VI. CONCLUDING REMARKS.

In this correspondence, it has been shown that the Hankel matrix (1) defined by Newton's sums can be used to obtain information on the location of zeros of a real polynomial \( f(x) \). In [6, pp. 208-236], certain other types of Hankel matrices constructed from Markov parameters of the real rational function \( R = p(x)/q(x) \) have been used in connection with the computation of the Cauchy index of \( R \) and to prove Routh-Hurwitz stability criterion. Interesting relationships between these matrices and the Bezoutian matrix, defined by two polynomials \( p(x) \) and \( q(x) \), have been established by Anderson in [1] and [2]. It would be quite interesting to see if these Hankel matrices of Markov parameters are also related to the Hankel matrix (1) of Newton sums used in this correspondence.

ACKNOWLEDGMENT

The author would like to express his thanks to the referee for drawing his attention to the paper of Anderson [1].

REFERENCES


Weak and Strong Max-Min Controllability

M. HEYMANN, M. PACTER, AND R. J. STERN

Abstract—Weak and strong max-min controllability for a two player (time-varying) linear control system are defined. It is proved that the two concepts are equivalent.

Consider the following linear system with dual controls:

\[
x(t) = A(t)x(t) + B_p(t)u(t) + B_e(t)v(t), \quad x(t_0) = x_0
\]

where \( x = x(t) \in \mathbb{R}^n \) is the state vector with \( x_0 \) specified initial state. The vectors \( u = u(t) \in \mathbb{R}^{n_u} \) and \( v = v(t) \in \mathbb{R}^{n_v} \) are regarded, respectively, as the pursuer and evader controls and are required to satisfy \( \int _{t_0}^{t} ||u(t)||^2 dt < \infty \) and \( \int _{t_0}^{t} ||v(t)||^2 dt < \infty \) on each compact interval \( I \subset [t_0, \infty) \). \( ||.|| \) denotes the Euclidean norm. The matrices \( A, B_p, B_e \) are assumed to have entries which are real and measurable on \( [t_0, \infty) \). For any pair of controls \( u \) and \( v \) we shall denote by \( x(t) \in \mathcal{Q}(t_0, t, u, v) \) the corresponding unique solution of (1) starting at \( x_0 \) at time \( t_0 \) \( (t \geq t_0) \).

Definition 1: An event \((t_0, x_0)\) in system (1) is weakly max-min controllable if for each announced evader control \( v \in [t_0, \infty) \), there exists a time \( i = i(v) \geq t_0 \) and a pursuer control \( u \) on \([t_0, i]\) such that \( x(t) = q(t, t_0, x_0, u, v) = 0 \). The event \((t_0, x_0)\) is strongly max-min controllable if for some \( i \geq t_0 \), it is strongly max-min controllable for some \( T \subset [t_0, \infty) \).

Obviously strong max-min controllability of an event implies weak max-min controllability. It will be shown in this note that the converse is also true. In [1] an extensive investigation of max-min controllability is presented.

It will be convenient to transform system (1) by a change of coordinates as follows: let \( \Phi(t, t_0) \) denote the fundamental matrix solution corresponding to system (1) and define the vector function

\[
z(t) = \Phi(t_0, t)x(t), \quad t \geq t_0
\]

Then, it is readily verified that \( z \) satisfies the following differential equation:

\[
z(t) = \tilde{B}_p(t)u(t) - \tilde{B}_e(t)v(t), \quad z(t_0) = x(t_0) = x_0
\]

where \( \tilde{B}_p(t) = \Phi(t, t_0)B_p(t) \) and \( \tilde{B}_e(t) = \Phi(t, t_0)B_e(t) \). Furthermore, it is easily noted that if \( z(t) = 0 \) if and only if \( z(t_0) = 0 \), whence system (3) is completely equivalent in respect to max-min controllability to system (1).

Define the controllability Grammians for the pursuer and evader by

\[
W_p(t_0, t) = \int _{t_0}^{t} \tilde{B}_p(s)\tilde{B}_p(s)^T ds, \quad t \geq t_0
\]

\[
W_e(t_0, t) = \int _{t_0}^{t} \tilde{B}_e(s)\tilde{B}_e(s)^T ds, \quad t \geq t_0
\]

where the prime denotes transpose. Clearly, \( W_p \) and \( W_e \) are symmetric nonegative definite \((n \times n)\) matrices and it is easily verified that their rank is a nondecreasing and left-continuous function of time (the latter holding because \( \mathbb{R}_+(F) \subset \mathbb{R}_+(F+G) \) for every pair of symmetric nonnegative matrices \( F,G \) where \( \mathbb{R}_+ \) denotes range). In [1] the following theorem is proved.

Theorem 1: Given system (3) with \( z_0 \neq 0 \), a necessary and sufficient condition for an event \((t_0, z_0)\) to be strongly max-min controllable in finite time \( T > t_0 \) is that the following conditions hold:

\[
z_0 \in \mathbb{R}(W_p(t_0, T))
\]

\[
\mathbb{R}(W_e(t_0, T)) \subset \mathbb{R}(W_p(t_0, T)).
\]

The remainder of this note will be devoted to proving the following theorem.

Theorem 2: Consider system (3). An event \((t_0, z_0)\) is strongly max-min controllable if and only if it is weakly max-min controllable.

We only have to prove that weak max-min controllability implies strong max-min controllability for some finite \( T > t_0 \).

Let

\[
T_1 = \inf _{t \geq t_0} \{ t : z_0 \in \mathbb{R}(W_p(t_0, t)) \}
\]

Under the assumption of weak max-min controllability we obviously have that \( T_1 \) is finite, and due to Grammian monotonicity, \( z_0 \in \mathbb{R}(W_p(t_0, t)) \) for all \( t > T_1 \). Hence, if \( T_2 \) is defined by

\[
T_2 = \sup _{t \geq t_0} \{ t : \mathbb{R}(W_e(t_0, t)) \subset \mathbb{R}(W_p(t_0, t)) \}
\]

it follows from Theorem 1 that \((t_0, z_0)\) is strongly max-min controllable if and only if \( T_2 > T_1 \). (It should be observed that when finite, \( T_1 \) and \( T_2 \)
are points of discontinuity of rank \( (W_p(t_0,i)) \) and of rank \( (W_p(t_0,i)) \), respectively. Also, when \( T_2 \) is finite, Grammian left-continuity implies that the sup in the definition of \( T_2 \) is attained as a max.) The proof of Theorem 2 will be accomplished by showing that if \( T_2 < T_1 \) (i.e., when strong max-min controllability fails to hold), then there exists a control \( v^* \) against which no \( u \) can drive \( z_0 \) to the origin, thereby showing that weak max-min controllability also fails to hold.

We shall require the following lemmas.

**Lemma 1:** Given system (3) with \( t_0+\alpha > 0 \), assume that \( T_1 \) is finite and that \( T_2 < T_1 \). Let \( (u,v) \) be a pair of control functions such that the corresponding solution of (3) satisfies \( z(t) = 0 \) for some \( i_0 < t < \infty \). Then \( i > T_2 \).  

**Proof:** Suppose \( i < T_2 \). If \( z(t) = 0 \) then the control \( v \) must satisfy  
\[
  z_0 - \int_{t_0}^{T_2} \hat{B}_i(t)v(t)dt \in \mathfrak{R}(W_p(t_0,i)).
\]  
(8)

Since \( i < T_2 \) and since \( \int_{t_0}^{T_2} \hat{B}_i(t)v(t)dt \in \mathfrak{R}(W_p(t_0,i)) \), it follows that  
\[
  z_0 - \int_{t_0}^{T_2} \hat{B}_i(t)v(t)dt \in \mathfrak{R}(W_p(t_0,i))
\]  
and  
\[
  \mathfrak{R}(W_p(t_0,i)) \subset \mathfrak{R}(W_p(t_0,i_0)).
\]  
(9)

From (8)-(10) we conclude that  
\[
  z_0 \in \mathfrak{R}(W_p(t_0,T_2))
\]  
(11)

which in view of Grammian left-continuity violates the assumption that \( T_2 < T_1 \).

**Lemma 2:** Given system (3), let \( T_1 < \infty \) and let \( T_2 \) denote the first discontinuity of rank \( W_p(t_0,i) \), \( i > T_2 \). Then for each positive \( e < T_2 - T \) there exists a measurable set \( I_e \) of positive measure, \( \mathfrak{R}(I_e) \subset \mathfrak{R}(W_p(t_0,T_2 + e)) \), such that  
\[
  \mathfrak{R}(\hat{B}_i(t)) \subset \mathfrak{R}(W_p(t_0,T_2 + e))
\]  
(12)

for almost all \( t \in [T_2,T_2 + e] \). Since  
\[
  \mathfrak{R}(\hat{B}_i(t)) \subset \mathfrak{R}(\hat{B}_i(t))
\]  
(13)

for all \( t \in [T_2,T_2 + e] \), and since  
\[
  \mathfrak{R}(\hat{B}_i(t)) \subset \mathfrak{R}(\hat{B}_i(t))
\]  
(14)

for all \( t \in [T_2,T_2 + e] \), it follows that  
\[
  \mathfrak{R}(\hat{B}_i(t)) \subset \mathfrak{R}(W_p(t_0,T_2 + e)).
\]  
(15)

thus contradicting the definition of \( T_2 \).

**Lemma 3:** Given system (3), let \( T > T_2 \) be any point of discontinuity of rank \( W_p(t_0,i) \), and let \( T_1^* \) be the first discontinuity of rank \( W_p(t_0,i) \), \( i > T \). Then for each positive \( e < T - T_1^* \) there exists a measurable set \( I_e \) of positive measure \( I_e \cap [T_2,T_2 + e] \), such that  
\[
  \mathfrak{R}(\hat{B}_i(t)) \subset \mathfrak{R}(W_p(t_0,T_2 + e))
\]  
(16)

for almost all \( t \in [T_2,T_2 + e] \).

The proof of Lemma 3 is similar to that of Lemma 2 and is therefore omitted.

**Lemma 4:** Let \( z(t) \) be a measurable \( R^m \) -valued function and let \( \mathfrak{R} \) be a proper subspace of \( R^m \). Let \( I_e \subset R^1 \) be a measurable set of positive measure such that \( (z(t)) \notin \mathfrak{R} \) for all \( t \in I_e \). Then there exists a vector \( h \) such that  
\[
  \langle h, v(t) \rangle = 0
\]  
(17)

for all \( t \in I_e \) and \( h(z(t)) \notin \mathfrak{R} \) for all \( t \in I_e \).  

**Proof:** Consider a hyperplane \( \mathfrak{R} \subset \mathbb{R}^m \). Each vector \( z(t) \) is either in one of the two open half spaces determined by \( \mathfrak{R} \) or in \( \mathfrak{R} \) itself. First assume that \( z(t) \notin \mathfrak{R} \) almost everywhere. Then choose \( h \notin \mathfrak{R} \). If the aforementioned assumption does not hold, one repeats the argument in an appropriate subspace of \( \mathfrak{R} \).

Assume \( T_2 < T_1 \). We shall now turn our attention toward the construction of a control \( v^* \) such that against \( v^* \) no \( u \) can force \( z_0 \) to 0 in finite time. This of course will complete our proof of Theorem 2. First, we shall replace \( \bar{B}_i(t) \) with a measurable \( (n \times p) \) matrix function \( \hat{B}_i(t) \) satisfying the following properties.

**Property 1:** \( \bar{B}_i(t) \equiv \hat{B}_i(t) \), for all \( t \in [t_0,T_2] \).

**Property 2:** \( \mathfrak{R}(W_p(t_0,i)) \subset \mathfrak{R}(\hat{B}_i(t)) \), for all \( t > t_0 \) where \( \hat{B}_i(t) \) denotes the Grammian associated with \( \hat{B}_i(t) \).

**Property 3:** \( \hat{B}_i(t) \) is constant between consecutive discontinuities \( T_1, T_2 \) of rank \( (W_p(t_0,i)) \).

**Property 4:** At each discontinuity \( \sigma \) of rank \( W_p(t_0,i) \), \( \sigma > T_2 \), we have  
\[
(18)
\]

and  
\[
(19)
\]

One can readily see that a \( \hat{B}_i \) satisfying Properties 1-5 can always be built—one first specifies \( \hat{B}_i \) and then defines an appropriate \( \hat{B}_i \). Details of this elementary recursive (but somewhat tedious) construction are left to the reader. Consider now the system  
\[
(3')
\]

Observe that if we construct a control \( v^* \) such that no \( u \) can drive \( z_0 \) to 0 in system (3') then (by Property 2) this is certainly the case for system (3), which is what we require. Upon defining \( T_1 \) for system (3) analogously to \( T_1 \) for system (3), it follows that \( T_2 < T_1 < T_1 \). The required control \( v^* \) is now constructed by the following procedure.

1) On \( (t_0,T_2) \) apply an arbitrary evader control. Lemma 1 guarantees the impossibility of \( z(t) = 0 \) at or before \( T_2 \).

2) Let \( T_2 \) denote the first discontinuity of rank \( W_p(t_0,i) \), \( T_2 > T_2 \). If \( T_2 < T_1 \), then take \( v^* = 0 \) on \( [T_2, T_3] \). Letting \( e = (T_3 - T_2)/2 \), it follows that in the presence of any \( u \) the solution of (3') satisfies \( z(t) = 0 \) on \( [T_2, T_3] \) for all \( t \in [T_2, T_3] \). Therefore, \( T_2 < T_1 \) implies that \( \hat{B}_i(t) \in \mathfrak{R}(\hat{B}_i(t) + e) \), and thus \( z(t) = 0 \) for all \( t \in [T_2, T_3] \). Now let \( T_2 = T_3 \). Suppose \( \mathfrak{R}(W_p(t_0,i) + e) \). Then for all \( t \), otherwise for (3) and (7) hold on some \( T, T_2 \) and there is nothing to prove. Due to Lemmas 2-4 it is readily seen that there exists a set \( I_e \), of positive measure, \( \mathfrak{R}(I_e) \subset \mathfrak{R}(W_p(t_0,i)) + e \), a measurable \( R^m \) -valued function \( v \) on \( [T_2, T_2 + e] \), and a vector \( h \) in \( \mathfrak{R}(W_p(t_0,i) + e) \). Clearly  
\[
(13)
\]

and  
\[
(14)
\]

Since  
\[
(15)
\]

then contradicting the definition of \( T_2 \).

**References**
