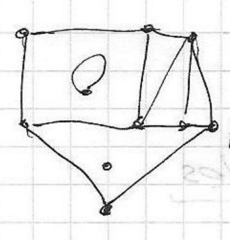
25.10.10
[illegible]
הרצאה 2

What can we say in FOL .. RA (DRC = FOL) about graphs?



$E(a,b)$ there is an edge between a and b

Set V

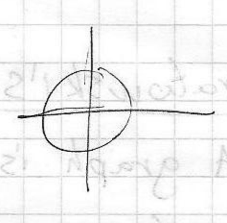
$E \subseteq V^2$ symetric

Query $\varphi(\bar{x})$ with free varibles

Boolean query Ψ without free varibles

$\{(x,y) : x^2+y^2=1\} \subseteq \mathbb{R}^2$

$\{(x,y) : x^2+y^2 \leq 1\}$



$\forall x\ (x^2 \geq 0)$ true or false

1. Connected
2. Given a, b, is there a path from a to b?
3. Is there a cycle?
4. Find longest path from a to b.
5. Given a, find all b reachable from a.
6. Given graph G find $\bar{G}$

a. G has no loops

$\neg\exists x\ E(x,x)$ eq $\forall x\ \neg E(x,x)$

b. There is no (2-cycle)

c. There is no (triangle)

$\neg\exists x_1 x_2 x_3 [E(x_1,x_2) \wedge E(x_2,x_3) \wedge E(x_3,x_1) \wedge \neg(x_1 \approx x_2) \wedge \neg(x_2 \approx x_3) \wedge \neg(x_3 \approx x_1)]$

Find a boolean query $\Psi_{no\text{-}cycle}$ such that (first order sentence)

For every finite V, and for every $E \subseteq V^2$ symmetric $G=(V,E) \models \Psi$ iff G has no cycles (planner)

פלאנרי - ניתן לצייר בלי לחתוך את הקשתות.

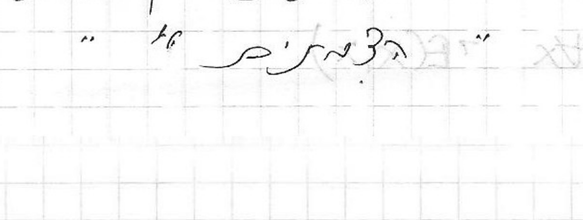
A graph is *plane* if it is given as a drawing in the euclidian plane $\mathbb{R}^2$ with crossing.

<u>Kuratowski's theoram</u> (1930 [illegible])

A graph is planner iff it ~~has~~ does not contain K_5 or $K_{3,3}$

דוגמאות:

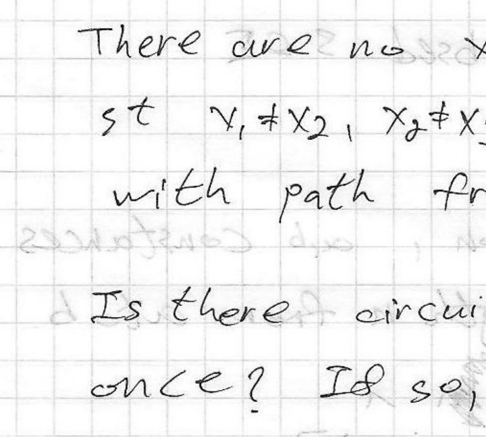
K_5 $K_{3,3}$



לכן ניתן להגדיר פורמולה שאומרת הוא פלאנרי אם אין לו בגרף $K_{3,3}$ או K_5, ובאורך של מספר קבוע וסופי של משתנים (ובמאות).

גרף אוילריאני: מעבר על כל הקשתות בדיוק פעם אחת.

גרף המילטוני: " " "כל הצמתים" " " .



V is very big

במקרה זה אין $K_{3,3}$, אבל יש כזה אם נעקוב למסלולים של הקשתות. מכאן משמעות המילה "contain" "מכיל".

There are no x_1, x_2, x_3 st $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_1$ with path from $x_1 x_2$ and $x_2 x_3$ and $x_3 x_1$

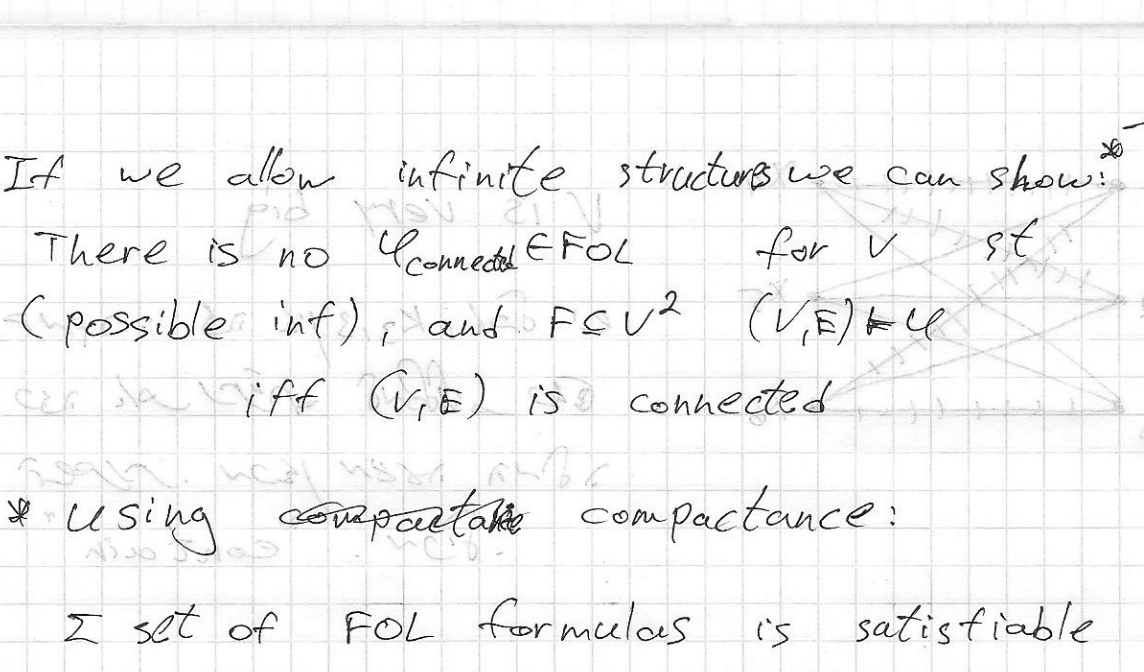
Is there circuit which uses every edge exactly once? If so, the graph is Eulerian.

<u>Theorem:</u> The following are not expresable in FOL

a. The universe has an even/odd number of elements.

b. A graph is connected.

Proof uses <u>Pebble games</u> (~~Eran~~ Fraïsse) E-F-games Ehrenfeucht



If we allow infinite structures we can show:*

There is no $\varphi_{connected} \in FOL$ for V st (possible inf), and $E \subseteq V^2$ $(V,E) \models \varphi$ iff (V,E) is connected

* using ~~compactable~~ compactance:

<u>Proof:</u> Σ set of FOL formulas is satisfiable iff for every finite subsed $S \subseteq \Sigma$ S is satisfiable

1. Σ-vocabilary, E-binary relation, a,b constances

There is no path of length n from a to b

$\varphi_n = \neg\exists x_1, \dots, x_{n-1} [a \neq x_i, b \neq x_i, x_i \neq x_j \wedge E(a,x_1) \wedge E(x_1,x_2) \wedge \dots \wedge E(x_{n-1},b)]$

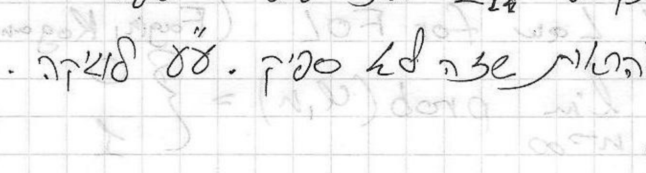
$\Sigma = \{\varphi_n : n \in \mathbb{N}\}$

Every $\langle V,E,a,b\rangle \models \Sigma$ is not connected.

2. Want to show:

Every finite subset $S \subseteq \Sigma$ has a connected model G_S

Fix S to be a finite and let n_S be the ~~biggest~~ largest index of $\{\varphi_n : n \in \mathbb{N}\}$ with $\varphi_{n_S} \in S$



$\{0,1,\dots,n_S+1\} = V_S$

$E_S = succ|_{V_S}$

$a=0$, $b=n_S+1$

הוספנו $\{\varphi_{connected}\}$ ע״פ לעיל, ניתן להראות

$\Sigma_1 = \Sigma \cup \{\varphi_{connected}\}$

$(\Sigma = \varphi_n : n \in \mathbb{N})$

Σ_1 is not satisfiable

(הטענה נכונה כאשר [illegible] דשורות, ע״י שילוב עם Σ נקבל Σ_1 שאין ע״פ משפט הקומפקטיות ניתן להראות שזה לא ספיק. ע״פ לוגיקה...)

<u>תרגיל</u>

$prob(\varphi, n) = ?$

How many structures $\langle [n], E\rangle$? (E binary relation)

- 2^{n^2} # of relations on $[n]$

$S_\varphi(n) :=$ # relations E on $[n]$ s.t. $\langle [n], E\rangle \models \varphi$

$\Rightarrow prob(\varphi, n) = \frac{S_\varphi(n)}{2^{n^2}}$

$prob(\varphi_{even}, 2n) = 1$

$prob(\varphi_{even}, 2n+1) = 0$

$\Rightarrow \lim_{n\to\infty} prob(\varphi_{even}, n) =$ does not exist.

$[n] = \{0, \dots, n-1\}$

$\varphi_{one_edge} := \exists x_1, x_2 [E(x_1,x_2) \wedge \forall y_1, y_2 (E(y_1,y_2) \rightarrow y_1 = x_1 \wedge y_2 = x_2)]$

$\Rightarrow prob(\varphi_{one_edge}) = \lim_{n\to\infty} \frac{\binom{n}{2}+n}{2^{n^2}} \longrightarrow 0$

$\binom{n}{2} + n = n^2$

<u>Theorem</u> 0-1 Law for FOL (Fagin, Kogan, ... 1970[illegible])

If $\varphi \in FOL$ $\lim_{n\to\infty} prob(\varphi, n) = \begin{cases} 0 \\ 1 \end{cases}$