Graph polynomials

Main theme

We deal with a purely **graph theoretical** problem:

Given a **regularly constructed** indexed family of graphs G_n such as
the paths P_n, the circles C_n, the wheels W_n, the cliques K_n, the grids $Grid_{m,n}$

and a graph polynomial $\mathfrak{P}$, such as
the matching , Tutte, clique, cover polynomial

compute all the values $\mathfrak{P}(G_n)$.

Often we have a (linear) recurrence relation, i.e. there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

When is this the case?

We shall see that methods from LOGIC help clarifying the situation.

Graph polynomials

Let $\mathfrak{G}$ be the class of all finite graphs, and

$\mathbb{Z}[\bar{X}]$ be a polynomial ring over $\mathbb{Z}$ with $\bar{X} = (X_1, \ldots, X_m)$.

A graph polynomial is a map

$$\mathfrak{P} : \mathfrak{G} \to \mathbb{Z}[\bar{X}]$$

which is invariant under graph isomorphisms.

There are obvious generalizations to

- vertex-labeled and edge-labeled (signed) graphs;
- hypergraphs; and to
- relational structures.
- Knot polynomials are defined on signed graphs, the shading graphs of knot diagrams, where invariance is additionally required under the Reidemeister moves.

Examples of graph polynomials

Well studied graph polynomials are:

- The chromatic polynomial;
  (W.T. Tutte, 1954)
- The Tutte polynomial and its colored versions
  (W.T. Tutte 1954, B. Bollobas and O. Riordan, 1999);
- The various matching polynomials;
  (O.J. Heilman and E.J. Lieb, 1972)
- Various clique and independent set polynomials
  (I. Gutman and F. Harary 1983)
- The Farrel polynomials,
  (E.J. Farrell, 1979)
- The various cover polynomials for digraphs
  (F.R.K. Chung and R.L. Graham, 1995)

The matching polynomial, I

For a graph G, the **matching polynomial** $\mu(G,X) \in \mathbb{Z}[X]$ is defined by

$$\mu(G,X) = \sum m_k(G) \cdot X^k$$

where $m_k(G)$ is the number of k-matchings of G.

We compute $\mu(P_n, X)$:

We use auxiliary polynomials

$$\mu^+(P_n,X) = \sum m_k^+(P_n) \cdot X^k$$

and

$$\mu^-(P_n,X) = \sum m_k^-(P_n) \cdot X^k$$

where $m_k^+(P_n)$ and $m_k^-(P_n)$ is the number of k-matchings of P_n which **includes**, respectively **excludes** the last vertex.

Clearly we have $m_k(P_n) = m_k^+(P_n) + m_k^-(P_n)$ hence

$$\mu(P_n,X) = \mu^+(P_n,X) + \mu^-(P_n,X)$$

The matching polynomial, II

It is easy to see that

$$\mu^{-}(P_{n+1}) = \mu^{-}(P_n) + \mu^{+}(P_n)$$
$$\mu^{+}(P_{n+1}) = X \cdot \mu^{-}(P_n)$$

For $\bar{\mu}_n = (\mu^{-}(P_n), \mu^{+}(P_n))^t$ we get

$$A\bar{\mu}_n = \bar{\mu}_{n+1}$$

with

$$a_{1,1} = 1, a_{1,2} = 1, a_{2,1} = X, a_{2,2} = 0$$

The characteristic polynomial of A is

$$det(\lambda\mathbf{1} - A) = \lambda^2 - \lambda - X$$

so we get the linear recurrence relation (independent of n)

$$\mu(P_{n+2}) = \mu(P_{n+1}) + X \cdot \mu(P_n)$$

The vertex-cover polynomial

For a graph G, the **vertex-cover polynomial** $vc(G, X) \in \mathbb{Z}[X]$ is defined by

$$vc(G, X) = \sum vc_k(G) \cdot X^k$$

where $vc_k(G)$ is the number of k-vertex-covers of G.

- $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$
- $vc(C_{n+1}, X) = X \cdot vc(C_n, X) + X^2 \cdot vc(C_{n-2}, X)$
- Let L_n be the graph which consists of n isolated loops.
  $vc(L_{n+1}, X) = X \cdot vc(L_n, X) = X^n$
- For the wheel graph W_n we have
  $vc(W_{n+1}, X) = X \cdot vc(W_n, X) + X^n = X \cdot vc(W_n, X) + X \cdot vc(L_n, X)$
  hence, using the characteristic polynomial of the matrix, $A = (a_{i,j})$ with
  $a_{1,1} = a_{1,2} = a_{2,2} = X$ and $a_{2,1} = 0$
  $vc(W_{n+1}, X) = 2X \cdot vc(W_n, X) - X^2 \cdot vc(W_{n-1}, X)$

F.M. Dong, M.D. Hendy, K.L. Teo and C.H.C. Little, The vertex-cover polynomial of a graph, Discrete Mathematics 250 (2002), 71-78

$\mathfrak{P}$-recursive families of graphs, I

Let $\mathfrak{P}$ be a graph polynomial and $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ be a family of graphs.

$\mathcal{G}$ is said to be **$\mathfrak{P}$-recursive** if there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

Let P_n be the path on n vertices. We get, for sufficiently large n,

- for the chromatic polynomials: $c(P_{n+1})(\lambda) = c(P_n, \lambda)(\lambda) \cdot (\lambda - 1)$.
- for the clique polynomials:
  $cl(P_{n+1})(X) = (1 + X) + cl(P_n)(X)$,
  $cl(K_{n+1})(X) = \sum_k^{n+1} \binom{n}{k} X^k = (X+1)^{n+1} = (X+1) \cdot cl(K_n)(X)$
- for the matching polynomials: $\mu(P_{n+1})(X) = X \cdot \mu(P_{n-1})(X) + \mu(P_n)(X)$,
- for the Tutte polynomials: $T(P_{n+1})(X, Y) = Y \cdot T(P_n)(X, Y)$.
- for the vertex-cover polynomials: $vc(P_{n+1}, X) = X \cdot vc(P_n, X) + X \cdot vc(P_{n-1}, X)$

Previous work, I

N.L. Biggs, R.M. Damerell and D.A. Sands, 1972

In 1972 N.L. Biggs, R.M. Damerell and D.A. Sands introduced recursive families of graphs.

These are our T-**recursive families of graphs**
where T is the Tutte polynomial.

They show that several families of graphs are recursive (in their sense).
Among them there are:

cycles, ladders and wheels

All these families have in common that they can be constructed from an initial graph by the repeated application of a fixed graph operation

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N.L. Biggs, R.M. Damerell and D.A. Sands,
Recursive families of graphs, J. Combin. Theory Ser. B 12 (1972), 123-131

Previous work, II

M. Noy and A. Ribó, 2004

In 2004 M. Noy and A. Ribó study
which graph families G_n,
constructed from an initial graph G_0,
by the repeated application of a fixed graph operation $F(G)$,
are T-**recursive families of graphs**.

They introduce a notion of **recursively constructible families of graphs**, and show that every such family is T-recursive.

Their notion is reminiscent of certain graph grammars.

M. Noy and A. Ribó, Recursively constructible families of graphs, Advances in Applied Mathematics 32 (2004) 350-363.

Our work

We use the finite model theory
of Monadic Second Order Logic (MSOL)
to extend these results in several ways:

- We prove that for every $\mathfrak{P}$ from a wide class of graph polynomials, the MSOL-definable graph polynomials, every recursively constructible family G_n is $\mathfrak{P}$-recursive.

- We extend the result to the class of iteration families of graphs which is proper extension of the class of recursively constructible families.

- We extend the result to signed graphs and knot diagrams and to various knot polynomials.

- We extend the result to hypergraphs and relational structures.

Recursively constructible families, I

In the absence of the formalisms of graph grammars Noy and Ribó give an adhoc definition of

repeated fixed succession
of elementary operations

which can be applied to a graph with a *context*, i.e. a labeled graph.

Let F denote such an operation.

Given a graph (with context) G, we put

$$G_0 = G, G_{n+1} = F(G_n)$$

Then the family

$$\mathcal{G} = \{G_n : n \in \mathbb{N}\}$$

is called **recursively constructible** using F, or an F-**iteration family**.

Recursively constructible families, II

Given a graph polynmial $\mathfrak{P}$,

the question now is to find

a characterization of those operations F,

for which a linear recurrence for the polynomials $\mathfrak{P}(G_n)$ holds.

M. Noy and A. Ribó give only a **suffient condition**
in the case of the **Tutte polynomial**.

General strategy

We proceed as outlined in the case of the matching and the vertex-cover polynmial.

To compute $\mathfrak{P}(G_{n+1})$, we try to find,
depending on $\mathfrak{P}$ and, possibly, on G_0),
but **independently of** n

- an $m \in \mathbb{N}$,
- auxiliary polynomials $\mathfrak{P}_i(G_{n+1}), i \leq m$,
- and a matrix $A = (a_{i,j}) \in \mathbb{Z}[\bar{X}]^{m \times m}$

such that

$$\mathfrak{P}_j(G_{n+1})(\bar{X}) = \sum_i a_{i,j}(\bar{X}) \cdot \mathfrak{P}_i(G_n)(\bar{X})$$

Then we use the **characteristic polynomial of** A

to convert this into a **linear recurrence** relation.

Where logic enters for the polynomials?

The polynomial is of the form

$$\mathfrak{P}(G) = \sum_{(V,E') \in K_1} \left(\prod_{E' \subseteq E} t(\bar{X}) \right)$$

or

$$\mathfrak{P}(G) = \sum_{(V',E|V') \in K_2} \left(\prod_{V' \subseteq V} t(\bar{X}) \right)$$

where $t(\bar{X})$ is a fixed term in the indeterminates $\bar{X}$
and K_1 or K_2 are definable in

Monadic Second Order Logic (MSOL).

We call them $MSOL$-**definable graph polynomials**.

There are more general versions.

$MSOL$-definable polynomials, I

The **vertex-cover polynomials** are of the form

$$\mathfrak{P}(G) = \sum_{(V,E,V') \in K_2} \left(\prod_{V' \subseteq V} X \right)$$

because saying that V' is a vertex-cover of (V, E) is $MSOL$-definable.

Rearranging the terms we get

$$vc(G) = \sum_{(V,E,V') \in VC} \left(\prod_{V' \subseteq V} X \right) = \sum_k vc_k(G) \cdot X^k$$

Note: The second order variable for V' is needed.

$MSOL$-definable polynomials, II

The **matching polynomial** is of the form

$$\mu(G) = \sum_{(V,E') \in Matching} \left(\prod_{E' \subseteq E} X \right)$$

However, being a matching is

- NOT $MSOL$-definable if graphs are represented as $G = (V, E)$.
- but IS $MSOL$-definable, if the graph is represented by its incidence graph $I(G) = (V \cup E, R)$.

For the **Tutte polynomial**, we have to add a linear order on the edges, to make it $MSOL$-definable, and note, that the Tutte polynomial is then indepent of the order on the edges.

Where logic enters in the operation F?

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures. We write $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$,
if $\mathfrak{A}$ and $\mathfrak{B}$ cannot be distinguished
by $MSOL(\tau)$-formulas of quantifier rank q.

An operation F on τ-structures is

$MSOL$-**smooth**

if whenever $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$, then also $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$.

The operation F should be $MSOL$-smooth for the presentation of the graphs, for which the polynomial is $MSOL$-definable.

For forming the cliques K_n we need the operation of adding a vertex connected to all previous vertices.

This is $MSOL$-smooth for $G = (V, E)$ but not for $I(G) = (V \cup E, R)$.

k-structures

A vocabulary τ is a set of relation symbols.

A τ-structure $\mathfrak{A}$
is an interpretation of the vocabulary τ over a non-empty universe A.

For $k \in \mathbb{N}$, a k-τ-structure is a τ-structure with k additional unary relations $C_1^A, \ldots C_k^A$, called colours.

We denote by τ_k the vocabulary $\tau \cup \{C_1, \ldots, C_k\}$.

Basic operations on k-τ-structures

$Add_{C_i}(\mathfrak{A})$**:** For $i \leq k$, add a new element to A of colour C_i.

$\rho_{i,j}(\mathfrak{A})$**:** For $i, j \leq k$, recolour all elements of A of colour i with colour j.

$\eta_{R,i_1,\ldots,i_m}(\mathfrak{A})$**:** For $R \in \tau$ an m-ary relation symbol
and for each $a_1 \in C_{i_1}^A, \ldots, a_m \in C_{i_m}^A$
add the tuple $(a_1, \ldots, a_m)$ to R^A.

$\delta_{R,i_1,\ldots,i_m}(\mathfrak{A})$**:** For $R \in \tau$ an m-ary relation symbol
and for each $a_1 \in C_{i_1}^A, \ldots, a_m \in C_{i_m}^A$
delete the tuple $(a_1, \ldots, a_m)$ from R^A.

Quantifierfree transductions:
For each $R \in \tau_k$ of arity $\alpha(R)$
let $\phi_R(x_1, \ldots, x_{\alpha(R)})$ be
a quantifierfree τ_k formula with free variables as indicated.
A quantifier free transduction redefines all the predicates R^A in $\mathfrak{A}$ by ϕ_R^A.

$MSOL$-elementary and $MSOL$-smooth operations

An operation F on τ_k-structures is $MSOL$-elementary if F is a finite composition of basic operations on τ_k-structures.

Proposition:

Let F be $MSOL$-elementary and $\mathfrak{A}$ and $\mathfrak{B}$ two τ_k structures with $\mathfrak{A} \equiv_q^{MSOL} \mathfrak{B}$, then $F(\mathfrak{A}) \equiv_q^{MSOL} F(\mathfrak{B})$,
Hence, F is a $MSOL$-smooth.

Proposition:

Let F be $MSOL$-elementary and $\mathcal{G}$ be an F-iteration family. Then $\mathcal{G}$ is of bounded clique-width.

Corollary:

$I(K_n)$, $Grid_{n,n}$ are not F-iteration families for any F which is $MSOL$-elemntary.

Main Result

THEOREM: Let

- F be an $MSOL$-smooth operation on τ_k-structures.
- $\mathfrak{P}$ be a τ-polynomial which is $MSOL(\tau)$-definable.
- $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be an F-iteration family of τ-structures.

Then $\mathcal{A}$ is $\mathfrak{P}$-recursive, i.e. there is $q \in \mathbb{N}$,
and polynomials $p_1, \ldots, p_q \in \mathbb{Z}[\bar{X}]$ such that for sufficiently large n

$$\mathfrak{P}(G_{n+q+1}) = \sum_{i=1}^{q} p_i \cdot \mathfrak{P}(G_{n+i})$$

Proof ingredients

- For fixed q and a fixed number of free variables, there are, up to logical equivalence, only finitely many $MSOL(\tau)$-formulas of quantifier rank q.

  Let $\bar{\mathfrak{P}} = (\bar{\mathfrak{P}}_1, \ldots \bar{\mathfrak{P}}_\alpha)$ be the vector of all $MSOL(\tau)^q$-definable polynomials.

- Feferman-Vaught Theorem for $MSOL$-definable graph polynomials

  J.A. Makowsky, Algorithmic uses of the Feferman-Vaught Theorem, Annals of Pure and Applied Logic, 126 (2004), 159-213

- Bilinear version of the Feferman-Vaught Theorem for graph polynomials.

  With an $MSOL$-elementary operation F and a fixed q there is a matrix M_F such that

$$\bar{\mathfrak{P}}(F(G)) = M_F \cdot \bar{\mathfrak{P}}(G)$$

- Use the characteristic polynomial of M_F.