromatic Polynomials

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The following presentation will focus on two main results:

1. The connection between the matching and chromatic polynomials
2. A formula for the matching polynomial of a general complement of a subgraph of a graph
Our Scope

- The graphs we will relate here are all finite, undirected and contain no loops or multiple edges

- The two basic polynomials mentioned here are the matching polynomial and the chromatic polynomial
The Matching Polynomial

• Let $G=(V,E)$ be a graph. A matching in $G$ is a spanning subgraph of $G$, whose components are nodes and edges (two connected nodes) only.

• Let $a_k$ denote the number of matchings in $G$ with exactly $k$ edges, and $n=|V|$. The matching polynomial is then:

$$M(G; w) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} a_k w_1^{n-2k} w_2^k$$
The Matching Polynomial 2

• 

\[ M(G; w) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} a_k w_1^{n-2k} w_2^k \]

• It can be noted that \( w_1 \) and \( w_2 \) are the weights (also called indeterminates) given to each node and edge, respectively.

• If we put \( w_1 = w_2 = w \), we get the following polynomial:

\[ M(G; w) = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} a_k w^{n-k} \]

which is called the simple matching polynomial
The Chromatic Polynomial

• The chromatic polynomial, noted \( P(G; \lambda) \), represent the number of colorings of the graph \( G \) with \( \lambda \) colors such that adjacent nodes receives different colors.
The Chromatic Polynomial 2

• In this lecture, we will assume:

\[ P(G; \lambda) = \sum_{k=0}^{n} c_k \cdot (\lambda)_{n-k} \]

where \( c_k \) is the number of ways to divide the nodes in \( G \) into \( n-k \) non-empty indistinguishable classes such that each class can be colored the same, and \( (\lambda)_{n-k} = \lambda \cdot (\lambda-1) \cdot \ldots \cdot (\lambda-(n-k)+1) \) is the number of ways of choosing the colors.

• That is called the complete graph basis.
Definitions

• Matching vector – the vector of the nonzero coefficients of $M(G; w)$, written in ascending powers of $w_2$. This vector is denoted by $\underline{m}(G)$

• Graphs $G$ and $H$ are matching equivalent iff $\underline{m}(G) = \underline{m}(H)$

• Chromatic vector – the vector of the nonzero coefficients of $P(G; \lambda)$, written in the complete graph basis. This vector is denoted by $\underline{c}(G)$

• Graphs $G$ and $H$ are chromatically equivalent iff $\underline{c}(G) = \underline{c}(H)$
Definitions 2

• Graphs $G$ and $H$ are co-matching iff $|V(G)| = |V(H)|$ (they have the same number of nodes) and $G$ and $H$ are matching equivalent.

• Graphs $G$ and $H$ are co-chromatic iff $|V(G)| = |V(H)|$ and $G$ and $H$ are chromatically equivalent.

• Graph $G$ is matching unique iff $M(G; w) = M(H; w)$ implies that $G$ is isomorphic to $H$.

• Graph $G$ is chromatically unique iff $P(G; \lambda) = P(H; \lambda)$ implies that $G$ is isomorphic to $H$. 
Basic Graphs

- $P_n$ - chain with $n$ nodes
- $K_n$ - complete graph with $n$ nodes
- $K_{m,n}$ - complete $m$ by $n$ bipartite graph
- $c_n$ - cycle with $n$ nodes
Connection Between the Matching and Chromatic Polynomials

• In the following slides we will obtain the connection between the matching and the chromatic polynomials

• This connection will be used to determine certain qualities of the graph based on those polynomials
Lemma 1

• Every proper coloring of a graph $G$ with $r$ colors induces a partition of $V(G)$ into $r$ parts such that nodes $x$ and $y$ belongs to the same part only if \( \{x, y\} \notin V(G) \)

• Proof: Let us assume that nodes $x$ and $y$ belongs to the same part but \( \{x, y\} \in V(G) \) in this case, we get two adjacent nodes with the same color \( \rightarrow \) the partition wasn’t made by a proper coloring
Lemma 2

• Let G be a graph. Then \( P(G; \lambda) = \sum_{k=0}^{n} a_k \cdot (\lambda)_{n-k} \) where \( a_k \) is the number of partitions of \( \mathbb{V}(G) \) into \( n-k \) non-empty parts such that two nodes \( x \) and \( y \) belongs to the same part only if \( \{x, y\} \notin V(G) \).

• Proof: notice that we need to prove that \( a_k = c_k \) that is, that \( a_k \) is also the number of the color partitions of G with exactly \( n-k \) non-empty parts.
Lemma 2: proof

• Since each partition counted in $a_k$ is also a proper color partition of $G$ with $n-k$ parts (since there are no two adjacent nodes in the same group, we can color the nodes of the group the same), we get $c_k \geq a_k$

• Since each proper color partition of $G$ fulfills the condition that two adjacent nodes cannot be in the same, we also get $c_k \leq a_k$

• This ends our proof that $c_k = a_k$
Lemma 3

- All color partitions of a graph $G$ (partition of the nodes of $G$ such that each group could be colored the same in a legal coloring) holds that all node groups contains no more than two nodes if and only if $\bar{G}$ is triangle free.
Lemma 3: proof

• Suppose that $\overline{G}$ is triangle free. Let us assume that a certain group of nodes in a legal color partition of $G$ contains more than two nodes. In this case, we know that the nodes aren’t adjacent (otherwise it wouldn’t be a legal coloring). Therefore, three of this nodes creates a triangle in $\overline{G}$ – in contradiction to our assumption.
Lemma 3: proof(2)

• $\iff$ suppose that each group of nodes in a legal color partition of $G$ contains no more than two nodes. Thus, $G$ doesn’t contain a legal coloring in which two nodes can be colored the same $\implies G$ doesn’t contain a set of three non-adjacent nodes $\implies \overline{G}$ is triangle free.
Theorem 2

Let $M(G; w) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_k w^{n-k}$ be the simple matching polynomial of graph $G$ and $P(\overline{G}; \lambda) = \sum_{k=0}^{n} a_k \cdot (\lambda)_{n-k}$ be the chromatic polynomial of its complement. Then $a_k \geq b_k$ for all values of $k$. Furthermore, $a_k = b_k$ for all values of $k$ iff $G$ is triangle free.
Theorem 2: proof

• Since $b_k$ is the number of matchings with n-k components, and $a_k$ is the number of color partitions with n-k components, we need to prove that each matching with n-k components in $G$ is counted as a color division with n-k parts in $\overline{G}$
Theorem 2: proof(2)

• Let there be such matching in $G$. For each component in $G$: if it is consisted of a single node, it is clear we can color it in a unique color in $\bar{G}$ (we can always color a node in a unique color)
Theorem 2: proof(3)

• If it is consisted of an edge, then the two nodes are non-adjacent in $\bar{G}$, and therefore can be colored in the same color
Theorem 2: proof(4)

• Given a color partition made in this way, the corresponding matching in G is given by taking two nodes in the same group as an edge, and singleton groups as a single node.

• That leads us to the conclusion that each matching in G can be translated into a unique coloring of $\overline{G}$, and therefore $a_k \geq b_k$. 
Theorem 2: proof(5)

- We will now show that if G is triangle free, than each coloring in G’ translates into an unique matching in G.
- This will prove that \( a_k \leq b_k \), and therefore \( a_k = b_k \).
- Since G is triangle free, and according to lemma 3, each nodes group in all color partitions of G’ contains no more than two nodes.
Theorem 2: proof(6)

• Therefore we have two cases for each group in the color partition: if the group contains a single node, we will take it as a single component in a matching in $G$. 

\[
\begin{align*}
\text{\textcolor{red}{\GG}} & \quad \text{\textcolor{blue}{\GG}} \\
\end{align*}
\]
Theorem 2: proof(7)

- Otherwise, the group must consist of two nodes. Since the nodes are in the same group, they can be colored the same. This implies that the two nodes are non-adjacent in $G'$ $\rightarrow$ they are adjacent in $G$.
- We will take the corresponding edge as a component in the matching in $G$. 

![Graph Diagram](image-url)
Theorem 2: proof(8)

- Since this process is reversible, we get a unique matching for each coloring.
- This proves that $a_k \leq b_k$ and since $a_k \geq b_k$ (by the first part of the theorem) we get that $a_k = b_k$ for all values of $k$.
- We are now left to prove that this only happens if $G$ is triangle free.
Theorem 2: proof(9)

- Let us assume $G$ isn’t triangle free, and prove that in this case $a_k > b_k$ for a certain value of $k$.
- The proof that $a_k = b_k$ for all values of $k$ was still valid if all color partitions of $G'$ had had no more than two nodes.
Theorem 2: proof(10)

• According to lemma 3, in addition to the color partitions with less than three nodes, there is a color partition which includes a group of three (or more) nodes.

• This color partition cannot be translated to a unique matching in G, since all the matching in G have been covered.

• Therefore, if this color partition have $l$ group of nodes, we get $a_k > b_k$ for $k = l$. 
Theorem 2: conclusions

• Let \( G \) be a graph. Then \( P(G; \lambda) = M(\overline{G}; w') \) (where \( w' \) means that \( w^k \) is replaced by \( (\lambda)_k \)) and dually, \( M(\overline{G}; w) = P(G; \lambda') \) where \( \lambda' \) means that \( (\lambda)_k \) is replaced with \( w^k \) or with \( w_1^{2k-n}w_2^{n-k} \) for the general matching polynomial iff \( \overline{G} \) is triangle free [theorem 1]

• \( m(G) = c(\overline{G}) \) iff \( G \) is triangle free [corollary 1.1]

• We can use the matching and chromatic polynomials to determine whether a graph contains triangle or not
Matching Polynomials of Complements of Graphs

• Let H be a graph and G a subgraph of G. A complement of G in H is a graph obtained from H by removing the edges of an isomorph of G.

• If we take H to be a labeled graph, the complement of G in H is unique, and will be denoted as $G_H$

• If $H = K_n$, then the complement of G in H also unique, and denoted as $\overline{G}_n$

• If $H = K_{m,n}$, the complement of G in H will be denoted as $\overline{G}_{m,n}$
Theorem 3

- Let $G$ be a graph with $e$ edges, and let $H$ be a labeled graph containing $G$ as a subgraph.

- Then: $M(G_h; w) = \sum_{k=0}^{e} (-1)^k w^k \sum M(H - V(S_k); w)$ where $S_k$ is a set of $k$ edges of $G$ belonging to a matching in $H$ ($V(G)$ denotes the group of vertices of graph $G$), and the summation is taken over all such matching in $H$, containing $k$ edges of $G$. 
Theorem 3: proof

• We will use the principle of Inclusion and Exclusion to prove the theorem.
• We need to find the contribution of the matchings in $G$ that doesn’t include any of the edges in $G$ – that is $M(H;\omega)$.
• Let $W(k)$ denote the contribution to $M(H;\omega)$ of the matchings that includes at least $k$ edges of $G$.
• Using the principle of Inclusion and Exclusion, we will get $M(G_H;\omega) = \sum_{k=0}^{\epsilon} (-1)^k W(k)$.
Theorem 3: proof(2)

• We are left to find $W(k)$
• Let there be a matching which includes a certain $k$ edges of $G$. The weight of those edges is $w_2^k$. Since the edges are part of a matching, they cover $2k$ nodes of $H$ – we are left to find the donation of the matching which covers $H - V(S_k)$. The contribution of this matching is therefore $w_2^k M(H - V(S_k); w)$
Theorem 3: proof(3)

• Since we need to find the contribution of all such matchings (which contains k edges of G), we get

\[ W(k) = \sum w_2^k M(H - V(S_k); w) = w_2^k \sum M(H - V(S_k); w) \]

• We are left to use the principle of Inclusion and Exclusion to get:

\[ M(G_H; w) = \sum_{k=0}^e (-1)^k W(k) = \sum_{k=0}^e (-1)^k w_2^k \sum M(H - V(S_k); w) \]
Corollary 3.1

• Let $G$ be a graph with $p$ nodes, and $n$ be a positive integer such that $n \geq p$

Also let $M(G; \omega) = \sum_{k=0}^{\lfloor p/2 \rfloor} a_k w_1^{p-2k} w_2^k$

Then: $M(\overline{G}_n; \omega) = \sum_{k=0}^{e} (-1)^k a_k w_2^k M(K_{n-2k}; \omega)$
Corollary 3.1: proof

• From theorem 3 we have:

\[ M(G_H; w) = \sum_{k=0}^{e} (-1)^{k} w_2^{k} \sum M(H - V(S_k); w) \]

• Since in our case \( H = K_n \), we get \( H - V(S_k) = K_{n-2k} \)

• Notice that the second summation is taken over all the matchings in \( K_n \) containing \( k \) edges of \( G \). There are exactly \( a_k \) of those (according to the definition of the matching polynomial). The terms of the summation are equal, and therefore:

\[ M(G_H; w) = M(G_H; w) = \sum_{k=0}^{e} (-1)^{k} a_k w_2^{k} M(K_{n-2k}; w) \]
Corollary 3.2

Let G be a bipartite graph with p nodes and let

\[ M(G; \underline{w}) = \sum_{k=0}^{\lfloor p/2 \rfloor} a_k w_1^{p-2k} w_2^k \]

Then:

\[ M(G_H; \underline{w}) = \sum_{k=0}^{e} (-1)^k a_k w_2^k M(K_{m-k,n-k}; \underline{w}) \]

where \( H = K_{m,n} \)
Corollary 3.2: proof

• The proof is similar to the proof of Corollary 3.1, when we take into consideration that \( H - V(S_k) = K_{m-k,n-k} \) (because \( H \) is a complete bipartite graph), that the terms in the second summation are equal, and that there are exactly \( a_k \) of those.
Corollary 3.2: conclusions

- We notice that the matching polynomial of all complements of a bipartite graph $G$ in $H = K_{m,n}$ is identical, although $G_H$ isn’t
- Conclusion: all complements of $G$ in $K_{m,n}$ are co-matching
- Example:
Theorem 4

• Let $G$ be a graph with $p$ nodes.

\[ M(G;w) = \sum_{k=0}^{\lfloor p/2 \rfloor} a_k w_1^{p-2k} w_2^k \iff M(\overline{G};w) = \sum_{k=0}^{e} (-1)^k a_k w_2^k M(K_{p-2k};w) \]

• Proof:

$\implies$ This direction follows from Corollary 3.1, if we notice that $\overline{G} = \overline{G}_p \quad (n = p)$

$\iff$ Let us assume that $M(\overline{G};w) = \sum_{k=0}^{e} (-1)^k a_k w_2^k M(K_{p-2k};w)$ and $M(G;w) = \sum_{k=0}^{\lfloor p/2 \rfloor} b_k w_1^{p-2k} w_2^k$. It follows that $M(\overline{G};w) = \sum_{k=0}^{e} (-1)^k b_k w_2^k M(K_{p-2k};w) \implies \sum_{k=0}^{e} (-1)^k b_k w_2^k M(K_{p-2k};w) = \sum_{k=0}^{e} (-1)^k a_k w_2^k M(K_{p-2k};w) \implies b_k = a_k$
Theorem 5

• If graphs G and H are matching equivalent pair and F is a complete graph or a complete bipartite graph, then $G_F$ and $H_F$ are matching equivalent also.

• Proof: G and H are matching equivalent, which means their coefficients vector is identical. According to Corollaries 3.1 and 3.2, the Coefficients vectors of $G_F$ and $H_F$ depends only in the coefficients vector of G and H and on $K_n / K_{m,n}$. Therefore, the coefficients vector is identical $\rightarrow$ $G_F$ and $H_F$ are also matching equivalent.
Theorem 6

• Graph $G$ is matching unique iff $\bar{G}$ is

• Proof: $\Rightarrow$ for every graph $H$ which holds that

$$M(\bar{G};w) = M(\bar{H};w)$$

we know that

$$M(G;w) = M(H;w)$$

(according to the formula for the complementary graph – theorem 4) which implies that $G$ is isomorphic to $H$ (from the uniqueness of $G$) $\Rightarrow \bar{G}$ is isomorphic to $\bar{H}$
Theorem 6: proof

• $\Longleftrightarrow$ The proof follows from the first direction – for every graph $H$, if we put $G = \overline{H}$ in the first sentence we will get that graph $\overline{H}$ is matching unique if $\overline{H} = H$ is. That is true for every graph $H$, and this completes our proof.
Lemma 4

Let G and H be matching equivalent graphs with p nodes and m nodes respectively. Then: $M(G; w) = w_1^{p-m} M(H; w)$

Proof: since G and H are matching equivalent, their matching vectors are equal. Therefore:

$$M(G; w) = \sum_{k=0}^{[p/2]} a_k w_1^{p-2k} w_2^k$$

and

$$M(H; w) = \sum_{k=0}^{[p/2]} a_k w_1^{m-2k} w_2^k$$

$$\Rightarrow w_1^{p-m} M(H; w) = w_1^{p-m} \sum_{k=0}^{[p/2]} a_k w_1^{m-2k} w_2^k = \sum_{k=0}^{[p/2]} a_k w_1^{p-2k} w_1^{m-2k} w_2^k = \sum_{k=0}^{[p/2]} a_k w_1^{p-2k} w_2^k = M(G; w)$$
Theorem 7

• Let $G$ be a graph which is matching equivalent graph $H$ which is triangle free. Then: $P(\bar{H}; \lambda) = M(G, w^*)$ where $w^*$ is the transformation which replaced by $w^k$ with $(\lambda)_{k+r}$, where $r = |V(H)| - |V(G)|$

And dually: $M(G, w) = P(\bar{H}; \lambda')$ where $\lambda'$ means that $(\lambda)_{k+r}$ is replaced by $w_1^{2r+2k-p}w_2^{p-r-k}$
Theorem 7: Proof

• From Theorem 1, and since H is triangle free, $P(\bar{H}; \lambda) = M(H; w')$, where $w'$ means that $w^k$ is replaced by $(\lambda)_k$.

According to Lemma 4, $w'_1 M(H; w') = M(G; w')$ where $r = |V(H)| - |V(G)|$, and therefore $P(\bar{H}; \lambda) = M(G; w^*)$ where $w^*$ means that $w^k$ is replaced by $(\lambda)_{k+r}$. 
Theorem 7: Proof(2)

• For the dual part: from theorem 1, since H is triangle free, we get $M(H;w) = P(\bar{H}; \lambda')$ where $\lambda'$ means that $(\lambda)_k$ is replaced by $w_1^{2k-n}w_2^{n-k}$ (where $n$ is the number of nodes in $H$).

From Lemma 4: $w_1^r M(H;w) = M(G;w)$ where $r = |V(H)| - |V(G)|$, and therefore $M(G;w) = P(\bar{H}; \lambda')$ where $w^*$ means that $(\lambda)_{k+r}$ is replaced by $w_1^{2k-n}w_2^{n-k}$
Theorem 8

• For any tree $T$, $m(T) = c(\overline{T}) = \Phi(T)$, where $\Phi(T)$ is the vector of absolute values of the coefficients of the characteristic polynomial of $T$.

• Proof: Since $T$ is a tree, it is triangle free. By corollary 1.1 (which is a conclusion of theorem 2) we get $m(T) = c(\overline{T})$. 
Theorem 8: Proof

• In the second lecture (example 18 in slide 19) we learned that if G is a tree then the characteristic and acyclic matching polynomial coincides (theorem 19, I. Gutman, 1977). Since the absolute values of the coefficients of the acyclic matching polynomial are $m(T)$ we get that $c(T) = \Phi(T)$
Deductions for Chromatic Equivalence and Uniqueness

• In the following slides we will show the connection between matching and chromatic polynomials in terms of equivalence, uniqueness, co-matchingness and co-chromaticness of triangle free graphs
Theorem 9

• Let G and H be a triangle free graphs. Then $\bar{G}$ is chromatically equivalent to $\bar{H}$ if and only if G is matching equivalent to H.

• Proof: $\Rightarrow$ Since $\bar{G}$ and $\bar{H}$ are chromatically equivalent we have $c(\bar{G}) = c(\bar{H})$. From corollary 1.1 we have $m(G) = c(\bar{G})$ and $m(H) = c(\bar{H})$ (because G and H are triangle free). Therefore, $m(G) = c(\bar{G}) = c(\bar{H}) = m(H) \Rightarrow G$ is matching equivalent to H.
Theorem 9: proof

• \( \iff \) assuming \( G \) is matching equivalent to \( H \), we get \( m(G) = m(H) \).

From corollary 1.1 we have \( m(G) = c(\overline{G}) \) and \( m(H) = c(\overline{H}) \) (because \( G \) and \( H \) are triangle free).

We now get: \( c(\overline{G}) = m(G) = m(H) = c(\overline{H}) \Rightarrow \overline{G} \) is matching equivalent to \( \overline{H} \).
Theorem 10

• Let $G$ and $H$ be a triangle free graphs. Then $\overline{G}$ and $\overline{H}$ are co-chromatic pair iff $G$ and $H$ are co-matching pair

• Proof: $\Rightarrow |V(\overline{G})|=|V(\overline{H})|$ and $\overline{G}$ is chromatic equivalent to $\overline{H}$ (by definition). From theorem 9 and since $G$ and $H$ are triangle free, $G$ is matching equivalent to $H$. By the definition of the complement of a graph, we know
Theorem 10: proof

• that $|V(G)|=|V(\bar{G})|$ and $|V(H)|=|V(\bar{H})|$ (the group of nodes stays the same).
  We therefore have $|V(G)|=|V(\bar{G})|=|V(\bar{H})|=|V(H)| \Rightarrow G$ and $H$ are co-matching pair.

• $\iff |V(G)|=|V(H)|$ and $G$ is matching equivalent to $H$ (by definition). From theorem 9, $\bar{G}$ is chromatic equivalent to $\bar{H}$. We saw that $|V(\bar{G})|=|V(G)|=|V(H)|=|V(\bar{H})| \Rightarrow \bar{H}$ and $\bar{G}$ are co-chromatic pair.
Theorem 11

- If $G$ is triangle free and $G$ is chromatically unique, then there is no other graph $H$ which is triangle free such that

- Proof: we will assume that there is a triangle free $H$ such that $M(\bar{G}, w) = M(H, w)$ and $H \not\cong \bar{G}$.

  According to theorem 2, since $G$ and $H$ are triangle free: $M(\bar{G}, w) = P(G; \lambda)$ and $M(H, w) = P(H; \lambda)$

  $\implies P(G; \lambda) = P(H; \lambda) \implies G \cong H$ (since $G$ is chromatically unique) $\implies \bar{G} \cong H$ in contradiction to our assumption