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Max-Min Control Problems: A System Theoretic Approach

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Abstract—In this paper we “max-min controllability” concept for a situation in which two linear control systems are in conflict is introduced and characterized. This concept is employed in solving a max-min linear-quadratic control problem with terminal state constraints and the relationship with differential game theory is discussed.

I. Introduction

Consider the following linear system with dual controls:

\[ \dot{x} = A(t)x + B_p(t)u + B_e(t)v, \quad x(t_0) = x_0 \]  

(1.1)

Here \( x = x(t) \) is the state vector in Euclidean space \( \mathbb{R}^n \), with \( x_0 \) a specified initial state at time \( t_0 \). The vectors \( u = u(t) \in \mathbb{R}^m \) and \( v = v(t) \in \mathbb{R}^m \), regarded respectively, as the pursuer and evader controls, are required to satisfy integral constraints on each compact interval \( I \subset [t_0, \infty) \), where \text{norm} denotes the Euclidean norm. The matrices \( A, B_p, \) and \( B_e \) are assumed to have entries which are real and measurable on \([t_0, \infty)\). For any pair of controls \( u \) and \( v \) we shall denote by \( x(t) = \phi(t,t_0,x_0,u,v) \) the corresponding unique solution of (1.1) emanating from \( x_0 \) at time \( t_0 \).

In situations in which the pursuer and evader are in competition, it is natural to seek a comparison between their control capabilities. Towards this end we introduce the following concepts.

Definition 1.1: An event \((t_0,x_0)\) in system (1.1) is strongly max-min controllable at time \( T \) (\( T \geq t_0 \)) if for each announced evader control \( v \) on \([t_0,T]\) there exists a pursuer control \( u \) on \([t_0,T]\) such that \( x(T) = \phi(t,T,x_0,u,v) = x_0 \). The event \((t_0,x_0)\) is strongly max-min controllable if it is strongly max-min controllable for some \( T \in [t_0, \infty) \).

Definition 1.2: An event \((t_0,x_0)\) in system (1.1) is weakly max-min controllable if for each announced evader control \( v \) on \([t_0, \infty)\), there exists a time \( \bar{T} = \bar{T}(v) \geq t_0 \) and a pursuer control \( u \) on \([t_0, \bar{T}]\) such that \( x(\bar{T}) = \phi(t_0,x_0,u,v) \).

Clearly strong max-min controllability of an event implies weak max-min controllability. That the converse is also true is not immediately evident since it is not clear that when weak max-min controllability holds there exists any one time \( T \) at which capture (i.e., \( x(T) = 0 \)) can be imposed by the pursuer in face of any evader control. This, however, is indeed the case as is shown in [1], and the two concepts of max-min controllability are actually equivalent. Henceforth, we will simply speak about max-min controllability referring to the simpler Definition 1.1.

It should be observed that max-min controllability generalizes the concept of controllability in linear control systems as expounded by Kalman (see, e.g., [2], [3]). While the existing “one player” controllability theory will be brought to bear on our development of the two player case, certain significant difficulties and interesting differences arise, as will be pointed out below.

Our results on max-min controllability will be employed in solving the following restricted end-point max-min control problem, denoted \((P)\).

\((P)\): We are given a linear dual control system (1.1) with \( x_0 \neq 0 \). The evader announces a control function \( v \), and the pursuer (if he has the capability) responds with a control function \( u \) such that \( x(T) = 0 \), where \( T > t_0 \) is a prespecified time. The players’ control choices are to be made in accordance with the optimization of the payoff functional

\[ P(u,v) = \int_{t_0}^{T} \left[ \|u(t)\|^2 - \|v(t)\|^2 \right] dt \]  

(1.2)

where it is understood that the evader is the maximizing player while the pursuer is the minimizer.
Let $\Phi(t,t_0)$ denote the fundamental matrix solution corresponding to system (1.1) and define the vector function

$$z(t) = \Phi(t,t_0)x(t), \quad t \geq t_0. \quad (1.3)$$

It is readily verified that $z$ satisfies the following differential equation:

$$\dot{z} = \tilde{B}_p(t)u - \tilde{B}_e(t)v \quad z(t_0) = x(t_0) = x_0. \quad (1.4)$$

where $\tilde{B}_p(t) = \Phi(t,t_0)B_p(t)$ and $\tilde{B}_e(t) = -\Phi(t,t_0)B_e(t)$; and $z(t) = 0$ if and only if $x(t) = 0$. Hence, system (1.4) is completely equivalent in respect to max-min controllability to system (1.1).

A more general type of max-min control situation can be described by a pair of linear control systems:

$$\dot{x}_p = A_p(t)x_p + B_p(t)u, \quad x_p(t_0) = x_{p0} \quad (1.5)$$

$$\dot{x}_e = A_e(t)x_e + B_e(t)v, \quad x_e(t_0) = x_{e0} \quad (1.6)$$

where (1.5) represents the pursuer dynamics and (1.6) represents the evader dynamics. Both $x_p$ and $x_e$ are in $R^n$ and capture is interpreted as an event $(t, x_p(t), x_e(t))$ such that $x_p(t) = x_e(t)$. Hence, max-min controllability for systems (1.5) and (1.6) refers to the existence of “capture events” as defined above, with Definitions 1.1 and 1.2 remaining otherwise intact. For a fixed $T > t_0$ define

$$z(t) = \Phi_p(t,T)x_p(t) - \Phi_e(t,T)x_e(t), \quad t_0 \leq t \leq T. \quad (1.7)$$

With (1.7), (1.5) and (1.6) yield the following differential equation for $z$:

$$\dot{z} = \tilde{B}_p(t)u - \tilde{B}_e(t)v, \quad z(t_0) = \Phi_p(T,t_0)x_{p0} - \Phi_e(T,t_0)x_{e0}. \quad (1.8)$$

where $\tilde{B}_p(t) = \Phi_p(T,t)B_p(t)$ and $\tilde{B}_e(t) = -\Phi_e(T,t)B_e(t)$ for $0 < t < T$; and $\Phi_p(T,t)$ and $\Phi_e(T,t)$ are the fundamental matrix solutions for (1.5) and (1.6), respectively. It is readily seen that $x_p(T) = x_e(T)$ if and only if $z(T) = 0$ and hence max-min controllability (in time $T$) for systems (1.5) and (1.6) is equivalent to max-min controllability (in time $T$) for (1.8).

Since system (1.8) is essentially the same as (1.4) we shall henceforth restrict our attention primarily to systems of the form (1.4) to which we shall refer as standard.

In [4] Ho, Bryson, and Baron applied the variational calculus to a linear-quadratic differential game of fixed duration (without terminal constraints). They then applied their results to problem (P) by means of penalty functions, and for a special case they obtained a solution as well as a sufficient condition for what we termed max-min controllability. While in the present paper we proceed from a system-theoretic rather than a variational viewpoint, our results on problem (P) are related to those in [4]. (In this regard see Remark 3.11 and also Section IV below.)

Finally, it was pointed out by a referee that problem (P) has various similarities with the so called “Sackelberg solution” of a game. In this regard, the interested reader is referred to [5] and [6].

The organization of the paper is as follows: In Section II we shall give an algebraic characterization of max-min controllability with some special attention to the autonomous case. In Section III problem (P) is solved, and in Section IV we compare our results with results and concepts concerning differential games with more elaborate information schemes as in Isaacs [7] and Hájek [8], [9], and we discuss the relation between max-min controllability and feedback.

II. MAX-MIN CONTROLLABILITY

For a standard system (1.4) defines the controllability Grammians for the pursuer and evader by

$$W_p(t_0,t) = \int_{t_0}^t \tilde{B}_p(\sigma)\tilde{B}_p(\sigma)^T d\sigma, \quad t \geq t_0 \quad (2.1)$$

$$W_e(t_0,t) = \int_{t_0}^t \tilde{B}_e(\sigma)\tilde{B}_e(\sigma)^T d\sigma, \quad t \geq t_0 \quad (2.2)$$

where the prime denotes transpose. Clearly $W_p$ and $W_e$ are symmetric nonnegative definite $(n \times n)$ matrices. Also, since for every pair $F$, $G$ of symmetric nonnegative matrices $F \subseteq (F + G)$ (where $\subseteq$ denotes range), it can be readily verified that the rank of these Grammians is a nondecreasing and left-continuous function of time. For the one player case, i.e., $\tilde{B}_e \equiv 0$, it is well known that the pursuer can drive the event $(t_0, z_0)$ to $(t_1, z_1)$ in system (1.4) if and only if

$$z_0 - z_1 \in \mathcal{R}(W_p(t_0,t_1)). \quad (2.3)$$

This result generalizes in the two player case to the following algebraic condition for max-min controllability.

Theorem 2.1: Given system (1.4) with $z_0 \not= 0$, a necessary and sufficient condition for an event $(t_0, z_0)$ to be max-min controllable in finite time $T > t_0$ is that the following conditions hold:

$$z_0 \in \mathcal{R}(W_p(t_0,T)) \quad (2.4)$$

$$\mathcal{R}(W_e(t_0,T)) \subseteq \mathcal{R}(W_p(t_0,T)). \quad (2.5)$$

**Proof:** Max-min controllability in time $T$ is equivalent to

$$z_0 - \int_{t_0}^T \tilde{B}_e(t)v(t)dt \in \mathcal{R}(W_p(t_0,T)) \quad (2.6)$$

for all evader controls $v$. Since

$$\bigcup_v \int_{t_0}^T \tilde{B}_e(t)v(t)dt = \mathcal{R}(W_e(t_0,T)),$$

it follows that (2.6) is equivalent to

$$z_0 + \mathcal{R}(W_e(t_0,T)) \subseteq \mathcal{R}(W_p(t_0,T)) \quad (2.7)$$

which in turn is equivalent to (2.4) and (2.5).
In a one player linear control system (i.e., $\tilde{B}_p(t) = 0$) it is well known that if an event $(t_0, z_0)$ in system (1.4) can be steered to $(T, 0)$, then it can also be steered to $(T', 0)$ for all $T' > T$. In the two player case this is no longer true in the sense that the fact that an event $(t_0, z_0)$ is max-min controllable in time $T$ does not imply that the same is also true for all $T' > T$. Indeed, there may exist an interval $(t_1, t_2)$ (with $t_0 < t_1 < t_2 < \infty$) such that $(t_0, z_0)$ is never max-min controllable in time $T$ for $T \notin (t_1, t_2)$. This fact is not surprising when one considers condition (2.5). It is then readily observed that max-min controllability has many similarities to (ordinary) controllability to a time-varying manifold, a situation which received essentially no attention in the literature. The above mentioned phenomenon is illustrated by the following simple example.

**Example 2.2:** Consider system (1.4) with dimensions $n = m_p = m_e = 2$. Let

$$
\tilde{B}_p(t) = \begin{pmatrix} 0 \\ b_p(t) \end{pmatrix} \quad \text{and} \quad \tilde{B}_e(t) = \begin{pmatrix} b_e(t) \\ 1 \end{pmatrix}
$$

where

$$
b_p(t) = \begin{cases} 0, & \text{for } t \in (0, 1] \\ 1, & \text{for } t \in (1, \infty) \end{cases} \quad \text{and} \quad b_e(t) = \begin{cases} 0, & \text{for } t \in (0, 2] \\ 1, & \text{for } t \in (2, \infty). \end{cases}
$$

It is easily seen that the initial event $(0, (0, 1))$ is max-min controllable at every $T \in I = (1, 2]$ but never for $T \in I$. For the one player case it is immediate from (2.3) that for any $t_0$, the set of states $z_0$ such that the event $(t_0, z_0)$ is controllable constitutes a subspace of $R^n$. The analog of this important geometric fact is also valid in the two player case, as we now show.

**Theorem 2.3:** Consider system (1.4). For any $t_0$, the set $C(t_0)$ of states $z_0$ such that the corresponding event $(t_0, z_0)$ is max-min controllable is a linear subspace of $R^n$.

**Remark 2.4:** It is of course an immediate consequence of Theorem 2.1 that the set of states $z_0$ for which the corresponding event $(t_0, z_0)$ is max-min controllable in (a fixed) time $T$ is a subspace. Theorem 2.3 claims that the set of states $z_0$ for which the event $(t_0, z_0)$ is max-min controllable in any time $T$ is also a subspace. This is less obvious especially when considering our preceding discussion in relation to the concept of weak max-min controllability.

**Proof of Theorem 2.3:** First observe that $0 \in C(t_0)$ trivially. Also, if $z_0$ satisfies (2.4) for some $T$ such that (2.5) holds, the same is also true for $az_0$, $a \in R$. Hence, $z_0 \in C(t_0)$ implies that $az_0 \in C(t_0)$ for every real $a$. Now let $z_0^1$ and $z_0^2$ be in $C(t_0)$ and let $T'$ denote the times satisfying (2.4) and (2.5) for $z_0^i$, $i = 1, 2$. If we assume (without loss of generality) that $T' > T$ then $z_0^1 + z_0^2 \in W(W(t_0, T'))$ and since (2.5) holds for $T'$ it follows that $z_0^1 + z_0^2 \notin C(t_0)$ and the proof is complete.

Consider now the pair of systems (1.5) and (1.6) and assume that $A_p$, $B_p$, $A_e$, and $B_e$ are constant matrices. Define the following “controllability matrices”:

$$
[A_p|B_p] \triangleq [B_p, A_p B_p, \cdots, A_p^{n-1} B_p]
$$

(2.8)

$$
[A_e|B_e] \triangleq [B_e, A_e B_e, \cdots, A_e^{n-1} B_e].
$$

(2.9)

We can then state the following result for autonomous systems (which has no analog in the time dependent case).

**Theorem 2.5:** Consider systems (1.5) and (1.6) with $A_p$, $B_p$, $A_e$, and $B_e$ constant matrices. Assume that $x_{p0} \in R([A_e|B_e])$. Then a necessary and sufficient condition that for any $t_0$ the event $(t_0, z_{p0}, x_{e0})$ be max-min controllable is that

$$
x_{p0} \in R([A_p|B_p])
$$

(2.10)

and

$$
R([A_p|B_p]) \subset R([A_e|B_e]).
$$

(2.11)

Moreover, if $(t_0, z_{p0}, x_{e0})$ is max-min controllable, then it is max-min controllable in time $T$ for every $T > t_0$.

**Proof:** Max-min controllability is equivalent to the existence of a time $T > t_0$ such that $T$ the reachable set of the pursuer system contains the reachable set of the evader system. Using the standard exponential notation, this is equivalent to

$$
e^{A_p T} x_{e0} + R([A_e|B_e]) \subset e^{A_p T} x_{p0} + R([A_p|B_p])
$$

(2.12)

which in turn is equivalent to (2.11) along with

$$
e^{A_p T} x_{p0} - e^{A_p T} x_{e0} \in R([A_p|B_p]).
$$

(2.13)

Now, $x_{p0} \in R([A_p|B_p])$ if and only if $e^{A_p T} x_{e0} \in R([A_e|B_e])$, and again applying the $A$-invariance property, but to the pursuer system, we conclude that (2.13) is equivalent to (2.10). Since $T > t_0$ can be chosen arbitrarily the proof is complete.

Theorem 2.5 states essentially that if the evader initial state is evader-controllable (in the evader control system) then max-min controllability holds if and only if the pursuer initial state is pursuer-controllable and the evader reachable subspace (from the origin) is also pursuer-reachable. In case $A_p = A_e$ we can subtract (1.6) from (1.5) letting $x = x_p - x_e$. This gives us system (1.1) with $B_e$ remaining intact and $B_p$ replaced by $-B_p$. In this case we can state the following corollary to Theorem 2.5, which is a specialization of Theorem 2.1 to the autonomous case.

**Corollary 2.6:** Consider system (1.1) with $A, B_p$, and $B_e$ constant matrices. A necessary and sufficient condition for max-min controllability of an event $(t_0, x_{e0})$ (for every $t_0$) is

$$
x_{e0} \in R([A|B_p])
$$

(2.12)

and

$$
R([A|B_e]) \subset R([A|B_p]).
$$

(2.13)
in which case $x_0$ can be steered to the origin in arbitrarily short time in the presence of any (announced) evader control.

Remark 2.7: While condition (2.13) is clearly satisfied whenever $\mathcal{R}(B_e) \subset \mathcal{R}(B_p)$, this condition is by no means necessary. However, in cases where (2.13) holds and $\mathcal{R}(B_e) \not\subset \mathcal{R}(B_p)$ we will see later that our information scheme is extremely restrictive and the evader may gain significant advantages by employing a feedback control rather than predesignated controls. This interesting situation is further discussed in Section IV.

III. Solution of Problem (P)

In this section we will focus our attention on the fixed duration restricted end-point max-min control problem $(P)$ which was introduced in Section I. We now rephrase problem $(P)$ as follows.

$(P)$: Given system $(1.4)$ with $z_0 \neq 0$, let $T > t_0$ be such that $(2.4)$ and $(2.5)$ are satisfied, i.e., $z_0$ is max-min controllable in time $T$. The evader announces a control $v$ and the pursuer responds with a control $u$ such that the associated solution of $(1.4)$ satisfies $z(T) = 0$, while both players make their control choices in accordance with the optimization (evader maximizing, pursuer minimizing) of the payoff functional $P(u_v, u_e)$ given by (1.2).

Let the evader specify a control $v$ on $[t_0, T]$. Then

$$
\int_{t_0}^{T} \tilde{B}_e(t)v(t)\,dt \in \mathcal{R}[\mathcal{W}_e(t_0, T)]
$$

and there exists $y \in \mathbb{R}^n$ such that

$$
\int_{t_0}^{T} \tilde{B}_e(t)v(t)\,dt = \mathcal{W}_e(t_0, T)y.
$$

Due to $(2.4)$ and $(2.5)$

$$
z(T) = 0 = z_0 + \int_{t_0}^{T} \tilde{B}_p(t)u(t)\,dt - \mathcal{W}_e(t_0, T)y
$$

$$
= z_0 - \mathcal{W}_p(t_0, T)y - \mathcal{W}_e(t_0, T)y
$$

for some $w = w(y) \in \mathbb{R}^n$. In view of the well-known minimum energy law (see, e.g., [2]) any pursuer control

$$
u_p(t) = -\tilde{B}_p^*w
$$

will drive $z_0$ to 0 and minimize $P(u_v, u_e)$ against the given $v$. Furthermore, as is easily verifiable,

$$
P(u_v, v) = w^T\mathcal{W}_p(t_0, T)y - \int_{t_0}^{T} \|v(t)\|^2\,dt.
$$

Assuming temporarily that $y$ is specified, a reapplication of the minimum energy rule for the evader gives

$$
v_e(t) = \tilde{B}_e^*(t)y
$$

as evader optimal control for a given $y$. Furthermore, for the fixed choice of $y$, $P(u_v, v_e)$ is given by the quadratic expression

$$
P(u_v, v_e) = w^T\mathcal{W}_p(t_0, T)y - y^T\mathcal{W}_e(t_0, T)y.
$$

The max-min solution will consequently be obtained as the solution (whenever it exists) to the following quadratic programming problem, which we denote $\mathcal{P}$:

$$
\begin{align*}
(\mathcal{P}) \quad \text{maximize} & \quad w^T\mathcal{W}_p(t_0, T)y - y^T\mathcal{W}_e(t_0, T)y \\
\text{subject to the constraint} & \quad \mathcal{W}_p(t_0, T)y = z_0 - \mathcal{W}_e(t_0, T)y.
\end{align*}
$$

We shall need to make use of the "generalized inverse" $M^+$ of a real matrix $M$ which is the (unique) solution of the so called "Penrose equations."

1) $MM^+M = M$
2) $M^+M = M$
3) $(MM^+)^T = MM^+$
4) $(M^+M)^+ = M^+M$.

Note that $M^+ = M^T$. Also, $MM^+$ is the projection onto $\mathcal{R}(M)$ along $\mathcal{N}(M^T)$ (where $\mathcal{R}$ denotes null space). In addition, we will make use of the fact that $\mathcal{R}(M^T) = \mathcal{R}(M)$ and of the fact that if $M$ is symmetric so is $M^+$. For a system of linear equations

$$
x = My
$$

the general solution is expressible as

$$
y = M^+x + \mathcal{N}(M).
$$

(For a general theory of generalized inverses and computational methods, the reader is referred to [11].)

In view of (3.10), the general solution to (3.8) is given by

$$
w = \mathcal{W}_p^+(z_0 - \mathcal{W}_e y) + \mathcal{N}(\mathcal{W}_p)
$$

when (2.4) and (2.5) hold. Here we have simplified notation temporarily and written $\mathcal{W}_p$ in place of $\mathcal{W}_p(t_0, T)$ and similarly for $\mathcal{W}_e$. Upon employing 1) and 2), the substitution of (3.11) into (3.7) yields the quadratic form

$$
P(u_v, v_e) = z_0^T\mathcal{W}_p^+z_0 - 2y^T\mathcal{W}_e \mathcal{W}_p^+z_0 + y^T(\mathcal{W}_e \mathcal{W}_p^+ \mathcal{W}_e - \mathcal{W}_e)y.
$$

Upon employing standard arguments of optimization theory it is readily verified that a necessary and sufficient condition for existence of a maximum of (3.12) is that

$$
\mathcal{W}_e \mathcal{W}_p^+ \mathcal{W}_e - \mathcal{W}_e \leq 0 \quad \text{(nonpositive definite).}
$$

When (2.5) holds, we also have, in view of the projection property of the generalized inverse, that

$$
\mathcal{W}_e = \mathcal{W}_p \mathcal{W}_e^+ \mathcal{W}_e.
$$

We will also make use of the following lemmas.
**Lemma 3.1.** Let (2.5) hold. Then
\[
\mathcal{R}(W_e^TW_p^TW_e - W_e) = \mathcal{R}(W_e) + \mathcal{R}(W_e - W_p). \tag{3.16}
\]

**Proof:** Clearly, \(\mathcal{R}(W_e) \subseteq \mathcal{R}(W_e^TW_p^TW_e - W_e)\). Upon employing (3.15) it is also evident that \(\mathcal{R}(W_e^TW_p^TW_e - W_e) \subseteq \mathcal{R}(W_e^TW_p^TW_e - W_e)\). Hence, the left side of (3.16) contains the right.

Conversely, assume \(g \in \mathcal{R}(W_e^TW_p^TW_e - W_e)\). Then, upon employing (3.15) it follows that \(g \in \mathcal{R}(W_e^TW_p^TW_e - W_e)\), where \(g = W_p^TW_p\). Again, with (3.15), we then obtain \(W_p^Tg = W_p^W\) and it is readily verified that \(g - g \in \mathcal{R}(W_e)\), and finally, \(g \in \mathcal{R}(W_e - W_p) + \mathcal{R}(W_e)\). Since \(g \) was chosen arbitrarily, this concludes the proof. \(\square\)

**Lemma 3.2.** Let (2.4) and (2.5) hold. A necessary and sufficient condition that (3.13) holds is that
\[
z_0 \in \mathcal{R}(W_e - W_p). \tag{3.17}
\]

**Proof:** Sufficiency follows directly from (2.5) and (3.15). To prove necessity assume that (3.13) holds. Then Lemma 3.1 implies that \(g^TW_p^TW_p^TW_e^Z_0 = 0\) for each \(g \in \mathcal{R}(W_e - W_p)\). But then \(W_p^TW_p^TW_e^Z_0 = 0\), and hence
\[
0 = g^TW_p^TW_p^TW_e^Z_0 = g^TW_p^TW_p^TW_e^Z_0 = g^ZW_p.
\]

where the last equality follows from (2.4). Thus, \(z_0\) is orthogonal to \(\mathcal{R}(W_e - W_p)^\perp\) which implies (3.17). \(\square\)

Assume now that (2.4), (2.5), and (3.17) hold. Upon differentiating (3.12) we find that any vector \(y^*\) satisfying
\[
(W_e^TW_p^TW_e - W_e)y^* = W_e^ZW_p^Z_0 \tag{3.18}
\]
maximizes (3.12). One particular solution of (3.18) is given by \(y^* = (W_e - W_p)^Z_0\). Indeed, we have
\[
(W_e^TW_p^TW_e - W_e)(W_e - W_p)^Z_0 = W_e^TW_p^TW_e - W_e
\]
where the first equality follows by (3.15), and the second equality is due to (3.17) and the fact that for any matrix \(M\) we have \(MM^T x = x\) for all \(x \in \mathcal{R}(M)\). Thus, recalling that by Lemma 3.1, \(\mathcal{R}(W_e^TW_p^TW_e - W_e) = \mathcal{R}(W_e) + \mathcal{R}(W_e - W_p)\), we have that the general solution of (3.18), and therefore to problem \((P)\), is given by
\[
y^* = (W_e - W_p)^Z_0 + \mathcal{R}(W_e) + \mathcal{R}(W_e - W_p). \tag{3.19}
\]

Upon substitution of (3.19) into (3.11) we obtain
\[
w^* = W_p^Z_0 - (W_e - W_p)^Z_0 + W_p\mathcal{R}(W_e - W_p) + \mathcal{R}(W_e). \tag{3.20}
\]

Substitution of (3.21) and (3.19) into (3.4) and (3.6), respectively, yields the following formulas for the optimal controls in problem \((P)\):
\[
u^*(t) = \tilde{B}_p(t)[W_p^TW_p((W_e - W_p)^Z_0 + \mathcal{R}(W_e - W_p)) + \mathcal{R}(W_e)] \tag{3.22}
\]
and
\[
\nu^*(t) = \tilde{B}_p(t)[(W_e - W_p)^Z_0 + \mathcal{R}(W_e) + \mathcal{R}(W_e - W_p)]. \tag{3.23}
\]

**Remark 3.3:** For the one-player case (i.e., \(B_e(t) = 0\)), formula (3.22) reduces to the optimal control law in [2] and [3] for the problem of driving the event \((t_0, Z_0)\) to \((T, 0)\) with minimum energy. Formula (3.23), however, has no analog in the one-player theory. Notice also that for existence of solutions in the one player minimum energy problems, conditions beyond controllability [such as (3.14)] do not arise since the individual Grammians are always semidefinite.

We shall now compute the optimal value \(P(u^*, v^*)\) of the performance criterion. First, upon employing (3.17) and (3.15) we observe that \(y^*(W_e^TW_p^TW_e - W_e)^y^* = y^*W_p^TW_p^Z_0\), and hence after substituting into (3.12), we obtain
\[
P(u^*, v^*) = z_0W_p^TW_p^Z_0 = z_0W_p^TW_p^Z_0. \tag{3.24}
\]

Next note that in view of (2.4) and (3.17) \(z_0W_p^TW_p^TW_e^Z_0 = z_0W_p^TW_e^Z_0 = z_0W_p^TW_p^Z_0 = P(u^*, v^*) = z_0W_p^TW_p^Z_0\), and hence we obtain \(P(u^*, v^*) = z_0W_p^TW_p^Z_0\), and finally
\[
P(u^*, v^*) = z_0W_p^TW_p^Z_0. \tag{3.25}
\]

It is interesting to observe that \(P(u^*, v^*) > 0\) for all \(z_0 \neq 0\). Indeed, choosing \(v = 0\) in (3.12) gives \(P(u_0, v_0) = z_0W_p^TW_p^Z_0\). Due to the fact that \(W_p > 0\), so is \(W_p^T\) and hence \(P(u^*, v^*) > P(u_0, v_0) > 0\). Furthermore, \(P(u_0, v_0) = 0\) only if \(z_0 = 0\) and \(z_0 \in \mathcal{R}(W_p)^\perp = \mathcal{R}(W_e)\). But by (2.5), \(\mathcal{R}(W_e) \subset \mathcal{R}(W_e)\) and hence, if \(P(u_0^*, v_0^*) = 0\) then \(z_0 \in \mathcal{R}(W_e - W_p)\) and we have that (3.17) implies \(z_0 = 0\). Thus, given that \(z_0 = 0\), the evader, by declaring \(v^*\), is assured that any pursuer control which steers \((t_0, Z_0)\) to \((T, 0)\) uses strictly more energy than \(\int_0^T ||v(t)||^2 dt\).}

Our observations are summarized in the following.

**Theorem 3.4:** Consider a standard system (1.4) with \(z_0 \neq 0\) and let \(T > t_0\) be such that (2.4) and (2.5) both hold. A necessary and sufficient condition that there exist optimal controls \(u^*\) and \(v^*\) in problem \((P)\) for the pursuer and evader, respectively, is that (3.14) and (3.17) hold. These controls are unique up to subspace translations and are given by (3.22) and (3.23). Formulas (3.22) and (3.23) uniquely determine \(P(u^*, v^*)\) via (3.25). Furthermore, \(P(u^*, v^*) > 0\). In formulas (3.14), (3.17), (3.22), (3.23) and (3.25) we have abbreviated \(W_e = W_e(t_0, T), W_p = W_p(t_0, T)\).
Remark 3.5: Given that (2.5) holds, one can readily verify that
\[ \mathcal{R}(W_p - W_e) \subseteq \mathcal{R}(W_p) \]  
(3.26)
and that
\[ \mathcal{R}(W_e - W_p) = \mathcal{R}(W_p) \]
(3.27)
if and only if in addition \( \mathcal{R}(W_e) \subseteq \mathcal{R}(W_p - W_e) \) holds.

Formula (3.26) implies that, given (2.5), the set of events \((t_0, z_0)\) for which there exists a solution to problem (P) is contained in the set of max-min controllable events, which of course was to be expected. However, (3.27) indicates that max-min controllability in general does not in itself guarantee existence of an optimal solution to the max-min control problem (P). Hence, there may exist events which are max-min controllable but for which an optimal solution does not exist. This interesting situation deserves some further investigation.

We now turn our attention toward deriving a more tractable characterization of the semidefiniteness condition (3.14). To this end we require the following lemmas.

Lemma 3.6: If \( A \) and \( B \) are symmetric and \( A > 0 \), \( B > 0 \) (i.e., positive definite matrices), then \( A - B > 0 \) holds if and only if \( A^{-1} - B^{-1} < 0 \).

Proof: There exist a real nonsingular matrix \( R \) such that \( RAR^T = \Lambda \) and \( RBR^T = I \) where \( I \) is the identity matrix and \( \Lambda \) is a real diagonal matrix (see, e.g., [13, p. 58]). If \( A > B \) then \( RAR^T - RBR^T = \Lambda - I > 0 \) implies that the diagonal elements of \( \Lambda \) are all \( > 1 \) and therefore, \( I - \Lambda^{-1} > 0 \). Thus, \( RAR^T - RAR^T \Lambda^{-1} > 0 \). Since \( RAR^T = B^{-1} \) and \( RAR^T \Lambda^{-1} = A^{-1} \), it follows that \( A^{-1} - B^{-1} < 0 \). \( \square \)

Lemma 3.7: The following hold.
1) \( W_p - W_e > 0 \) implies (3.14).
2) \( W_p - W_e > 0 \) implies \( W_p^T - W_e^T < 0 \) if and only if \( \mathcal{R}(W_p) = \mathcal{R}(W_e) \).
3) An immediate consequence of 1) and 2).

Combining Theorem 3.4 and Lemma 3.7-3) yields the following result upon noting that \( W_p - W_e > 0 \) implies (2.5).

Corollary 3.8: Consider system (1.4) with \( z_0 \neq 0 \) and let \( T > t_0 \) be such that (2.4) holds. In addition assume that \( \mathcal{R}(W_p) = \mathcal{R}(W_e) \) (where \( W_e \) and \( W_p \) are abbreviations as in Theorem 3.4). Then a sufficient condition for the existence of an optimal solution to problem (P) is that (3.17) holds and \( W_p - W_e > 0 \).

In Lemma 3.7 we were able to replace condition (3.14) with the simpler condition \( W_p - W_e > 0 \), provided we assumed \( \mathcal{R}(W_p) = \mathcal{R}(W_e) \). In the next lemma we will prove that the same simplification can be accomplished if instead we assume that \( \mathcal{R}(W_p) = R^a \).

Lemma 3.9: Assume \( \mathcal{R}(W_p) = R^a \) (i.e., \( W_p > 0 \)). Then (3.14) holds if and only if \( W_p - W_e > 0 \).

Proof: There exists a nonsingular real matrix \( R \) such that \( RWR = \Lambda \) and \( RWR = I \) where \( \Lambda \) is a diagonal matrix with nonnegative entries (recall the proof of Lemma 3.6). Now \( W_p - W_e = R^{-1}(I-\Lambda)R^{-1} \). Also, \( W_pW_p - W_eW_e = R^{-1}(\Lambda^2 - \Lambda)R^{-1} \). The proof is completed upon noting that \( \Lambda^2 - \Lambda < 0 \) if and only if \( I - \Lambda > 0 \). \( \square \)

In view of the previous lemma we have the following additional corollary to Theorem 3.4.

Corollary 3.10: Consider system (1.4) with \( \mathcal{R}(W_p) = R^a \) (where \( W_e \) and \( W_p \) are abbreviations as in Theorem 3.4). Then the following hold.
1) A necessary and sufficient condition that there exists an optimal solution to problem (P) for every event in \( (t_0, R^a) \) is that \( W_p - W_e > 0 \), in which case the optimal pursuer control [as given by (3.22)] is unique.
2) If \( W_p - W_e \) is singular, then a necessary and sufficient condition that there exists an optimal solution to problem (P) for an event \((t_0, z_0)\) is that \( z_0 \in \mathcal{R}(W_p - W_e) \) and \( W_p - W_e > 0 \).

Remark 3.11: In [4] problem (P) was solved via a variational penalty function method for the special case where \( W_p - W_e > 0 \). The authors of [4] suggested that \( W_p - W_e > 0 \) meant that the pursuer was more controllable than the evader, conveying the intuitive idea that a kind of (max-min) controllability property exists. Indeed, \( W_p - W_e > 0 \) implies (2.5). From the development in the present paper, however, we see that max-min controllability can hold in the absence of definiteness conditions on \( W_p - W_e \).

Consider, for example, \( W_p = I \) and \( W_e = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \).

Remark 3.12: Due to the Cauchy–Binet theorem (see, e.g., [16, p. 15]), \( W_p - W_e > 0 \) implies \( \text{det}(W_p) > \text{det}(W_e) \).

IV. FURTHER REMARKS

In this section we shall address ourselves to the comparison of our results in the previous sections with certain results and concepts in the existing body of differential game literature.
We have refrained from calling \((P)\) a differential game due to the information structure which we have imposed on the problem. While this "open-loop" structure finds application in many engineering and economic models of competitive situations, open-loop strategies are specializations of more general types of strategies which can be found in the literature. Two specific approaches made reference to below are Hájek [8], [9], and Isaacs [7].

Hájek in [8] considered strategies to be "snap decision rules"; i.e., a player's control depends instantaneously upon the opposing player's control. Hájek considered a question which might be termed "strategic max-min controllability," or the pursuer's ability to steer \(x_0\) to 0 in a variant of system (1.4) in the presence of any evader strategy. For the case of (possibly time varying) control and state constraints he showed that strategic max-min controllability of an event is equivalent to conventional controllability in a certain associated one-player linear system. Controllability in this system depends on the Pontryagin difference of the players' control constraints, and after some interpretation it can be shown, as one would expect, that this system's controllability implies max-min controllability in our sense. Extensions to other types of strategies and targets as well as strategy design may be found in [8] and [9]. A study of the relationship between the results of Hájek and those given in the present paper may prove worthwhile, in view of the overall desirability of bringing control theoretic tools to bear on differential games.

If we allow the evader some measurements of the state, the conditions derived in Section II may no longer be sufficient for the pursuer to force the initial state to the origin in system (1.4) even in the autonomous case. To see that this is the case consider the following simple example.

**Example 4.1:** Consider system (1.1) with

\[
A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_e = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The initial event is \( \left(0, \left(\frac{1}{2}\right)\right) \) and it is easily seen that conditions (2.12) and (2.13) hold. Hence, the initial event is max-min controllable in system (1.1). If, however, the evader employs the linear feedback rule \( u(t) = Kx(t) \) with \( K = (0,1) \), then no pursuer control can steer the initial state to the origin in finite time.

Example 4.1 indicates that under certain conditions, the evader has the capability, by employing a constant linear feedback control, to destroy the max-min controllability property and hence prevent the pursuer from ever being able to force capture. We will now show that this capability is, indeed, quite general.

Consider system (1.1) with \( A, B_p \) and \( B_e \) constant real matrices and assume

\[
\mathcal{R}\left( [A|B_e] \right) \subset \mathcal{R}\left( [A|B_p] \right).
\] (4.1)

We now state the following problem: Under what conditions does there exist an \( m_e \times n \) constant real matrix \( F \) such that

\[
\mathcal{R}\left( [A + B_eF|B_e] \right) \not\subset \mathcal{R}\left( [A + B_eF|B_p] \right). \tag{4.2}
\]

Clearly, if the evader can find \( F \) such that (4.2) holds then the pursuer cannot force capture for any initial state \( x_0 \neq 0 \). We shall require the following lemma (see [14, p. 45] for details).

**Lemma 4.2:** Let \( X \) and \( Y \) be finite-dimensional linear spaces, and let \( G : Y \to X \) be a linear map. Then for a linear map \( D : X \to Y \) there exists a map \( C : X \to Y \) such that \( D = GC \) if and only if \( \mathcal{R}(D) \subset \mathcal{R}(G) \).

In view of Lemma 4.2 we can rephrase the aforementioned question as follows: given that (4.1) holds, characterize the existence of a constant \((n \times n)\) matrix \( A_e \) such that

\[
\mathcal{R}(A_e) \subset \mathcal{R}(B_e)
\] (4.3)

and

\[
\mathcal{R}\left( [A + A_e|B_e] \right) \not\subset \mathcal{R}\left( [A + A_e|B_p] \right). \tag{4.4}
\]

**Theorem 4.3:** Given system (1.1) with \( A, B_p, B_e \) constant real matrices such that (4.1) holds and \( B_p \neq 0 \). A necessary and sufficient condition for the existence of a matrix \( A_e \) such that (4.3) and (4.4) hold is

\[
\mathcal{R}(B_e) \subset \mathcal{R}(B_p). \tag{4.5}
\]

**Proof:** Clearly \( \mathcal{R}(A + A_e|B_e) = \mathcal{R}(A|B_e) \) for every \( A_e \) which satisfies (4.3). If \( \mathcal{R}(B_e) \subset \mathcal{R}(B_p) \), then \( \mathcal{R}(A_e) \subset \mathcal{R}(B_p) \) and consequently \( \mathcal{R}(A + A_e|B_e) = \mathcal{R}(A|B_p) \) and hence \( A_e \) cannot be found to satisfy (4.4). Conversely, assume there exists a subspace \( \mathcal{V} \subset \mathcal{R}(B_e) \) such that \( \mathcal{V} \neq 0 \). \( \forall \mathcal{V} \cap \mathcal{R}(B_p) = 0 \). Write \( \mathcal{V} = \mathcal{V} \cap \mathcal{R}(B_e) \) for a subspace \( \mathcal{V} \) which satisfies \( \mathcal{R}(B_p) \subset \mathcal{V} \). Let \( P \) be the projection of \( \mathcal{R}^n \) on \( \mathcal{V} \) along \( \mathcal{W} \), and define \( A_e = -PA \). This \( A_e \) clearly satisfies (4.3) and \( A + A_e = (I - P)A \) is such that \( \mathcal{R}(A + A_e) \subset \mathcal{V} \). Hence, since \( \mathcal{R}(B_p) \subset \mathcal{W} \), it follows that \( \mathcal{R}(A + A_e|B_p) \subset \mathcal{W} \). Since \( \forall \mathcal{V} \subset \mathcal{W} \), (4.4) follows.

**Example 4.4:** Let the system matrices be given by

\[
B_e(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{for all } t > 0,
\]
while

\[
B_p(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad t \in [0,1]
\]

and

\[
B_p(t) = B_e(t), \quad \text{for all } t \in (1,2]
\]

and

\[
B_p(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{for all } t > 2.
\]

One can readily check that \( \mathbb{R} [W_p(0,3)] = R^3 \), and hence all events \((0,z_0)\) are max-min controllable in time 3. Note, however, that \( \mathbb{R} [W_p(\tau,3)] \subseteq \mathbb{R} [W_p(\tau,2)] \) for \( \tau > 2 \). Hence, there are no max-min controllable events of the form \((2,z)\). Thus, if we allow the evader to make one measurement of the state at time 2 the pursuer cannot force the solution of (1.4) to the origin.

Notice that the property exhibited in Example 4.4 is strictly a "nonautonomous" phenomenon. To see this, one can readily verify that a sufficient condition [beyond (2.4) and (2.5)] for preclusion is

\[
\forall \tau \in J_{t_0}(T) \quad \mathbb{R} [W_p(\tau,3)] \subseteq \mathbb{R} [W_p(\tau,2)]
\]

for all \( t \in (3,4) \) and \( (2,z) \). One can readily check that the chase is not possible for \( \alpha > \infty \) and \( \beta > \infty \), but it is possible for \( \alpha < \infty \) and \( \beta < \infty \).

We shall conclude by noting the connection between our results in Section III, the results of Ho, Bryson, and Baron [4], and Isaacs' approach to differential games. Consider a differential game with dynamics (1.4), payoff

\[
P(u,v) = \alpha \|z(T)\|^2 + \int_{t_0}^T \|v(t)\|^2 \, dt - \int_{t_0}^T \|u(t)\|^2 \, dt
\]

(4.8)

where \( T > t_0 \) is given, \( \alpha > 0 \), and where there are no terminal restraints. Following [4], we see that a sufficient condition for the solvability of the Isaacs' equation (and therefore for the existence of value and a closed-loop saddle point) for every \( \alpha > 0 \) is that \( W_p(t_0,T) - W_e(t_0,T) > 0 \). Upon comparing the formulas in [4] for the optimal controls in the above differential game with formulas (3.22) and (3.23), we see that the latter controls are true limiting cases as \( \alpha \to \infty \) of the former. Hence, by adding the "penalty" term \( \alpha \|z(T)\|^2 \) to (1.2) and letting \( \alpha \to \infty \) in the free endpoint game, we have that the value \( V(\alpha) \) converges to \( P(u^*,v^*) \), our max-min. The approximating games have closed-loop saddle points, while (P), in general, has not even a value in Isaacs' sense. Arguing similarly, one can also prove that if \( v \) is required to lie in a ball \( \{ w \in R^n: ||w|| \leq \beta \} \) then there exists a feedback law \( u(z) \) for the pursuer such that against any admissible evader feedback law the associated solution of (1.4) satisfies \( ||z(T)|| \leq \gamma \), where \( \gamma > 0 \) depends on \( \beta \). As is seen in Example 4.1, however, for the case \( \gamma = 0 \) there might not be such a pursuer feedback law for any \( \beta > 0 \).

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References

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