ON STABILIZATION OF DISCRETE-EVENT PROCESSES

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Abstract

Discrete-Event processes are modeled by state-machines in the
Ramadge-Wonham framework with control by a feedback event-
disability mechanism. In this paper concepts of stabilization of
discrete-event processes are defined and investigated. We examine the
possibility of driving a process (under control) from arbitrary initial
states to a prescribed subset of the state set and then keeping it there
indefinitely. This stabilization property is studied also with respect to
'open-loop' processes (i.e., uncontrolled processes) and their asymptotic
behavior is characterized. To this end, we use known classical
concepts of dynamics as invariant-sets and attractors are defined and
characterized in the discrete-event control framework. Finally, we
provide polynomial time algorithms for verifying various types of
attraction and for the synthesis of attractors.

1. Introduction

This paper is a preliminary investigation of the concepts of
stabilization of discrete-event processes (DEP). We adopt a slightly
modified version of the framework proposed by Ramadge and Wonham
[1-3] for the study of DEP. Our model is thus a state machine with a
means of external control: a feedback event-disability mechanism.
Unlike [4-6], we consider a state model describing the possible order of
elementary events but not their exact timing.

In most of the works concerning supervisory control of DEP (e.g.,
[7-10]) it is assumed that the initial state of the process is fixed, known
apriori and one of the 'legal' states of the process. The control problem is
then to synthesize a supervisor which confines the behavior of the
process, initialized at the prespecified initial state, to within legal
bounds. However, there are cases in which either the initial state is not
one of the legal states of the process or it is unknown apriori. In such
cases the question of stabilization is of great interest.

In this paper we study the ability of a process to reach a set of
target states from an arbitrary initial state and then remain there
indefinitely. This stabilization property is examined under different
control strategies. To this end, the classical concept of attraction [11] is
reformulated and characterized in our framework. Polynomial time
algorithms are provided for the verification of different types of
attraction.

This paper is organized as follows. In the remainder of this section
we give some terminology and notation. Invariant sets of states and
realizable processes are defined in section 2. In section 3, the notion of
strong attraction is introduced and examined with respect to processes
without external control. Further, an efficient algorithm for computing
the asymptotic behavior of such processes is proposed. Section 4
develops control strategies under which strong attraction can be
achieved. To this end, a weaker form of attraction is introduced. An
efficient algorithm for computing the region of weak attraction is
provided in section 5. In section 6, an illustrative example is given and
the relation between attraction and recovery of failure is mentioned.

1.1 Processes

Let $\Sigma$ be a finite alphabet (event set). A process over $\Sigma$ is modeled
as a finite (directed) graph $G = (V, E)$ where $V$ is a set of states
(vertices) and $E \subseteq V \times \Sigma \times V$ is a set of edges. An edge of $G$ is thus an
ordered triple $e = (v, \alpha, u) \in E$ and it is said to be directed from $v$ to $u$.
The state $v$ is called the start-state of $e$, the state $u$ is called the end-
state of $e$ and $\alpha \in \Sigma$ is the event associated with $e$. If $(v, \alpha, u) \in E$ we
say that $v$ is a predecessor of $u$ and $u$ is a successor of $v$. Edges with
the same start state and the same end state are called parallel. It is
assumed that there are no two edges going out of the same state
associated with the same event, that is, for each pair of edges in $E$

$[ (v, \alpha, u), (v, \beta, w) \in E \text{ and } \alpha = \beta ] \implies u = w$.

We interpret $G$ as a device that starts its execution at an arbitrary state
$v \in V$ (may be determined by a nondeterministic mechanism in $G$ or
forced externally) and thereafter executes a sequence of state transitions
as permitted by $E$.

A path is a finite sequence of edges $e_1, e_2, \ldots, e_n$ such that the end
state of $e_{i-1}$ is the start state of $e_i$. The number of edges in a path is
called the length of a path. The start of the path is the start state of $e_1$
and its end is the end state of $e_n$. To each path $(r_0, \Omega_1, v_1), (v_1, \Omega_2, v_2), \ldots, (v_{n-1}, \Omega_n, v_n)$ there corresponds a unique
(state) trajectory $r_0, v_1, \ldots, v_n$. Further, if $v_0 = v_n$ the trajectory is said to
be closed. A closed trajectory in which no state (except the start and end
states) appears more than once is called a cycle. A graph without cycles is
called acyclic.

A state $v$ is reachable from a state $u$ if there exists a path from $u$ to $v$.
A state $v$ is said to be reachable from a subset of states $A$ if $v$ is
reachable from at least one state in $A$. The reach of $A$ in $G$, denoted
$\rho_G(A)$, is defined as the set of all states in $G$ that are reachable from $A$.

Let $G = (V, E)$, $\emptyset \neq A \subseteq V$. We say that a state $v \in V - A$ is
connected to $A$ if there is a path from $v$ to a state in $A$. Further, $G$ is
called $A$-connected if each $v \in V - A$ is connected to $A$. A process
$G' = (V', E')$ is called a subprocess of the process $G = (V, E)$, denoted
$G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$.

The union of two processes $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is
another process $G_3$ (written $G_3 = G_1 \cup G_2$) whose state set is
$V_3 = V_1 \cup V_2$ and whose edge set is $E_3 = E_1 \cup E_2$. If $v$ is a state in $G$
then $G - v$ denotes the subprocess of $G$ obtained by deleting $v$ from $G$.
Deletion of a state always implies the deletion of all edges incident on
that state. If $e$ is an edge in $G$, then $G - e$ is a subprocess of $G'$ obtained by
deleting $e$ from $G$. Deletion of an edge does not imply deletion of its
end states.

A subprocess $G' = (V', E')$ of $G = (V, E)$ is called an induced
subprocess if $E'$ contains all the edges of $E$ whose end points are in $V'$.
In this case we say that $G'$ is induced by $V'$. The process induced by $V'$
is denoted $\rho_G(V')$.

1.2 Supervisors

As in [1], we assume that $\Sigma$ consists of two disjoint subsets $\Sigma_0$
and $\Sigma_u$: uncontrolled and controlled events. Events in $\Sigma_0$ can be
disabled by external control while events in $\Sigma_u$ cannot be prevented from occurring.
Clearly, this classification of $\Sigma$ induces a similar classification of $E$, that is,$E = E_0 \cup E_u$ where $E_0 = E \cap (V \times \Sigma_0 \times V)$ and $E_u = E - E_0$.

A supervisor for $G$ is a map $S : V \rightarrow 2^{\Sigma_u}$. For each state $v \in V$ the
supervisor specifies a subset of controlled events that must be disabled.
The concurrent operation of the process G and a supervisor S, denoted (S,G) and called the closed-loop process, is defined as the subprocess (V,E') of (V,E) satisfying the condition that for all e = (v,u) ∈ E.

2. Invariant Sets of States and Realizable Processes

Let G = (V,E) be a process and let A ⊆ V, E' ⊆ E. We say that A is E'-invariant iff

\[ (\forall (v,u) \in E') \, v \in A \Rightarrow u \in A. \]

That is, there is no edge in E' leading out of A. We remark that the important special case where A is E_invariant has been discussed in [2], in connection with a modular-approach solution for the problem of maintaining a predicate on V invariant.

A subprocess G' = (V',E') of the process G = (V,E) is called realizable iff

\[ (\forall (v,u) \in E) \, v \in A \Rightarrow (v,u) \in E'. \]

That is, a subprocess G' ⊆ G is realizable iff every uncontrolled edge going out of a state in G' is an edge of G'. Moreover, it is easily seen that a subprocess G' = (V',E') is realizable iff there exists a supervisor S such that the closed-loop process (S,G) and the subprocess G' have the same "behavior" in the sense that for each state v ∈ V', the set of all paths starting at v is the same in G' and (S,G). In fact, the notion of a realizable subprocess is closely related to the concept of controllable language [1].

3. Strong Attraction

In this section we examine some properties of 'open-loop' processes, i.e., processes without external control. First we introduce the concept of strong attraction.

3.1 Strong attractors

Let G = (V,E) be a process and let A,B ⊆ V such that ∅ ̸= A ⊆ B. We say that A is a strong attractor for B with respect to G, denoted A ≡ B, iff the following conditions are satisfied:

(a) A is E-invariant.
(b) For each state v ∈ rG(B) there is a path that starts at v and ends in A.
(c) There are no cycles of G in rG(B) - A.

Thus, if A ≡ B then whenever the process G is initialized at state v ∈ B it always reaches A within a finite number of state transitions and remains in A.

We show now that for each nonempty E-invariant subset A of V there exists a unique largest subset for which A is a strong attractor. To this end, let A ≡ A ⊆ V be E-invariant, and define T_G(A) to be the (finite) class of all subsets of V for which A is a strong attractor; that is,

\[ T_G(A) = \{ B ⊆ V \mid A ⊆ B \land A ≡ B \}. \]

Proposition 3.1

The class T_G(A) is nonempty and closed under set union.

Most of the proofs of the Propositions that appear in this paper will be omitted for the limitation of space.

An immediate consequence of proposition 3.1 is that T_G(A) has a unique maximal element. The maximal set B for which A ≡ B is denoted A_G(A) and called the region of strong attraction of A with respect to G.

For a subset A which is not E-invariant we will say that A_G(A) = ∅. If A_G(A) = V we say that A is a global strong attractor with respect to G (denoted A ≡ G). In cases of no confusion we shall not mention the underlying process and write, e.g., that A is a global strong attractor. It is readily verified that in the case of global strong attraction conditions (a2) and (a3) can be written as

(a2') G is A-connected.
(a3') G - A is acyclic.

3.2 Asymptotic behavior

The meaning of a subset A ⊆ V being a global strong attractor is that there exists a number N ≤ |V - A| such that every trajectory of G of length greater than N ends in A. Further, the subset A is reachable from each state in V. In other words, initializing the process at an arbitrary state v ∈ V causes the process to reach a state in A in a finite number of state transitions. Once the process reaches a state in A it remains in A.

A natural question that arises is whether we can maximally restrict the state domain in which the process, initialized at an arbitrary state, can be "found" after a sufficient large number (bounded by |V|) of state transitions. That is, we are interested in the asymptotic "behavior" of the process.

Thus, let G = (V,E) and let g(G) be the (finite) class of all subsets of V that are global strong attractors with respect to G. That is,

\[ g(G) = \{ A ⊆ V \mid A ≡ G \}. \]

First we need the following obvious observations.

Observation 3.2

The state set of G is a global strong attractor (w.r.t. G).

Observation 3.3

Let C be a cycle of G. Then C is a cycle of G for every A ∈ g(G).

Using observations 3.2 and 3.3, the following proposition is readily proved.

Proposition 3.4

Let G = (V,E). Then g(G) ̸= ∅, and if A_1, A_2 ∈ g(G) then

\[ ∅ ̸= A_1 ∩ A_2 ∈ g(G). \]

Proposition 3.4 implies that the finite class g(G) contains a unique infimal element w.r.t. inclusion, which is denoted inf\{g(G)\}. Further, this infimal element satisfies the condition that

\[ inf\{g(G)\} = \{ A \mid A ∈ g(G) \}. \]

For an effective computation (i.e., a polynomial time algorithm) of the minimal global strong attractor we need the following proposition.

Proposition 3.5

Let G = (V,E) and v ∈ V. Then v ∈ inf\{g(G)\} if and only if either

(i) v is reachable from a state of a cycle in G;
(ii) v has no successors.
Proof

For abbreviation let \( W = \inf_g(G) \).

(II). Clearly, every global strong attractor of \( G \) contains all the states in \( G \) which are 'dead-end', namely without outgoing edges. Otherwise, condition (a2\') cannot be satisfied. Thus condition (ii) is a sufficient one. As regards condition (i), we note that conditions (a1) and (a3) imply that every cycle of \( G \) is contained in every global strong attractor. Moreover, since every global strong attractor \( A \) is \( E \)-invariant it follows that every state reachable from a state in \( A \) must also be in \( A \). So we conclude that every state reachable from a state of a cycle in \( G \) is in \( W \), which is one of the global strong attractors of \( G \).

(Only if). Fix \( v_0 \in W \) and suppose, towards a contradiction, that \( v_0 \) does not satisfy conditions (i) and (ii), that is,

(iii) \( v_0 \) has at least one successor ; and

(iv) the subset \( X \) of all states in \( V \) from which \( v_0 \) is reachable satisfies the condition that every state in \( X \) is not a state of a cycle in \( G \).

We shall show now that \( W = X \) is a global strong attractor, contradicting our assumption that \( W \) is the minimal one.

Let \( Y \) be the set of all states in \( W \) from which \( v_0 \) is reachable, i.e., \( Y = W \cap X = W - Y \). Clearly, \( Y \) is not empty (since \( v_0 \in W \cap X = Y \) and \( W - X = W - Y \)). First we claim that \( W - Y \) is \( E \)-invariant. To see this, suppose \( W - Y \) is not \( E \)-invariant, and that for some \( w \in W - Y \) there exists an edge \( (v, \sigma, w) \in E \) such that \( w \in W - Y \). By the definition of \( W \), it is clear that \( W = \emptyset \), and thus the \( E \)-invariance of \( W \) implies \( w \in W \). Since

\[
( w \in W \text{ and } w \in W - Y ) \Rightarrow w \in X,
\]

it follows that \( v_0 \) is reachable from \( w \). Consequently, \( (w, \sigma, w) \in E \) implies \( v_0 \) is reachable also from \( u \), i.e., \( u \in Y \), contradicting our assumption that \( u \in W - Y \). So \( W - Y \) is \( E \)-invariant.

Next we have to show that \( G = (W - Y) \)-connected, that is, there exists a path from each state in \( V \) \((W - Y) \) to a state in \( (W - Y) \). First we consider the state \( v_0 \). Since \( v_0 \in Y \subseteq W \) and \( W \) is \( E \)-invariant then every successor \( v_0 \) of \( v_0 \) is in \( Y \) (by assumption (ii)), \( v_0 \) has at least one successor). Further, \( v_0 \in Y \subseteq X \) since otherwise \( v_0 \) is reachable from \( v_0 \), meaning that \( Y \) is a state of a cycle, in contradiction to assumption (iv). Thus, \( v_0 \in W - Y \) and \( v_0 \) is connected to \( W - Y \). Moreover, by the definition of \( X \), \( v_0 \) is reachable from every state in \( X \) and thus each state in \( X \) is connected to \( W - Y \). Finally, \( W \) is a global strong attractor and thus, by condition (a2\'), every state in \( V \) \(W - Y \) is connected either to \( W - Y \) or to \( Y \subseteq X \), which is connected to \( W - Y \).

It remains to be shown that \( G = (W - Y) \) is acyclic, namely that every cycle of \( G \) intersects \((W - Y) \). To this end, let \( C \) be a cycle of \( G \). Since \( G \) is acyclic and \( W \) is \( E \)-invariant then \( C \) is contained in \( W \). Further, by assumption (iv), every state in \( Y \subseteq X \) is a state of \( C \) and thus \( C \) must be contained in \((W - Y) \). That is, \( G = (W - Y) \) is acyclic.

To summarize, we have shown that \( W = X \) is also a global strong attractor w.r.t. \( G \), contradicting our assumption that \( W = \inf_g(G) \).

\[\square\]

Using proposition 3.3 and the transitive closure of \( G \) (i.e., the directed graph in which there is an edge from \( v \) to \( w \) if and only if there is a nonempty path from \( v \) to \( w \) in \( G \) [12, Ch. 1]), the infimal global strong attractor \( \inf_g(G) \) can be computed in polynomial time.

4. Weak Attraction

In this section we introduce a weaker form of attraction which can be obtained under a suitable control.

4.1 Weak attractors

Let \( G = (V, E) \), \( \emptyset \neq A \subseteq B \subseteq V \). The subset \( A \) is called a weak attractor for \( B \) w.r.t. \( G \), denoted \( A \leftrightarrow B \), if there exists a supervisor \( S \) such that \( A \leftrightarrow B \).

Clearly, strong attraction implies weak attraction but the converse is in general not true. Further, it is easily seen that a necessary condition for a subset \( A \) to be a weak attractor for another subset is that \( A \) be \( E \)-invariant.

Necessary and sufficient conditions for an \( E \)-invariant subset \( A \) to be a weak attractor for \( B \) are given by the following proposition.

Proposition 4.1

Let \( G = (V, E) \), \( \emptyset \neq A \subseteq B \subseteq V \), such that \( A \) is \( E \)-invariant. Then \( A \leftrightarrow B \) if and only if there exists a subprocess \( G' = (V', E') \) of \( G \) such that \( B \subseteq V' \) and the following conditions are satisfied:

(b1) \( G' \) is \( G \)-connected.

(b2) \( G' \) is realizable.

(b3) \( G' \sim A \) is acyclic.

Corollary 4.1

If \( A \) is \( E \)-invariant and \( G' = (V', E') \) satisfies condition (b1)-(b3) then \( A \leftrightarrow V' \).

Proposition 4.1 provides necessary and sufficient conditions for the solvability of the Weak Attraction Problem (WAP), namely given a process \( G = (V, E) \) and subsets \( A \subseteq B \subseteq V \), verify whether WAP is solvable or not. Notice that if \( \Sigma_E = \emptyset \) (i.e., every edge of \( G \) is controlled) then WAP is solvable iff each state in \( B \) is connected to a state in \( A \). However, if the former condition does not hold (i.e., \( \Sigma_E \neq \emptyset \)) then WAP is not necessarily solvable even if the latter condition is satisfied.

So far we considered only \( E \)-invariant subsets of \( V \) as candidates for weak attractors. Clearly, this is a necessary condition. Suppose, however, that we are given two subsets \( A \) and \( B \), such that \( A \subseteq B \subseteq V \) and \( A \) is not \( E \)-invariant. An interesting question is whether there exists a subset \( A' \subseteq A \) such that \( A' \sim B \). That is, find (if it exists) a subset \( A' \) of \( A \) for which a supervisor \( S \) can be synthesized, so that from each initial state \( v \in B \), the closed-loop process \( G/S \) reaches \( A' \) in finite number of state transitions and remains in \( A' \).

The following intuitive proposition states that the problem above is solvable iff the maximal \( E \)-invariant subset of \( A \) is a weak attractor for \( B \). The fact that every subset \( A \subseteq V \) contains a unique maximal \( E \)-invariant subset, denoted \( A^* \), can be easily verified (cf. [2, sec. 7]).

Proposition 4.2

Let \( G = (V, E) \), \( \emptyset \neq A \subseteq B \subseteq V \). There exists a subset \( A' \subseteq A \) such that \( A' \sim B \) if and only if \( A^{*} \sim B \).

An effective computation of \( A^* \) is provided in [2, sec. 7], based on a fixed point characterization of \( A^* \). The verification whether \( A^* \) is a weak attractor for \( B \) can be accomplished by using the algorithm presented in section 5.

4.2 Region of weak attraction

Let \( G = (V, E) \) be a process. In a previous section we showed that for every \( E \)-invariant subset \( A \) there is a (unique) maximal subset \( B \) for which \( A \sim B \), and thus the notion of the region of strong attraction is well defined. In this section we examine whether an analogous notion can be defined for weak attraction. That is, given a nonempty subset \( A \subseteq V \), we want to know whether the class of subsets that are weakly attracted by \( A \) is closed under set union, and hence has a maximal element.

Let \( A \) be \( E \)-invariant and define the class of subsets \( W(A) \) according to

\[
W(A) = \{ B \subseteq V \mid A \subseteq B \text{ and } A \sim B \}.
\]
Proposition 4.3:

Let \( A \) be an \( E_u \)-invariant subset of \( V \). Then the class \( \Omega_G(A) \) is nonempty and closed under set union.

Since \( W_G(A) \) is finite and closed under set union it follows that \( W_G(A) \) contains a unique supremal element w.r.t. inclusion, denoted \( \Omega_G(A) \) and called the region of weak attraction of \( A \) w.r.t. \( G \). If \( A \) is not \( E_u \)-invariant we say that \( \Omega_G(A) = \emptyset \). Further, if \( \omega_G(A) = V \) we say that \( A \) is a global weak attractor w.r.t. \( G \), denoted \( A \Leftrightarrow \).

It is easily seen that

\[ \Lambda_G(A) \subseteq \Theta_G(A) \]

for every \( A \subseteq V \).

ALGORITHM

Input: A process \( G = (V,E) \) and a subset \( A \subseteq V \).
Output: A subprocess \( P \) whose state set is \( \Theta_G(A) \).

1. Let \( P_0 \) s.t. (\( U_0, D_0 \)) = 〈A ≥ G , j = 0〉.
2. If there are no \( P_j \)-attractable states in \( V - U_j \) then \( P = P_j \), stop.
3. Let \( v \in V - U_j \) be a \( P_j \)-attractable state.
Define \( P_{j+1} : (U_{j+1}, D_{j+1}) \) as

\[ U_{j+1} = U_j \cup \{ v \} \]

\[ D_{j+1} = D_j \cup \{ (v, σ, u) \in E \mid u \in U_j \} \]

\[ j := j + 1 \]

5. Computation of \( \Omega_G(A) \)

Fix \( G = (V,E), \emptyset \not \subseteq A \subseteq V \). In this section we propose an algorithm for the computation of the region of weak attraction \( \Omega_G(A) \). A by-product of this algorithm is a subprocess of \( G \) satisfying conditions (b1)-(b3). Further, the question of whether \( A \) is a weak attractor for a subset \( B \subseteq A \) is equivalent to the question of whether \( B \subseteq \Omega_G(A) \). Thus, the algorithm provides a constructive method for verifying weak attractor.

Throughout this section we assume that \( A \) is \( E_u \)-invariant, for otherwise \( \Omega_G(A) = \emptyset \).

We derive now an intuitive consequence of proposition 4.1 concerning the region of weak attraction. Since, by definition, \( A \subseteq \Omega_G(A) \), it follows by proposition 4.1 that there exists a subprocess \( G' = (V', E') \) of \( G \) such that \( \Omega_G(A) \subseteq V' \) and \( G' \) satisfies condition (b1)-(b3). Moreover, \( G' \) must satisfy the condition that \( V' \subseteq \Omega_G(A) \). Otherwise the process \( G' \) would have been a contradiction to the assumption that \( \Omega_G(A) \) is the largest subset for which \( A \) is a weak attractor.

We have proved:

Proposition 5.1

Let \( G' = (V', E') \subseteq G \) be a subprocess such that \( \Omega_G(A) \subseteq V' \). If \( G' \) satisfies (b1)-(b3), then

\[ V' = \Omega_G(A) \]

The subprocess \( G' = (V', E') \) in proposition 5.1 is not necessarily unique. However, its state set \( V' \) is unique. The result of the algorithm below for computing \( \Omega_G(A) \) is a subprocess of \( G \) that satisfies conditions (b1)-(b3) and whose state set is \( \Omega_G(A) \). But first we need the following definition.

Let \( G' = (V', E') \subseteq G = (V,E) \) be a process satisfying conditions (b1)-(b3), that is, \( G' \) is realizable and \( A \) connected and \( G' - A \) is acyclic. We say that a state \( v \in V - V' \) is \( G' \)-attractable iff \( v \) is a predecessor of a state in \( V' \) and every uncontrolled edge of \( G \) leaving \( v \) ends in \( V' \), that is, \( v \in V - V' \) is \( G' \)-attractable iff

\[ (\exists (v, σ, u) \in E) \quad u \in V' \quad \text{and} \]

\[ (\forall (v, σ, u) \in E'_j) \quad u \in V' \]

Now we are ready for the following

Theorem 5.1

Let \( P = (U,D) \) be the process obtained in step (2). Then

(i) \( P \) satisfies conditions (b1)-(b3).

(ii) \( U = \Omega_G(A) \).

For the proof of Theorem 5.1 we need the three following propositions. Intuitively, the first proposition states that \( A \) is a weak attractor for the state set of each process \( P_j \). Formally, we have the following

Proposition 5.2

For every iteration \( j \), the process \( P_j \) satisfies conditions (b1)-(b3), that is, \( P_j \) is realizable and \( A \) connected and \( P_j - A \) is acyclic.

The second proposition clarifies the role of attractable states.

Proposition 5.3

Let \( P = (U,D) \subseteq G \) such that \( A \subseteq U \) and \( P \) satisfies conditions (b1)-(b3). Then every \( P \)-attractable state \( v \in V - U \) is in the region of weak attraction of \( A \), i.e.,

\[ v \in \Omega_G(A) \]

The final proposition required for the proof of theorem 5.1 characterizes the class of subprocesses of \( G \) whose state set is the region of weak attraction of \( A \).
Proposition 5.4
Let \( P = (U, D) \subseteq G \) such that \( A \subseteq U \) and \( P \) satisfies conditions (b1)-(b3). Then
\[
U = \Omega_0(A)
\]
if and only if there are no \( P \)-attractive states in \( V - U \).

Proof of proposition 5.4
Let \( P = (U, D) \subseteq G = (V, E) \) such that \( A \subseteq U \) and \( P \) satisfies conditions (b1)-(b3).

Let \( X \) denote the set of all states in \( V - U \) that are predecessors of a state in \( U \), i.e.,
\[
X = \{ x \in V - U \mid \exists (x', x, u) \in E \} \cup U
\]
and suppose that every state in \( X \) is not \( P \)-attractive. We have to prove that \( U = \Omega_0(A) \).

First notice that \( P \) satisfies conditions (b1)-(b3) and thus, by corollary 4.1, \( U \subseteq \Omega_0(A) \). For the reverse inclusion we shall show first that none of the states in \( X \) can be in the region of weak attraction of \( A \), i.e.,
\[
X \cap \Omega_0(A) = \emptyset.
\]

For this let \( x \in X \subseteq V - U \) and suppose, towards a contradiction, that \( x \in \Omega_0(A) \). According to proposition 4.1, if \( (U \cup \{ x_1 \}) \subseteq \Omega_0(A) \) then there exists a subprocess \( G' = (V', E') \) of \( G \) such that \( (U \cup \{ x_1 \}) \subseteq V' \) and \( G' \) satisfies conditions (b1)-(b3). Since none of the states in \( X \) is \( P \)-attractive then there exists an edge \( e_1 = (x_1, x, v_1) \in E_0 \) such that \( v_1 \notin U \). The edge \( e_1 \) is uncontrolled and thus \( v_1 \), as well as \( v_1 \), must be included in \( G' \). Otherwise \( G' \) could not be realized (condition (b2)). Moreover, \( G' \) is \( A \)-connected and thus it must contain a trajectory from \( v_1 \) to a state in \( U \) (notice that every state in \( U \) is connected to \( A \)). Now, since \( v_1 \notin U \) there are two possibilities: either \( v_1 \notin V - U \) or \( v_1 \in X \).

(i) If \( v_1 \notin V - U \) then every trajectory of \( G' \) from \( v_1 \) to a state in \( U \) must include at least one state in \( X \) (this is because every predecessor of a state in \( U \) is in \( X \)). Let \( t \) be such a trajectory, namely a trajectory connecting \( v_1 \) to \( U \), and let \( x_1 \) be the first state in \( X \) traversed by \( t \). Subsequently, denote by \( t_1 \) the subtrajectory of \( t \) connecting \( v_1 \) to \( x_2 \), i.e., \( t_1 = (x_1, x_2) \). Notice that none of the states of \( t_1 \) is in \( U \) (written \( t_1 \cap U = \emptyset \)). Also, the condition \( x_2 \notin A \) must be satisfied in order that \( G' \) is \( A \)-connected and thus it must contain the cycle \( x_1 \), \( x_2 \).

(ii) If \( v_1 \in X \) then \( x_2 = v_1 \) and \( t_1 \) is the empty trajectory.
Since \( t_1 \) is a trajectory of \( G' \) then \( x_2 \) is also a state of \( G' \). So we conclude that
\[
x_1 \in V' \implies x_2 \in V'.
\]

Following the argument of the previous paragraph we get that \( G' \) must contain a trajectory, say \( t_2 \), connecting \( x_2 \) to \( x_3 \), where \( x_3 \notin \{ x_1 \} \) and \( x_3 \in V' \).

Continuing this procedure we end up with the following conclusions regarding the process \( G' \) : \( x_1 \) is connected to \( x_2 \) by \( t_1 \), \( x_2 \) is connected to \( x_3 \) by \( t_2 \), \( x_3 \) is connected to \( x_4 \) by \( t_3 \), \( x_4 \) is connected to \( x_5 \), \( 1 \leq j \leq n \), by \( t_j \) and
\[
x_1, x_2, \ldots, x_n \in V',
\]
where \( n \) is number of states in \( X \) and \( x_i \neq x_j \), \( i \neq j \).

It is readily verified that the trajectory \( t_1, t_2, \ldots, t_n \) forms a cycle in \( G' \) (note that \( A \subseteq U \) and \( t_1 \cap U = \emptyset \), \( 1 \leq i \leq n \)). Thus we conclude that the assumption \( x_1 \in V' \cap \Omega_0(A) \) implies \( X \cap \Omega_0(A) = \emptyset \).

As regards the rest of the states in \( V - U \) : since every path from a state in \( V - U \) to a state in \( A \) must traverse at least one state in \( X \) it is clear that
\[
(V - U) \cap \Omega_0(A) = \emptyset. \tag{5.2}
\]
From (5.1) and (5.2) we get that \( \Omega_0(A) \subseteq U \), which concludes the "if" part of this proof, i.e.,
\[
U = \Omega_0(A).
\]
(Only if). Suppose \( U = \Omega_0(A) \) and assume there exists a state \( v \in V - U \) such that \( v \) is \( P \)-attractive. However, by proposition 5.3 it follows that \( v \in \Omega_0(A) \), contradicting our assumption that \( U \) is the region of weak attraction of \( A \).

Proof of Theorem 5.1
Let \( P = (U, D) \subseteq G = (V, E) \) be the process obtained in step (2) of the algorithm. By proposition 5.2 it is clear that \( P \) satisfies conditions (b1)-(b3). Further, since every state in \( V - U \) is not \( P \)-attractive (according to the condition of step (2)) then by proposition 5.4
\[
U = \Omega_0(A).
\]

6. Example
Let \( G = (V, E) \) be a process as displayed below:

![Diagram](image-url)

The state set of \( G \) is \( V = \{ 0, 1, \ldots, 7 \} \), and the edge set is \( E = \{ u_i \} \cup \{ c_i \} \). The edges denoted \( u_i \) are uncontrolled while \( c_i \) denotes a controlled edge.

First we comment that the subset \( A_1 = \{ 1, 2 \} \) cannot be a strong attractor for any subset of \( V \) (since \( A_1 \) is not \( E \)-invariant). Nevertheless, \( A_1 \) is \( E_0 \)-invariant and thus it has a potential to become a weak attractor (e.g., by the deletion of \( c_4 \)).

Next we consider the subset \( A = \{ 0, 1, 2 \} \). Clearly \( A \) is \( E \)-invariant, and if \( B_0 = A \cup \{ 3 \} \) then \( A \) is a strong attractor for \( B_0 \) w.r.t. \( G \). It is easily seen that \( B_0 \) is the maximal subset of \( V \) which is strongly attracted by \( A \). That is, \( \Lambda_0(A) = B_0 \). Further, we remark that the region of strong attraction \( \Lambda_0(A) \) can be computed in polynomial time by using the transitive closure of \( G \) (see at the end of section 3).
We examine now the weak attraction problem, namely given two subsets $A, B$ of $V$, decide whether there exists a supervisor $S$ such that $A$ is a strong attractor for $B$ w.r.t. $(S, G)$. To this end, let $A = \{0, 1, 2\}$, $B_1 = A \cup \{3, 4\}$ and $B_2 = A \cup \{7\}$. Recall that we defined weak attraction $A \rightarrow B$ as the possibility of driving $G$ (under control) from every initial state in $B$ to some state in $A$. Consequently, the deletion of the controlled edge $c_5$ implies $A \rightarrow B_1$. Furthermore, it is readily verified that the subprocess $< B_1 > G_2$ (i.e. the subprocess induced by the states of $B_1$) satisfies the conditions of weak attraction, as stated in proposition 4.1.

As regards $B_2$, it can be shown that no subprocess of $G$, whose state set contains $B_2$, satisfies the conditions of proposition 4.1 (i.e., (b1)-(b3)). Thus we conclude that $A$ is not a weak attractor for $B_2$. Intuitively, this result can be explained as follows: Suppose $G$ is initialized at state $7 \in B_2$. Then either $G$ reaches state 0 (and then is captured in $A$) or it executes $u_1$ and reaches state 6. Since the edge $u_2$ is uncontrolled and (thus cannot be removed from $G$) it follows that the edge $c_9$ must not be deleted from $G$. Otherwise the subset $A = \{0, 1, 2\}$ is not reachable from state 6. However, the latter conclusion and the fact that $u_1$ is uncontrolled imply the existence of the cycle $C = 6, 5, 6$. The cycle $C$ prevents the guaranteed attraction of state 7 to a state in $A$, i.e., if $G$ is initialized at state 7 then no control strategy can assure that $G$ (under control) will reach the subset $A$ after executing a finite number of state transitions.

The existence of a subprocess $G'$ as required in proposition 4.1 can be effectively verified by using the algorithm of section 5 for computing the region of weak attraction. If we apply the algorithm to this example we obtain the following steps:

1. Start with the subprocess $P_0 = < A > G = (\{0, 1, 2\}, \{u_2, u_4, c_1, c_5\})$ (step (i)).

2. A candidate state for the next step is any predecessor of a state in $A$ which is $P_0$-attractable. Since the uncontrolled edges $u_4$, $u_2$ and $u_4$ lead to a state in $V \backslash A$, none of the states 7 or 5 is $P_0$-attractable. Thus, choose for example state 3 and construct (step (3)) the subprocess

   $P_1 = (\{0, 1, 2, 3\}, \{u_2, u_4, c_4, c_5, c_2, u_1\})$

   \[
   \rightarrow < \{0, 1, 2, 3\} > G .
   \]

3. Only state 4 is $P_1$-attractable and thus construct (step (3) again) the subprocess

   $P_2 = (\{0, 1, 2, 3, 4\}, \{u_2, u_4, c_4, c_5, c_2, u_1, u_2, c_1\})$

   \[
   \rightarrow < \{0, 1, 2, 3, 4\} > G .
   \]

4. There are no $P_2$-attractable states and thus the algorithm terminates (step (2)).

By theorem 5.1 we conclude that $Q_0(A) = \{0, 1, 2, 3, 4\} = B_1$, and that $P_2$ satisfies the conditions of proposition 4.1. Based on $P_2$, a control pattern achieving weak attraction of $B_1$ by $A$ is readily synthesized (see the proof of proposition 4.1).

As was explained in the paragraph following proposition 5.1, the resulting process in (step 2) is not unique. For example, if we had interchanged steps (ii) and (iii) we would have ended up with the process $P_2 - c_1$. Nevertheless, the region of weak attraction of $A$ is yet $B_1$, since the state set of $P_2 - c_1$ is $B_1$. This illustrates the consequence of proposition 4.3, namely that the region of weak attraction is well defined.

Our intuitive conclusion that $A$ is not a weak attractor for $B_2$ is now an immediate consequence of the fact that $B_2$ is not a subset of $Q_0(A)$.

We end this example by pointing out the close relation between attraction properties and the problem of recovery from control failures. For example, suppose $A = \{0, 1, 2\}$ is the legal state set of $V$, and that a control failure may cause $G$ to reach the illegal state 7. Since $A$ is not a weak attractor for $B_2$, no control strategy can assure a guaranteed recovery (i.e. a guaranteed return of $G$ (under control) to a legal state in $A$) from this control failure. On the other hand, $A \rightarrow B_1$ implies the existence of a supervisor achieving guaranteed recovery from control failures causing $G$ to reach states 3 or 4. Such a supervisor is readily synthesized by using the output of the algorithm in section 5.

7. Conclusion

The paper has presented the concept of strong attraction which plays a key role in the investigation of the following problems. The first one is the ability of a process to reach a set of target states from an arbitrary state and then remain there indefinitely. Another problem, which is closely related to the former, is the recovery from control failures. Finally, a special kind of asymptotic behavior of a process has been characterized as its minimal strong attractor. The first two problems were examined also under control, and an efficient procedure for synthesizing controllers that improve the attraction ability of processes has been proposed. The properties of such controllers and the extension of the above results for other representations of discrete event processes are interesting topics for further research.

REFERENCES


