THREE TREE-PATHS

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Abstract

Itai and Rodeh [IR] have proved that for any 2-connected graph $G$ and any vertex $s \in G$ there are two spanning trees such that the paths from any other vertex to $s$ on the trees are disjoint. In this paper the result is generalized to 3-connected graphs.
1. Introduction.

A graph $G=(V(G),E(G))$ consists of a set of vertices $V(G)$ and a set $E(G)$ of unordered pairs of $V(G)$ called edges. An $x-y$ path $\pi[x,y]$ in a graph $G$ is a sequence of distinct vertices of $V(G)$ $(x = v_0,v_1,\ldots,v_m = y)$ such that $(v_{i-1},v_i) \in E(G)$ ($i=1,\ldots,m$). Two paths $\pi_1[x_1,y_1]$ and $\pi_2[x_2,y_2]$ are disjoint if they have no common vertices except perhaps $\{x_1,y_1\}\cap\{x_2,y_2\}$. Let $\kappa(x,y;G)$ denote the maximum number of disjoint $x-y$ paths in $G$. A graph $G$ is $k$-connected if between any two vertices of $G$ there are at least $k$ disjoint paths. The connectivity, $\kappa(G)$, of $G$ is the maximum integer $k$ such that $G$ is $k$-connected, i.e., $\min\{\kappa(x,y;G) : x,y \in V(G)\} = k$. If $v \in V(G)$ we denote by $\Gamma(v)$ the set of all vertices adjacent to $v$ while $\Gamma_E(v)$ denotes the set of edges adjacent to $v$. The degree of $v$, $\deg(v;G)$, is equal to the cardinality of $\Gamma_E(v)$. A vertex $v$ is a leaf if $\deg(v;G)=1$.

We abuse the notation and say that an edge or a vertex belongs to a graph $(e \in G$ or $v \in G$ instead of $e \in E(G)$ or $v \in V(G)$). Also, $G-e$ denotes the graph from which $e$ was deleted from the edge set (likewise $G+e$). For all other notations refer to Bondy & Murty ([BM76]).

Let $T$ be a spanning tree of $G$ and $v,w \in V(G)$ then $T[v,w]$ is the path from $v$ to $w$ on $T$. The following result appears in [IR]:

**Theorem 1.1**

If $G$ is a 2-connected graph and $s \in V(G)$, then there exist two spanning trees of $G$, $T_1, T_2$, such that for every vertex $v \in G$ the paths $T_1[v,s]$ and $T_2[v,s]$ are disjoint.

Itai and Rodeh [IR] used this result to develop distributed algorithms which are resilient to the failure of a single line (or processor). To increase reliability more spanning trees are needed. The purpose of this paper is to prove the following:

**Theorem 1.2**

If $G$ is a 3-connected graph and $s \in V(G)$, then there exist three spanning trees of $G$, $T_i$ ($i=1,2,3$), such that for every vertex $v \in G$, the paths $T_i[v,s]$ ($i=1,2,3$) are disjoint.
We start the paper by proving general properties of 3-connected graphs (Section 2). To prove the theorem, in Section 3 we define a property, Extended 3-Tree Path property (E3TP), which implies the existence of the three trees. The remainder of the paper shows that 3-connectivity implies E3TP. To that end, we investigate the structure of a minimal counterexample (MCE). In Section 4 we show that in every MCE, $deg(s)=3$. In Section 5 it is shown that an MCE cannot contain certain subgraphs. The proof is completed in Section 6. While Section 7 contains the conclusions.

Our proof methodology is to obtain from an MCE, $G$, a smaller graph $G'$, then show that $G'$ is 3-connected, thus by definition of MCE, $G'$ satisfies E3TP. Thus $G'$ contains three trees, which we modify to get three trees in $G$.

2. Connectivity

Let $U \subseteq V(G)$ then $E(U;G)$ consists of all the edges of $G$ both endpoints of which belong to $U$ and the subgraph induced by $U$ is the graph $G[U]=(U,E(U;G))$. A set $S \subseteq V(G)$ is a separation set of $G$ if its deletion from $G$ increases the number of connected components. A separation set is a minimum separation set if its cardinality is minimum.

The concatenation $\pi \sigma$ of two edge-disjoint paths $\pi=[v,x]$ = $(v=v_0,...,v_n=x)$ and $\sigma=[x,w]$ = $(x=v_n,v_{n+1},...,v_m)$ is the $v-w$ path obtained by deleting cycles from the sequence $(v=v_0,...,v_m=w)$.

Lemma 2.1

Let $G$ be a 3-connected graph, $S=\{v_1,v_2,v_3\}$ a separation set of $G$ and $e=(v_1,v_2)$. Then for all vertices $x,y \in V(G)-\{v_1,v_2\}$, $\kappa(x,y;G-e)=3$.

Proof

Case 1: $v_3 \{x,y\}$: Thus, $x,y \in V(G)-S$. If $x$ and $y$ belong to different connected components of $G-S$ then there exist three $x-y$ paths which do not use $e$. Thus, assume $x$ and $y$ are in the same component. At most one $x-y$ path uses $e$. In this path, $e$ can be replaced by a $v_1-v_2$ path which except for its endpoints lies entirely in a component not containing $x$ and $y$. 
Case 2: \( v_3 \in \{x, y\} \): W.l.g., \( v_3=y \). In \( G \) there are three \( x-v_3 \) paths. At most one of them, say \( \pi_1 \), uses \( e \). (If none uses \( e \), we are done.) The other two paths lie entirely in the connected component of \( G-S \) which contains \( x \). Let \( \pi_1 = \pi_1[x,v_i](v_i,v_{3-i})\pi_i[v_{3-i},v_3] \), for \( i=1 \) or \( i=2 \). Let \( \sigma \) be a \( v_i-v_3 \) path in the connected component of \( G-S \) not containing \( x \). Then replace \( \pi_1 \) by the path \( \pi_1'=\pi_i[x,v_i]\sigma \) to obtain three \( x-v_3 \) paths in \( G-e \).

The following two lemmas follow from Menger’s Theorem,

**Lemma 2.2:**

Let \( v \in V(G) \) and \( \text{deg}(v;G)\geq k \). If for every two vertices \( x,y \neq v \), \( \kappa(x,y;G)\geq k \) then \( \kappa(G)\geq k \).

**Proof**

If \( \kappa(G)<k \) there exists a set \( W \) of cardinality \( \leq k-1 \) separating \( v \) from some other vertex, say \( x \). Since \( \text{deg}(v)\geq k \), \( v \) has a neighbor \( y \in W \). Thus \( W \) separates \( y \) from \( x \), which implies \( \kappa(x,y;G)\leq k-1 \).

An edge \( e \) is **subdivided** when it is deleted and replaced by a path of length two connecting its endpoints, the internal vertex of this path being a new vertex. The graph \( G \) is a **subdivision** of \( H \) if \( G \) can be obtained from \( H \) by a series of edge subdivisions. Let \( h(G) \) be the minimum size graph \( H \) such that \( G \) is a subdivision of \( H \).

**Lemma 2.3**

Let \( G \) be a 3-connected graph, \( S=\{v_1,v_2,v_3\} \) a separating set and \( e=(v_1,v_2) \). Then \( H=h(G-e) \) is 3-connected.

**Proof**

It is sufficient to show that every two vertices \( x,y \in V(H) \) are connected by three disjoint paths. By Lemma 2.1, \( \{x,y\}\cap\{v_1,v_2\}\neq\emptyset \). Thus, w.l.g., \( x=v_1 \).

**Case 1:** \( \text{deg}(v_1;G)=3 \): we are done since \( \text{deg}(v_1;G-e)=2 \), implying that \( v_1H \).

**Case 2:** \( \text{deg}(v_1;G)>3 \): If \( \text{deg}(y;G-e)=2 \) then \( y=v_2 \) and by reversing the roles of \( x \) and \( y \) we return to Case 1. Thus, \( \text{deg}(y;G-e) \geq 3 \). Let \( W=\{w_1,w_2\} \subseteq V(H) \) separate between \( v_1 \) and \( y \) in \( G-e \) (thus \( v_1 \)
and \( y \) are not connected in \( G - e \). Since \( \deg (v_1; G - e) \geq 3 \), in \( G - e \) \( v_1 \) has a neighbor \( u_1W \cup \{ y \} \). If \( y \neq v_2 \), \( W \) separates between \( u_1 \) and \( y \) contrary to Lemma 2.1. Otherwise, \( y = v_2 \) has a neighbor \( u_2W \cup \{ v_1 \} \) and \( W \) separates between \( u_1 \) and \( u_2 \) again contradicting Lemma 2.1.

Remark: Figure 2.1 shows that the condition \( e \in E(S) \) is necessary.

Let \( e = (x,y) \) be an edge of \( G \). The contraction of \( e \) in \( G \) (\( G / e \)) is the graph resulting from \( G \) by replacing the vertices \( x,y \) by a single vertex \( xy \) and connecting it to \( \Gamma (x; G) \cup \Gamma (y; G) - \{ x,y \} \).

**Corollary 2.4:**

Let \( G \) be a 3-connected graph and \( e = (v_1, v_2) \in E(G) \). Then either \( G / e \) or \( h(G - e) \) is 3-connected.

**Proof**

If \( G / e \) is only 2-connected then \( v_1 \) and \( v_2 \) belong to some minimum separating set \( S = \{ v_1, v_2, v_3 \} \), to which the previous lemma may be applied.

We conclude this section with a lemma to be used in Section 6.

Let \( x \in G \) and \( S \subseteq V(G) \). An \( x \)–\( S \) fan is a set of disjoint paths \( \pi_1[x,y_1], \ldots, \pi_k[x,y_k] \) were all the vertices \( y_i \) are distinct vertices of \( S \); \( k \) is the size of the fan.

**Lemma 2.5**

Let \( B \subseteq V(G) \) be a set of vertices such that \( \kappa(x,y; G) = k \) for all \( x,y \in B \). If for every other vertex \( v \) there is a \( v \)–\( B \) fan of size \( k \), then \( \kappa(G) = k \).

![Figure 2.1](image-url)
Proof

Let \( v, w \) be two vertices of \( G \) and \( S \) a separation set of \( G \) which separates \( v \) from \( w \), such that \( |S| < k \). By our assumption \( S \) cannot separate all \( v-B \) paths nor all \( w-B \) paths. Thus, for some \( x, y \in B \) there are two paths \( \pi[v, x] \) and \( \sigma[w, y] \) which do not pass through vertices of \( S \). As \( x, y \in B \) there are \( k \) disjoint \( x-y \) paths, and \( S \) cannot separate all of them. Let \( \tau[x, y] \) be an \( x-y \) path not passing through \( S \). We conclude that \( \rho = \pi[v, x] \tau[x, y] \sigma[y, w] \) is a \( v-w \) path not passing through \( S \) (\( \rho \) is a path since by our definition of concatenation of paths cycles are deleted); thus \( S \) does not separate \( v \) from \( w \).

3. Extended three tree path property.

Let \( S \) be a set. The relation \( \subseteq S \times S \) is an order on \( S \) if \( \subseteq \) is irreflexive, antisymmetric, transitive and total, i.e., for all \( u, v \in S \) either \( v \not\subseteq u \) or \( u \not\subseteq v \).

To prove Theorem 1.2 we need the following definition of an \( s-t \) ordering which is equivalent to the \( s-t \) numbering found in [Ev] and used in [IR]. Let \( e=(s,t) \) be an edge in a graph \( G \). An \( s-t \) ordering of \( G \) is an order \( \subseteq \) on the vertices of \( G \) such that:

(S1) \( s \) is minimum and \( t \) is maximum, i.e. for all \( v \neq s, t \) \( s \not\subseteq v \).  
(S2) For every vertex \( v \in V(G) - \{s,t\} \) there are two vertices \( v_1, v_2 \in \Gamma(v) \) such that \( v_1 \not\subseteq v \) and \( v \not\subseteq v_2 \).

In [Ev] it is proven that such a ordering exists for all 2-connected graphs.

Let \( \subseteq \) be an order on the vertices of the graph \( G \). A path \( \pi=(v_1, \ldots, v_n) \) \( \subseteq \)-increases (\( \subseteq \)-decreases) if \( v_1 \not\subseteq v_2 \) \( \ldots \) \( v_n \not\subseteq v_1 \) (\( v_n \not\subseteq v_{n-1} \) \( \ldots \) \( v_1 \not\subseteq v_{n-1} \)). Let \( s \) be the minimum vertex of \( \subseteq \). A pair of spanning trees, \( (T_1,T_2) \) are ordered by \( \subseteq \) if for every vertex \( v \) the path \( T_1[v,s] \) \( \subseteq \)-decreases and the path \( T_2[v,s] \) \( \subseteq \)-increases. (In this case \( T_2[v,s] \) \( \subseteq \)-increases and \( T_1[v,s] \) \( \subseteq \)-decreases.) The motivation for this definition is that when \( u \not\subseteq v \), \( T_1[u,s] \) and \( T_2[v,s] \) are disjoint. The following lemma is useful in proving that an order is an \( s-t \) ordering.
Lemma 3.1

If the pair of spanning trees \((T_1, T_2)\) is ordered by \(\leq\) which satisfies (S1) for some edge \((s,t)\) then \(\leq\) is an \(s\to t\) ordering.

Proof

Let \(v \neq s, t\) be a vertex of \(G\). Since \((T_1, T_2)\) are ordered by \(\leq\), the path \(T_1[v,s] = (v,v_1)T_1[v_1,s]\) is \(\leq\)-decreasing. Therefore \(v_1\) is a neighbor of \(v\) satisfying \(v_1 \leq v\). Considering \(T_2[v,t]\) provides us with another neighbor \(v_2\) satisfying \(v \leq v_2\).

We now show a simple application of this lemma to be used in the proof of our last theorem.

Lemma 3.2

Let \(T_1, T_2\) be spanning trees of a graph \(G\) and \(\leq\) an \(s\to t\) order of \(G\). If \((T_1, T_2)\) is ordered by \(\leq\) and \(\Gamma(s;T_1) = \{v\}\) then there exists an \(s\to v\) order \(\leq'\) such that \((T_2, T_1)\) is ordered by \(\leq'\).

Proof

To define \(\leq'\), let \(s\) be the \(\leq'\)-minimum and \(v\) the \(\leq'\)-maximum. For all other vertices \(u, w\) if \(u \leq w\) then define \(w \leq' u\). Since a \(\leq\)-increasing (decreasing) path is \(\leq'\)-decreasing (increasing), \((T_2, T_1)\) is ordered by \(\leq'\), so by the previous lemma \(\leq'\) is an \(s\to v\) ordering.

Let \(s \in V(G)\), and \(e_1, e_2 \in \Gamma_E(s)\). The quadruple \((G, s, e_1, e_2)\) satisfies the Extended 3-Tree Path property \((E3TP)\) if there are three trees \((T_1, T_2, T_3)\) such that:

(E1) \(T_i\) are spanning trees of \(G\) \((i = 1, 2, 3)\).

(E2) For all \(v \in G\) \(T_3[v,s]\) is disjoint of \(T_1[v,s]\) and \(T_2[v,s]\).

(E3) \(E(T_i) \cap \Gamma_E(s) = \{e_i\}\) for \(i = 1, 2\).

(E4) If \(e_2 = (s,t)\) then there exists an \(s\to t\) ordering \(\leq\) of \(G\) such that \((T_1, T_2)\) is ordered by \(\leq\).

Note that by (E4) \(T_1[v,s]\) and \(T_2[v,s]\) are disjoint, and in fact \(\leq\) was introduced to facilitate proving that.
Lemma 3.3

Let $G$ be a graph. If for every $s \in V(G)$ there exist $e_1, e_2 \in \Gamma_E(s)$, such that $(G, s, e_1, e_2)$ satisfies E3TP then $G$ is 3-connected.

Proof

We show that between every $v, w \in V$ there exist three disjoint paths. Let $e_1, e_2 \in \Gamma_E(v)$. Since $(G, v, e_1, e_2)$ satisfies E3TP construct the three trees. The desired paths are $T_i[w, v]$ $(i=1,2,3)$.

Note that some 2-connected graphs $G$ may satisfy E3TP for some $s, e_1$ and $e_2$ (Figure 3.1).

Our goal will be to prove the converse of this lemma, namely, for every 3-connected graph $G$, $s \in V(G)$ and $e_1, e_2 \in \Gamma_E(s)$, $(G, s, e_1, e_2)$ satisfies E3TP. To this end, $(G, s, e_1, e_2)$ is a counterexample if $G$ is a 3-connected graph, $s \in V(G)$, $e_1, e_2 \in \Gamma_E(s)$ and $(G, s, e_1, e_2)$ does not satisfy E3TP. A minimum-counter-example (MCE) is a counterexample for which $|V(G)| + |E(G)|$ is minimum. A graph $G$ is an MCE if there exist $s, e_1$ and $e_2$ s.t. $(G, s, e_1, e_2)$ is an MCE. To satisfy our goal of proving Theorem 1.2, we show that there exists no MCE.

4. Reducing the degree of the root of an MCE

4.1 Edge contraction in MCE.

In this and the following section we investigate the structure of an MCE. Here we consider contracting an edge incident with $s$, and in subsection 4.2 we delete such an edge.

\[
\begin{align*}
&\text{Figure 3.1} \\
&T_1 \\
&T_2 \\
&T_3 \\
&\mathcal{S} = \{s, v\}
\end{align*}
\]
Lemma 4.1:

Let \( e = (s, p) \neq e_1, e_2 \). If \( \kappa(G/e) = 3 \) then \((G,s,e_1,e_2)\) is not an MCE.

Proof

For the sake of contradiction assume that \((G,s,e_1,e_2)\) is an MCE. By hypothesis \(G/e\) is 3-connected. If \( e_i=(s,v_i) \) then let \( e_i'=(s_0,v_i) \). By the minimality of \( G \), \((G/e,s_0p,e_1',e_2')\) satisfies E3TP. Let \( T_i' \) \((i=1,2,3)\) be the corresponding trees in \(G/e\) and \( t'\) the \( s_0-p\)-ordering. As \( \deg(p;G) \geq 3 \), \( p \) has two neighbors \( p_1,p_2 \neq s \). Assume, w.l.g., that \( p_1 \not\leq p_2 \).

Define now \( T_i = T_i'−e_i'+e_i+(p,p_i) \) \((i=1,2)\). Note that \( T_i[p,s]=(p,p_i)T_i[p_1,s] \). The tree \( T_3 \) is obtained from \( T_3' \) by adding the edge \((s,p)\) and replacing each edge of the form \((v,s_0p)\) by the corresponding edge (either \((v,s)\) or \((v,p)\)) of \( G \).

To define the \( s-t \) ordering in \( G \) let \( \preceq \) be any extension of \( \prec' \) to all of \( V(G) \) such that \( p_1 \not\leq p \not\leq p_2 \).

We prove now that the new trees satisfy E3TP in \( G \).

\(<E1>\) The subgraph \( T_i \) is a spanning tree of \( G \) since \( |V(T_i)| = |V(G/e)| + 1 = |V(G)| \) and \( |E(T_i)| = |E(T_i')| + 1 = (|V(G/e)| − 1) + 1 = |V(G)| − 1 \).

\(<E2>\) We show that \( T_1[v,s] \) and \( T_3[v,s] \) are disjoint, the proof for \( T_2[v,s] \) is identical. For \( v \neq p \):
\[ T_1[v,s] = T_1'[v,s_0p]−e_1 + e_i \]. Thus, since \( T_1'[v,s_0p] \) and \( T_3'[v,s_0p] \) are disjoint,
\[ T_1[v,s] \cap T_3[v,s] \subseteq \{v,s,p\} \]. Since \( p \) is a leaf of \( T_1 \) and \( v \neq p \), \( pV(T_1[v,s]) \) and so \( T_1[v,s] \) and \( T_3[v,s] \) are disjoint.

As for \( p \), by the construction \( T_1[p,s]=(p,p_1)T_1'[p_1,s_0p] \) and \( T_3[p,s]=(p,s) \). Therefore, also the path \( T_1[p,s] \) is disjoint of \( T_3[p,s] \).

\(<E3>\) Actually no edges were removed from \( T_i' \) \( i=1,2 \) and the only added edge is \((p,p_i)\) which is not adjacent to \( s \).

\(<E4>\) By definition \( \preceq \) is a total order and as \( p_1 \not\leq p \not\leq p_2 \) (S1) holds. By Lemma 3.1 it remains to show that \((T_1,T_2)\) are ordered by \( \preceq \). For \( v \neq p \), \( T_i'[v,s_0p]=T_i[v,s] \) \((i=1,2)\). Since \( T_i'[v,s_0p] \) is \( \preceq \)-
decreasing then so is \( T_1[v,s] \). Thus, it remains to check only for \( p \). \( T_i[p,s] = (p,p_1)T_i'[p_1,s] \) and as \( p_1 \parallel p \) and \( T_i'[p_1,s] \) is \( 1 \)-decreasing so is \( T_i[p,s] \).

The proof that \( T_2[p,t] \) is \( 1 \)-increasing is identical.

4.2 Edge removal.

Let \( e = (s,p) \neq e_1,e_2 \). If \( G-e \) is 3-connected then obviously \( (G,s,e_1,e_2) \) is not an MCE. In this section we generalize this observation to the case where \( \deg(p;G-e) = 2 \).

Lemma 4.2

Let \( s \in G \) with \( \deg(s;G) \geq 3 \) and \( e \in \Gamma_e - \{e_1,e_2\} \). If \( H=h(G-e) \) is 3-connected then \( (G,s,e_1,e_2) \) is not an MCE.

Proof

Assume that \( (G,s,e_1,e_2) \) is an MCE. Let \( e = (s,p) \). If \( \deg(p;G) \geq 3 \) then \( H=G-e \). Since by the hypothesis \( H \) is 3-connected, \( (H,s,e_1,e_2) \) is a smaller counterexample.

Therefore, \( \deg(p;G) = 3 \). Let \( \Gamma(p) = \{p_1,p_2,s\} \). If \( p \in V(H) \) then since \( \deg(p;H) = 2 \), \( H \) is not 3-connected. (This happens only if \( (p_1,p_2) \in E(G) \).) Thus \( |V(H)| < |V(G)| \) and since \( (G,s,e_1,e_2) \) is an MCE, \( (H,s,e_1,e_2) \) satisfies E3TP. Let \( T'_i \) \( (i=1,2,3) \) be the corresponding trees, and \( t' \) the \( s-t \) ordering. Assume, w.l.g., that \( p_1 \parallel p_2 \). If the edge \( (p_1,p_2) \) belongs to \( T'_3 \) then assume, w.l.g., that \( T'_3[p_1,s] = (p_1,p_2)T_3[p_2,s] \) and define \( T_3 = T'_3-(p_1,p_2)+p_1+p_s \). Otherwise, \( T_3 = T'_3+p_s \). (Thus we may have shortened some paths of \( T_3 \).) Now we construct the other two trees: If \( (p_2,p_1) \in E(T'_1) \) then \( T_1 = T'_1-(p_2,p_1)+(p_1,p) \). Otherwise, \( T_1 = T'_1+p_1 \). If \( (p_1,p_2) \in E(T'_2) \) then \( T_2 = T'_2-(p_1,p_2)+(p_2,p_1) \). Otherwise, \( T_2 = T'_2+p_2 \).

To define the \( s-t \) ordering in \( G \) let \( \parallel \) be any extension of \( t' \) to all vertices of \( G \) such that \( p_1 \parallel p \parallel p_2 \).

We prove that the trees \( T_i \) \( (i=1,2,3) \) satisfy E3TP in \( G \).

(E1) and (E2)

By our construction, replacing the edge \( (p_1,p_2) \) by the path \( (p_1,p,p_2) \) leaves the trees connected without introducing any cycles. If \( (p_1,p_2) \in E(T'_i) \) then \( (p_1,p_2)T_i[v,s] \) \( (i=1,2) \),
$T_3[v,s] \subseteq T_3'[v,s] + (p_1,p) + (p,s)$ and thus is disjoint of $T_i[v,s]$. Otherwise, $T_3[v,s] = T_3'[v,s]$ and therefore, (E1) and (E2) hold for $T_j$ $(j=1,2,3)$. Thus, it remains to prove these conditions only for $p$. But $T_1[p,s] = (p,p_1)T_1'[p_1,s]$, $T_2[p,s] = (p,p_2)T_2'[p_2,s]$ and $T_3[p,s] = (p,s)$ are disjoint.

(E3) No edges adjacent to $s$ were added or removed from $T_1'$ and $T_2'$. Thus this condition remains unchanged.

(E4) First we prove that $\hat{\lambda}$ is an $s$–$t$ ordering. Since $\hat{\lambda}$ satisfies condition (S1), by Lemma 3.1 it remains to show that $(T_1,T_2)$ are ordered by $\hat{\lambda}$. For $p$, $T_1[p,s] = (p,p_1)T_1'[p_1,s]$ $p_1 \downarrow p$ and the path $T_1'[p_1,s]$ $\downarrow$-decreases. Similarly, since $p \downarrow p_2$, the path $T_2[p,t] = (p,p_2)T_2'[p_2,t]$ $\downarrow$-increases. For $v \neq p$ then we have to check only the case where $(p_2,p_1) \in T_1'[v,s]$ (and similarly for $T_2'[v,t]$). As $p_1 \downarrow p_2$, $p_1$ appeared in $T_1'[v,s]$ after $p_2$. To get $T_1[v,s]$ we replaced the edge $(p_2,p_1)$ by the path $(p_2,p,p_1)$, but $p_1 \downarrow p \downarrow p_2$, thus the order on $T_1[v,s]$ is preserved.

4.3 The root-degree of an MCE.

In this section we prove the following.

Lemma 4.3

If $(G,s,e_1,e_2)$ is an MCE then $\deg(s) = 3$.

Proof

Let $e \in \Gamma_E(s) - \{e_1,e_2\}$. By Corollary 2.4 either $G/e$ or $h(G-e)$ is 3-connected. If $G/e$ is 3-connected then by Lemma 4.1 $(G,s,e_1,e_2)$ is not an MCE. Thus, $\kappa(h(G-e)) \geq 3$. If $\deg(s) > 3$ then by Lemma 4.2 $(G,s,e_1,e_2)$ is not an MCE.

In the remaining sections we assume $\deg(s) = 3$.

Lemma 4.4

Let $\Gamma_E(s) = \{s_1,s_2,s_3\}$ and $(G,s,(s,s_1),(s,s_2))$ satisfy E3TP. Let $T_1,T_2,T_3$ be the corresponding trees. Then for $1 \leq i,j \leq 3$ and $i \neq j$, $s_i$ is a leaf of $T_j$. 
Proof

Assume, for example, that \( s_1 \) is not a leaf of \( T_2 \), then there exists a vertex \( x \) such that \( T_2[x,s]=T_2[x,s_1]T_2[s_1,s] \). Since, \( T_1[x,s]=T_1[x,s_1](s_1,s) \), \( T_1[x,s] \cap T_2[x,s]=\{s_1\} \), i.e., the two paths are not disjoint, contrary to the hypothesis.

5. Illegal subgraphs in an MCE.

This section deals with an MCE for which \( \text{deg}(s)=3 \) and shows several subgraphs which cannot be contained in an MCE. The proofs of these lemmas are very similar to the proofs of Lemmas 4.1 and 4.2, so we will be more brief.

Lemma 5.1

Let \( F_1 \) be the graph of Fig 5.1. If \( G \) contains the graph \( F_1 \) as a subgraph and \( \text{deg}(v;G)=3 \) then for \( e_2 \neq (s,p) \), \( (G,s,e_1,e_2) \) is not an MCE.

Remark: The vertices \( s,p \) and \( q \) are connected to \( G-F_1; (p,q) \) may also belong to \( G \).

Proof

If \( G \) is an MCE then by Lemma 4.3 \( \text{deg}(s;G)=3 \). Let \( \Gamma_E(s;G)=\{e_1,e_2,e_3\} \), then since by definition \( e_1=(s,v) \) and \( e_2 \neq (s,p) \), we have \( e_3=(s,p) \). By the minimality of \( G \), \( H=h(G-b) \) is not an MCE. We first show that \( \kappa(H)=3 \), then use the three spanning trees of \( H \) to construct trees for \( G \).

Claim: \( \kappa(H)=3 \).

Proof of Claim: Obviously, \( \Gamma(s;G)=\{u,v,p\} \) is a separating set of \( G \). Applying Lemma 2.1 to \( \Gamma(s;G) \) shows that except perhaps for \( x=v \) or \( y=p \) \( \kappa(x,y;G-b)=3 \). We need not prove the connectivity.
for $v$ since $\deg(v; G-b)=2$ and thus it does not belong to $H$. If $\deg(p; G)=3$ then $p$ does not belong to $H$. Thus, for all $x, y \in V(H)$, $\kappa(x, y; G-b)=3$, from which the claim follows. Otherwise, $\deg(p; H) \geq 3$; thus by Lemma 2.2 $\kappa(H)=3$. (For both cases we have equality since $\deg(s; H)=3$.)

Define $e^H_2$ to be the sole member of the set $\Gamma_E(s; H)-\{(s, q), e_3\}$. Since $G$ is an MCE, $(H, s, (s, q), e^H_2)$ satisfies E3TP. Let $T^H_i$ $(i=1, 2, 3)$ be the corresponding trees and $\ell^H$ the $s-t$ ordering. Note that $s \ell^H q$ and $q \ell^H w$ for all $w \in V(H)-\{s, q\}$.

**Case 1:** $\deg(p; G)>3$. In this case $p \in H$, so $e^H_3 = (s, p)$. The $s-t$ ordering for $G$ is defined as $s \ell^H q$ and agrees with $\ell^H$ on all other vertices. The trees are

$$T_1 = T^H_1 - (s, q) + (s, v) + (v, q),$$
$$T_2 = T^H_2 + (v, q),$$
$$T_3 = T^H_3 + (p, v).$$

**Case 2:** $\deg(p; G) = 3$.

The $s-t$ ordering for $G$ is defined as $s \ell^H p \ell^H q$ and agrees with $\ell^H$ on all other vertices. Let $\Gamma(p) = \{s, v, p_1\}$. Then by definition, $e^H_3 = (s, p_1)G$. Define

$$T_1 = T^H_1 -(s, q) + (s, v) + (v, q) + (v, p),$$
$$T_2 = T^H_2 (v, q) + (p_1, p),$$
$$T_3 = T^H_3 - (s, p_1) + (s, p) + (p, p_1) + (p, v).$$

**Lemma 5.2**

Let $F_2$ and $F_3$ be the graphs of Fig 5.2. If $G$ contains either $F_2$ or $F_3$ and $v, w$ have no additional neighbors, then for $e \neq (s, p) (G, s, e_1, e_2)$ is not an MCE.

**Proof**

$G-b$ is 3-connected. Thus $G$ cannot be an MCE.

**Lemma 5.3**

Let $F_4$ be the graph of Fig 5.3. If $G$ contains $F_4$, $e \neq (s, p)$ and $v, w$ have no additional neigh-
bors, then \((G,s,e_1,e_2)\) is not an MCE.

**Remark:** W.l.g., \((p,q)G\), since otherwise we have the graph \(F_3\).

**Proof**

It is easy to show that \(H = h(G-b)\) is 3-connected. Thus, if \((G,s,e_1,e_2)\) is an MCE, \((H,s,(s,q),e_2)\) satisfies E3TP. Let \(T_i^H\) \((i=1,2,3)\) be the corresponding trees and \(\{\}^H\) the \(s-t\) ordering. By Lemma 4.4 \(p\) is a leaf of \(T_1^H,T_2^H\) and \(q\) is a leaf of \(T_2^H,T_3^H\). W.l.g., \(T_1^H[p,s] = (p,q,s)\). As \((p,q) \in T_1^H\) then

\[ T_1 = T_1^H -(q,s) +(q,v)+(w,v)+(v,s) +(p,w), \]

and \(T_2 = T_2^H +(v,q)+(w,q)\).

W.l.g., \(T_3^H[q,s] = (q,p,s)\) then let

\[ T_3 = T_3^H -(q,p) +(q,w)+(w,p) +(v,w) \]

\(\{\}\) is defined as the extension of \(\{}^H\) to \(G\) which satisfies \(s \{v \{w \{z \mid z \in V(G)-s \} ordered by \(\{}^H\). \]

We now come to the main lemma of this section.

**Lemma 5.4**

Let \((G,s,e_1,e_2)\) be an MCE such that \(\{s,p,q\}\) is a separation set with \((s,p) \in G\) and

\[ F_4 \]
Then each component of $G \setminus \{s, p, q\}$ contains more than two vertices.

**Proof**

Let $B$ be a component of $G \setminus S$. If $|V(B)| = 1$ then the subgraph spanned by $S$ and $B$ is the graph $F_1$. If $|V(B)| = 2$ then the subgraph spanned by $S$ and $B$ is one of the graphs $F_i$, $i = 2, 3, 4$. In both cases $G$ is not an MCE.

6. E3TP and 3-connected graphs.

We give now the main result of this paper which implies Theorem 1.2.

**Theorem 6.1**

Let $G$ be a 3-connected graph, $s \in V(G)$ and $e_1, e_2 \in \Gamma_E(s)$, then $(G, s, e_1, e_2)$ satisfies E3TP.

**Proof**

As shown in Fig 6.1 if $|V(G)| = 4$, E3TP is satisfied. Thus, $|V(G)| > 4$. Assume the theorem is false and let $(G, s, e_1, e_2)$ be an MCE. By Lemma 4.3, $\deg(s) = 3$. Let $\Gamma_E(s) = \{e_1, e_2, e_3\}$, where $e_3 = (s, p)$. By Lemma 4.1, $\kappa(G/e_3) = 2$. Therefore, there is a vertex $q$ such that $S = \{s, p, q\}$ is a separating set. Since $s$ must have an edge in each component, $G \setminus S$ contains only two components $G_1, G_2$ and since $\deg(s; G) = 3$ and $(s, p) \in G$, $(s, q)E(G)$. By Lemma 5.4 $|V(G^i)| > 2 \ (i = 1, 2)$.

**Case 1:** $(p, q)E(G)$.

As there are three disjoint paths from $q$ to $s$ we may assume, w.l.o.g., that at least two of them pass through $G^2 + s + p$. Since $\deg(s) = 3$ one of these paths contains $e_3$. Also, there is always at least one path through $G^2 + s + p$. 

![Figure 6.1](image-url)
in $G^1$. Let $G^2 = G/G^1$ (the contraction of $G^1$ to a single vertex $u$) and $G^1$ the graph resulting from $G$ by replacing $G^2$ by $F_4$ (see Fig. 6.2). I.e. $V(G^1) = V(G) - V(G^2) + \{v, w\}$ and $E(G^1) = E(G) - E(G^2) + (v, w) + (s, v) + (v, q) + (w, p) + (w, q)$; ($u$, $v$, and $w$ are new vertices.) Now we prove that $G^1$ and $G^2$ are 3-connected.

**Claim 1:** $G^1$ is 3-connected.

Since there is at least one path between any two vertices of $s, p, q$ containing only vertices of $G^1$, any two vertices $x, y \in \{s, p, q\}$ are 3-connected. For $x \neq s, p, q$ there exists an $x - \{s, p, q\}$ fan. Thus, by Lemma 2.5, $G^1$ is 3-connected.

**Claim 2:** $G^2$ is 3-connected.

As it can be verified that the connectivity between any two vertices $\neq q$ is 3, the claim follows by Lemma 2.2.
Case 1.1 \( t \in G^2 \):

As \( |V(G')| > 2 \) \( (i=1,2) \), \( |V(G)| < |V(G)| \) therefore \((G^1,v,e_{1}(s,v))\) and \((G^2,s,v,e_{2})\) satisfy E3TP. (Note that in \( G^1 \) \( v \) plays the role of \( t \).) Let \( T_1^1,T_2^1,T_3^1 \) be the trees in \( G^1 \), \( \{1\} \) an \( s-v \) ordering and \( T_1^2,T_2^2,T_3^2 \) the trees in \( G^2 \) with an \( s-t \) ordering \( \{2\} \). For \( i=1, 2, 3 \), define:

\( T_i = T_i^1 \cup T_i^2 - \{u,v,w\} \). (Note that \( (s,p) \) belongs to both \( T_3^1 \) and \( T_3^2 \).)

Assume \( s \{1\} x_1 \{1\} ... x_{i-1} \{1\} q \) are the vertices of \( G^1 \cup S \) and \( s=y_1 \{2\} \ldots \{2\} y_{i}=t \) are the vertices of \( G^2 \). Define \( \{1\} \) to agree with \( \{1\} \) on vertices of \( G^1 \cup S \) and with \( \{2\} \) on vertices of \( G^2 \) also if \( x \in G^1 \cup S \) and \( y \in G^2 \) then \( x \{1\} y \). We prove now that \( T_1,T_2,T_3 \) satisfy E3TP.

\textbf{(E1)} It is easy to verify that for each vertex \( x \) there exists a \( x-s \) path on each \( T_j \), thus each of the \( T_i \)’s is connected and span \( G \).

To show that \( T_i \) contains no cycles we check the number of edges of \( T_i \). For example:

\[
|E(T_i)| = |E(T_i^1)| - 2 + |E(T_i^2)| - 3 = |V(G^1)| - 1 + |V(G^2)| - 1 - 5 = |V(G^1)| + |V(G^2)| + 7 = |V(G)| + 6 - 7 = |V(G)| + 1.\]

Thus, \( T_i \) is a spanning tree.

\textbf{(E2)} If \( x=p \) then \( T_3[p,s]=(p,s)=(T_3^1[p,s]=T_3^2[p,s]) \), which is disjoint of \( T_i[p,s] \) \( (i=1,2) \).

If \( x \in G^1 \):

\[
T_1[x,s] = T_1^1[x,s], \tag{1}
\]

\[
T_2[x,s] = T_2^1[x,q]T_2^2[q,s] \text{ and } \tag{2}
\]

\[
T_3[x,s] = T_3^1[x,p]T_3^2[p,s]. \tag{3}
\]

By E3TP in \( G^1 \), \( T_3[x,s] \) is disjoint of \( T_1[x,s] \) and of \( T_2^1[x,q] \). It remains to show that \( T_3[x,s] \) is disjoint of \( T_2^2[q,s] \). This follows because the former lies entirely in \( G^1+s \) and the latter in \( G^2+q+s \).

The remaining cases \( (x \in G^2 \text{ and } x=q) \) are similar.

\textbf{(E3)} By the construction, only edges belonging to the original graph \( G \) appear in \( T_i \). Therefore, for each tree \( T_i \) \( (i=1,2) \) \( \Gamma_{E}(s;G) \cap T_i = \{e_i\} \).
To prove that \( \{ \) is a \( s-t \) ordering we first show condition (S1), i.e. that \( s \) is the minimal vertex of \( \{ \). The maximal vertex of \( \{ \) is equal to the maximal vertex of \( \{^2 \). Note that by E3TP in \( G^2 \), \( \{^2 \) does not obtain that maximum on \( q \) or \( p \). By Lemma 3.1 it remains to show that \((T_1, T_2)\) are ordered by \( \{ \). We show now that \( T_1[x,s] \) is \( \{ \)-decreasing for \( x \in G \). Assume \( x \in G^1 \), then \( T_1[x,s] = T_1^1[x,s] \). As \( \{ \) and \( \{^1 \) agree on \( G^1 \), \( T_1[x,s] \) is \( \{ \)-decreasing. If \( x \in G^2 \) then \( T_1[x,s] = T_1^2[x,q]T_1^1[q,s] \). By E3TP on \( G^2 \), \( T_1^2[x,q] \) is \( \{^2 \)-decreasing and as before \( T_1^1[q,s] \) is \( \{^1 \)-decreasing. Now \( \{ \) agrees with \( \{^1 \) on \( G^1 \) and with \( \{^2 \) on \( G^2 \). Therefore, \( T_1[x,s] \) is \( \{ \)-decreasing. The proof that \( T_2[x,s] \) is \( \{ \)-increasing is similar.

**Case 1.2** \( t \in G^1 \):

As before, \((G^1,s,e_2,(s,v))\) and \((G^2,s,(s,u),e_1)\) satisfy E3TP. We proceed as in case 1.1 to obtain three trees \( T_1, T_2, T_3 \subseteq G \) satisfying (E1-E4). In particular, there exists an \( s-v \) order \( \{ \) ordering \( (T_1,T_2) \). Since \( \Gamma(s; T_1) = \{ t \} \), we may use Lemma 3.2 to obtain an \( s-t \) order \( \{' \) such that \( (T_2,T_1) \) is ordered by \( \{' \). Thus \( (T_2,T_1,T_3) \) are three trees which demonstrate that \((G,s,e_1,e_2)\) satisfies E3TP.

**Case 2:** \( (p,q) \in E(G) \)

In this case we construct \( G^1 = G/G^2 \) and \( G^2 = G/G^1 \). The rest of the proof is identical to case 1.

\[
\]

7. Conclusions.

The results of this paper and [IR] suggest the following conjecture:

**Conjecture**

If a graph \( G \) is \( k \)-connected and \( s \in G \) then there exist \( k \) spanning trees of \( G \) such that the paths from any vertex \( v \) to \( s \) on the spanning trees are disjoint.

In this paper we used \( s-t \) ordering to ensure that the paths \( T_1[v,s] \) and \( T_2[v,s] \) are disjoint. We have not been able to extend this method to \( k>3 \). Also, since our proof is much more complicated that
that of $k=2$, new ideas are required.

A similar conjecture can be stated for edges: Namely, if the graph is $k$-edge connected then for every vertex $s$ there exist $k$ spanning trees such that the paths from any vertex $v$ to $s$ on the spanning trees are edge-disjoint.

Both conjectures were proven for $k=2$ [IR]. It seems that methods similar to those of this paper are sufficient to prove the edge-conjecture for $k=3$. However, we feel that it is more promising to show that for all $k$ the vertex-conjecture implies the edge-conjecture (or vice versa).
References.


