The Maximum Carpool Matching Problem

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CSR 2017

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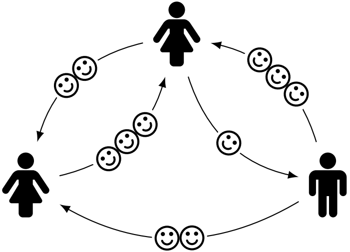

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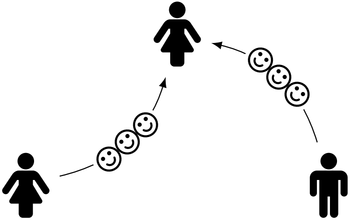

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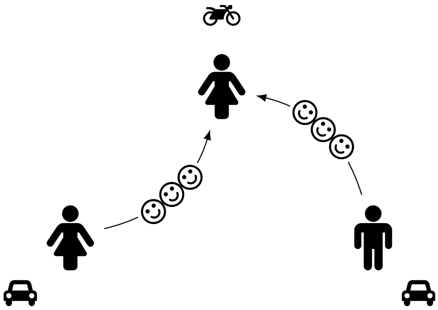

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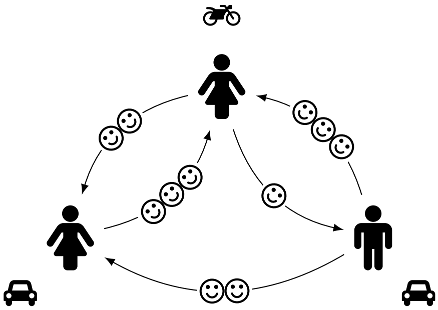

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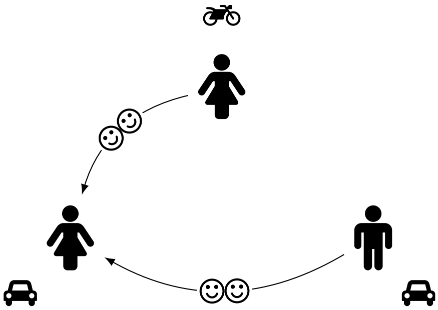
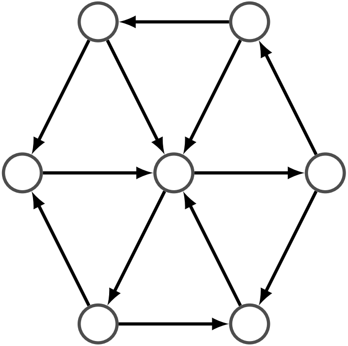


Formal Definition

Input: (D, w, c)

- $D = (V, A)$



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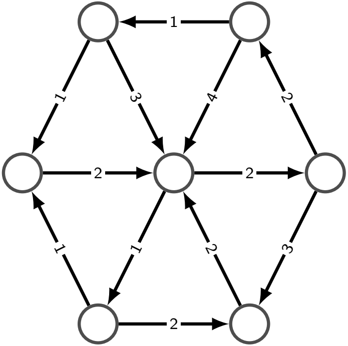

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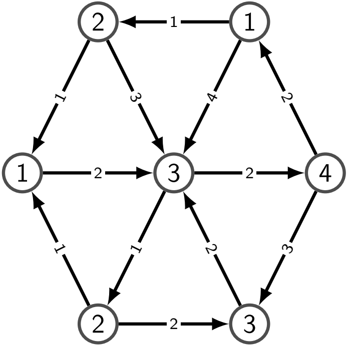

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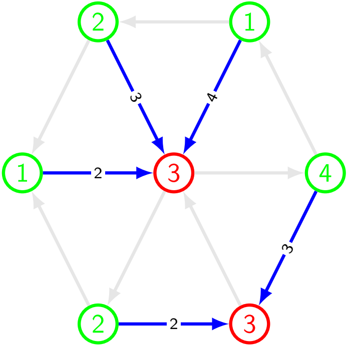

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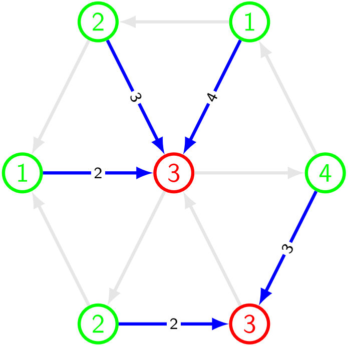

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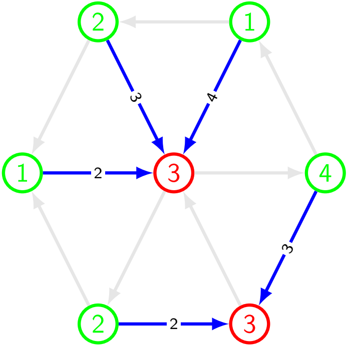

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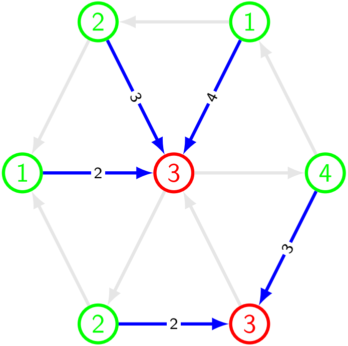

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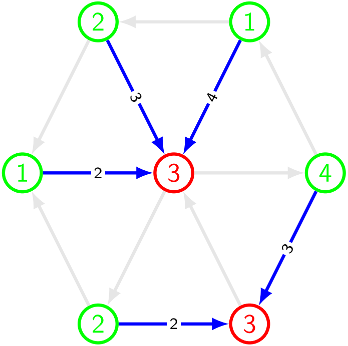

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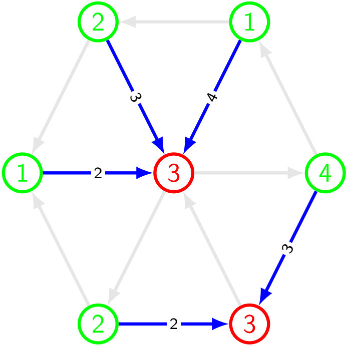
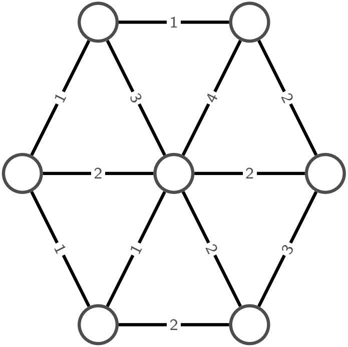


We seek for a matching that maximizes

$$w(M) = \sum_{e \in M} w(e)$$

Maximum Spanning Star Forest (MSSF)

- A special case of the MAXIMUM CARPOOL MATCHING problem (MCM)



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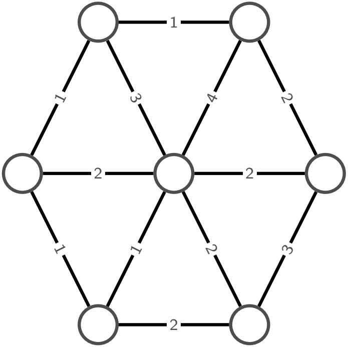

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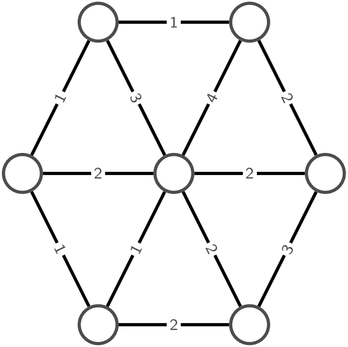

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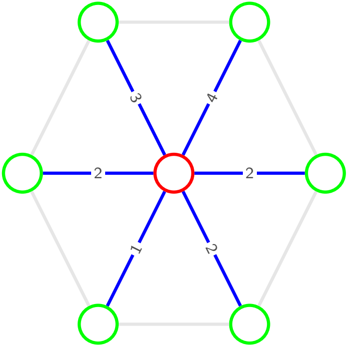
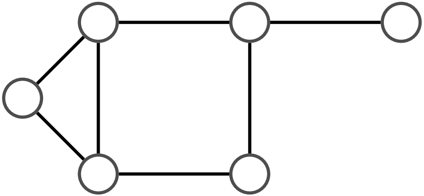


Hardness

- MSSF (and thus MCM) is APX-hard



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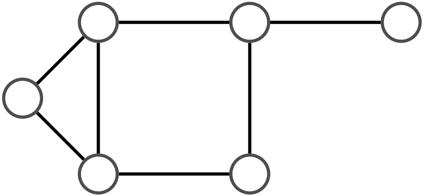
- Unweighted MSSF (uniform weight)



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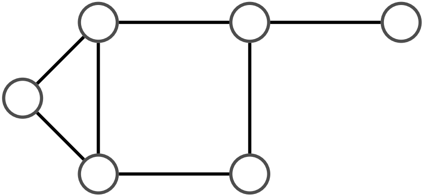
  - Maximize the number of leaves



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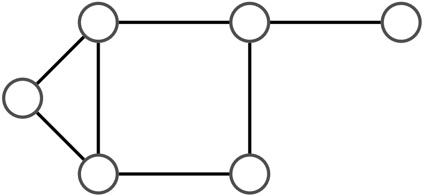
  - Minimize the number of centers



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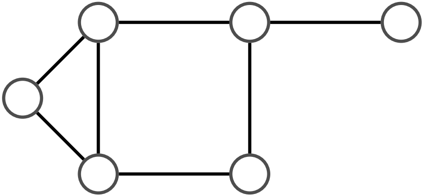
  - Minimize the number of centers
    - (Minimum Dominating Set problem)



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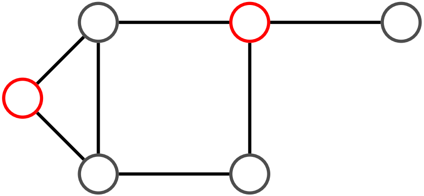

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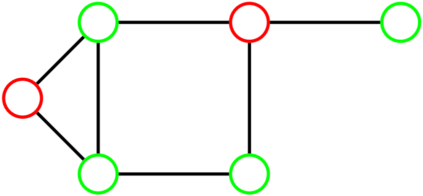

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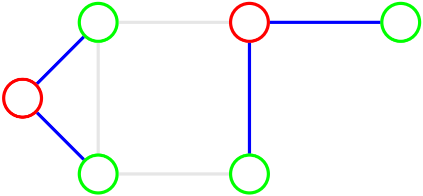
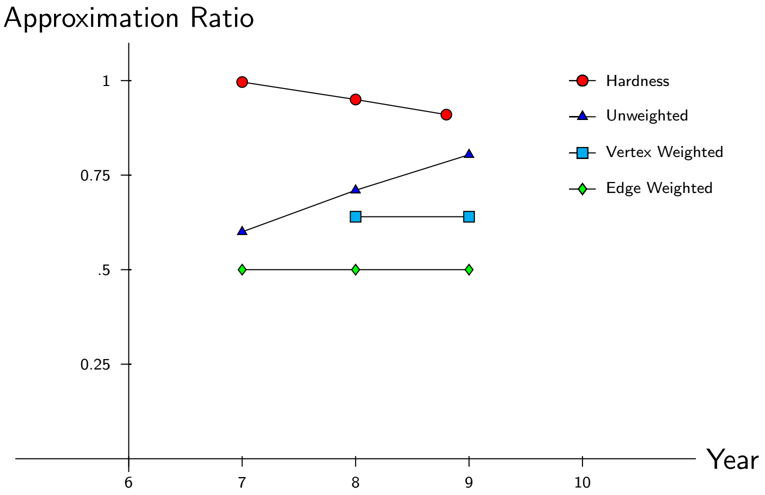


Previous Work (MSSF)



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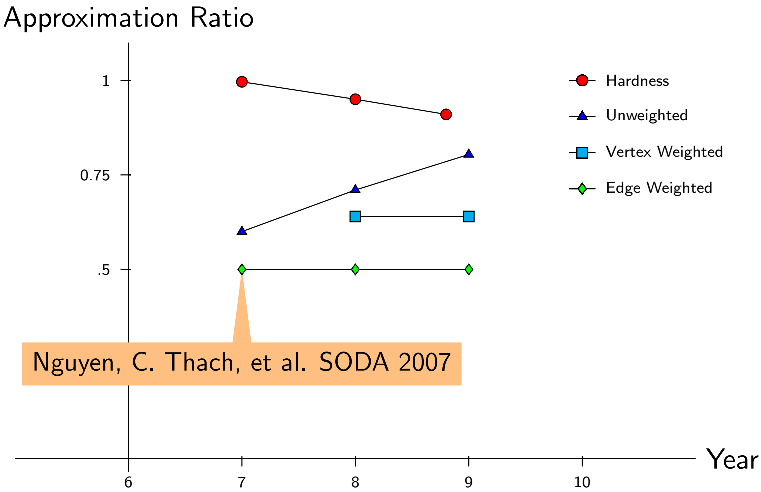

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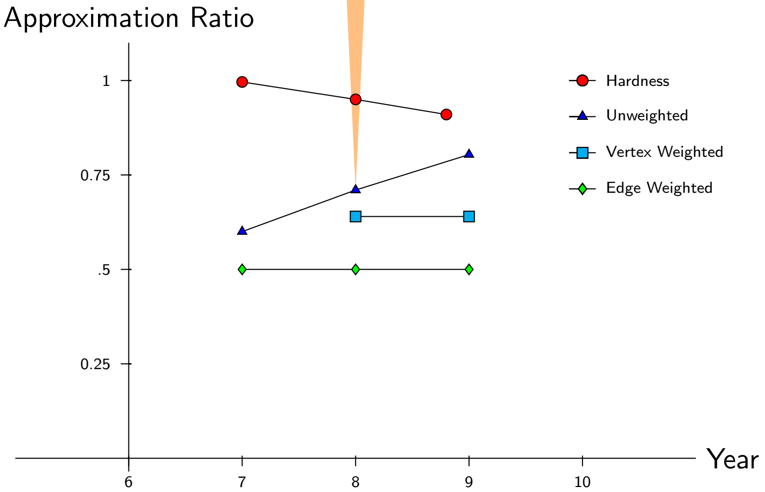
Chen, Ning, et al. APPROX 2007



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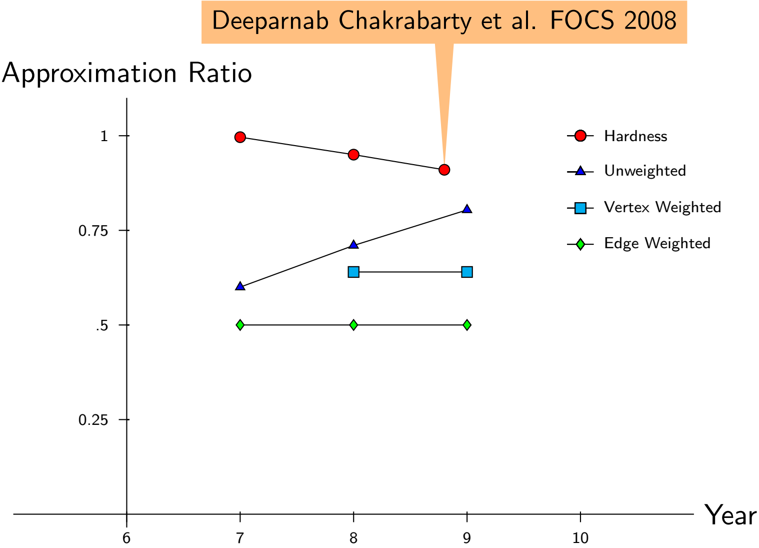

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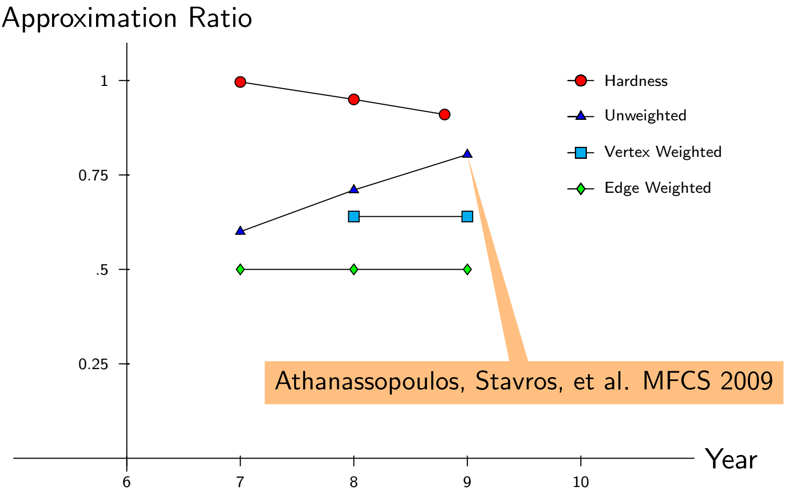

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- The algorithms for MSSF do not generalize to MCM

Our Result (MCM)

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  - $1/2$-approximation (unweighted)

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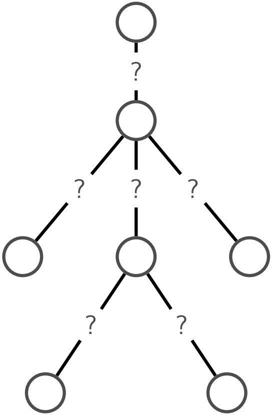
  - 1/3-approximation (weighted)

Maximum Carpool Matching

MSSF on Trees

- Optimally solvable (using DP)



MSSF on Trees

- Optimally solvable (using DP)
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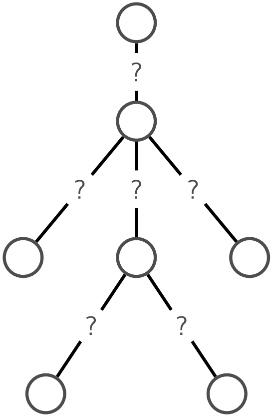

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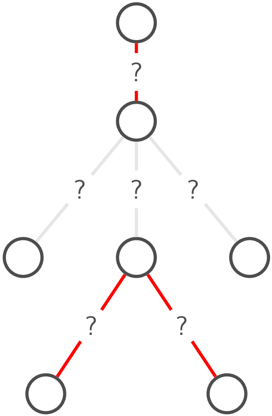

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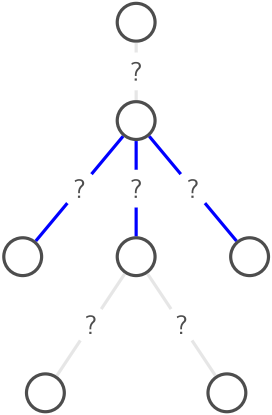

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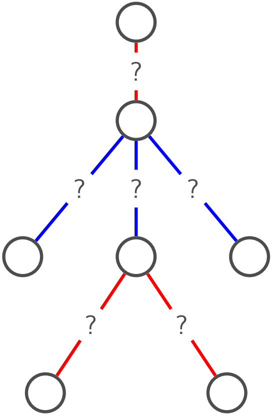

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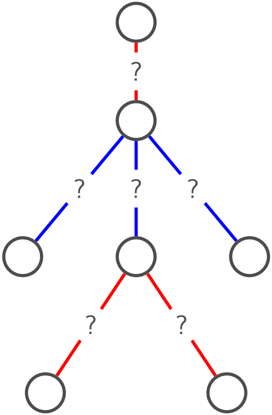
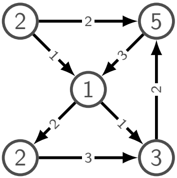
- 1/2-approximation algorithm to MSSF (Max Spanning Tree $\geq$ OPT)



Relaxed Matching

- Every vertex can be a driver and a passenger at the same time



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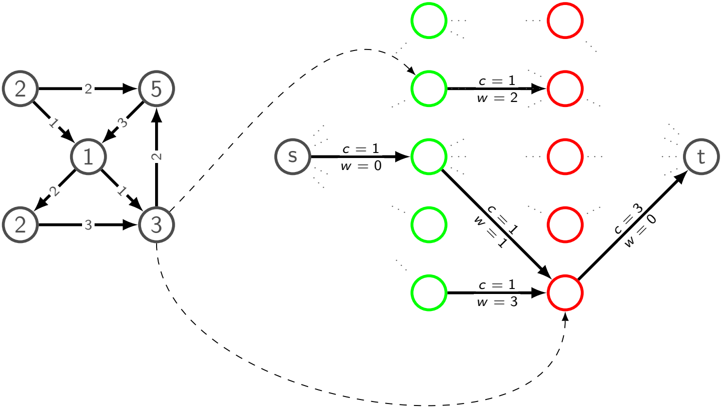
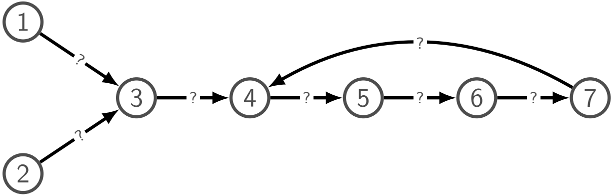


Relaxed Matching (Observations)

- A relaxed matching is a pseudo-forest (out degree ≤ 1)



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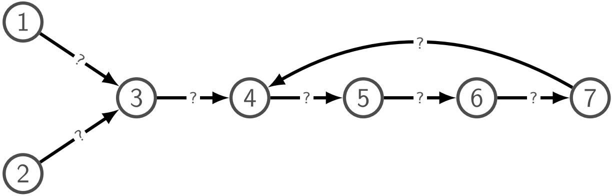

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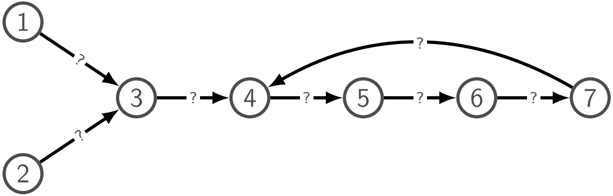

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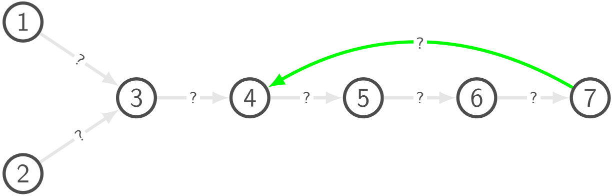

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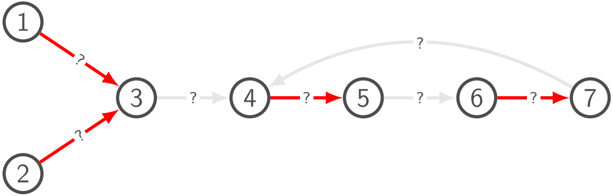

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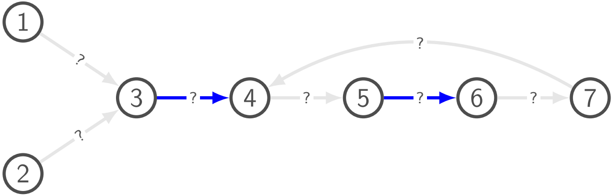

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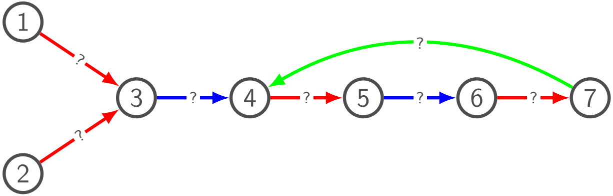

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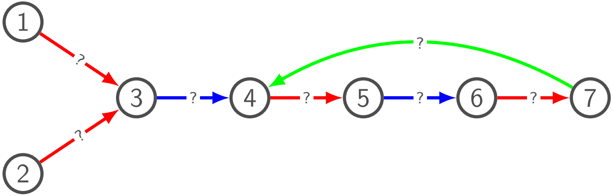
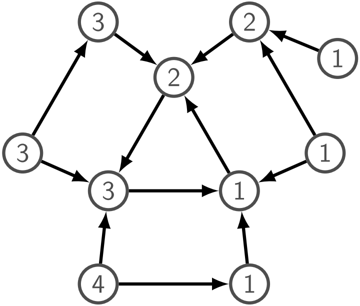


Maximum Carpool Matching

Unweighted

Unweighted MCM (1/2-approximation algorithm)

- Start with any maximal solution (bad in general)



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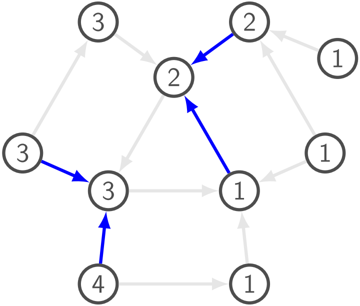

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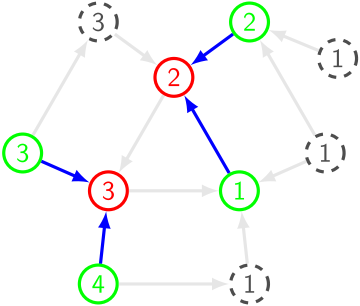

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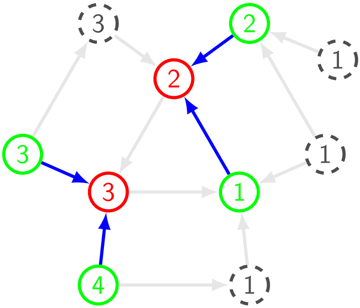
- Improve $\iff$ a passenger has two incoming arcs from unmatched vertices



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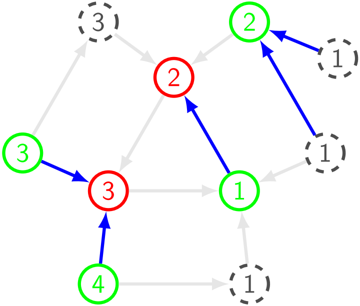

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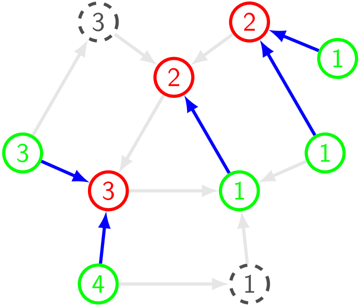

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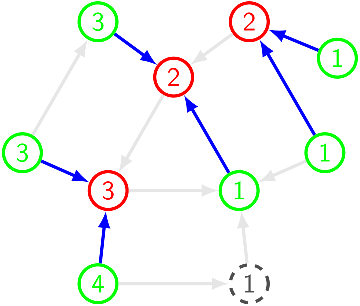

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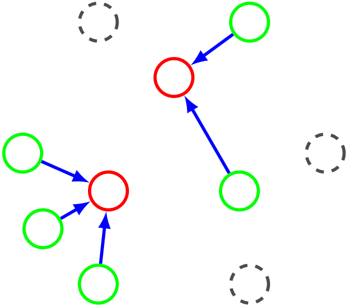

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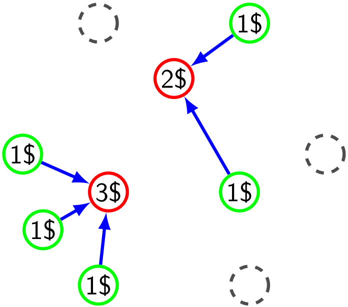

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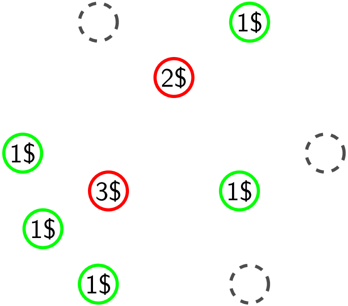

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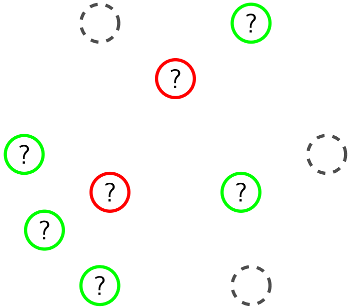

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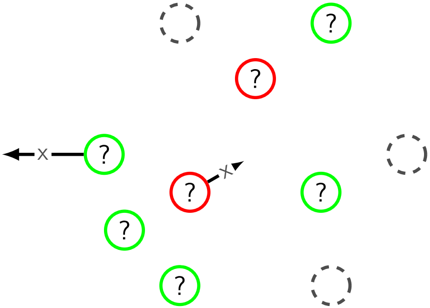

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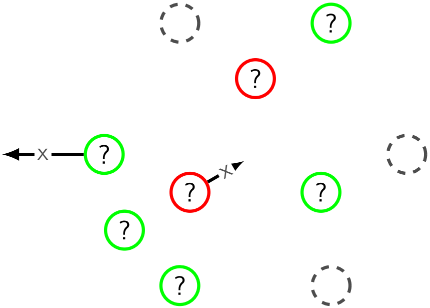

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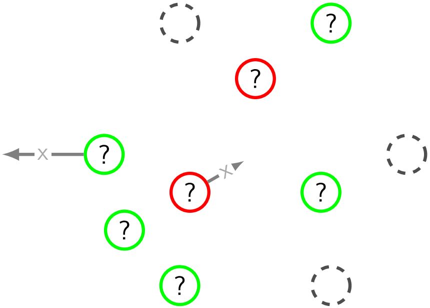

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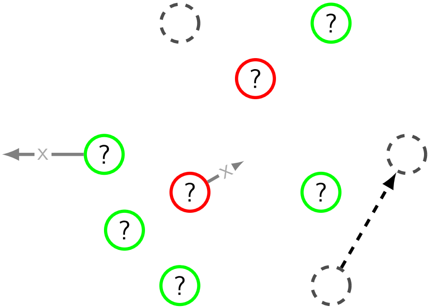

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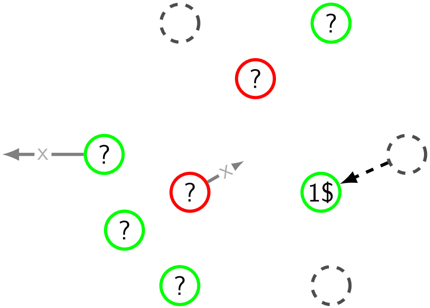

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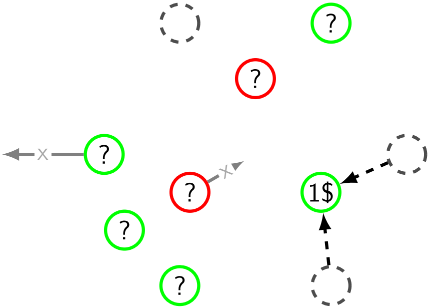

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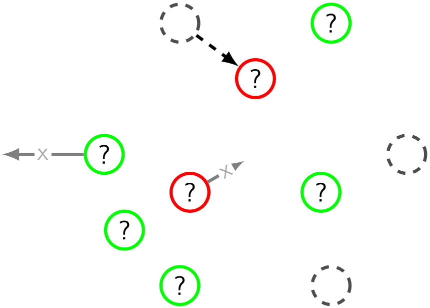

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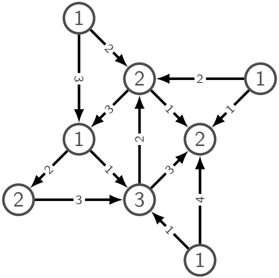
Followups

Fixed Matching

- P and D are given (tractable)



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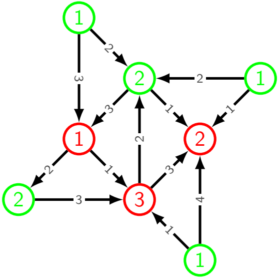

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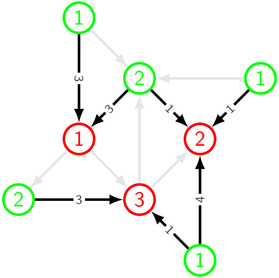

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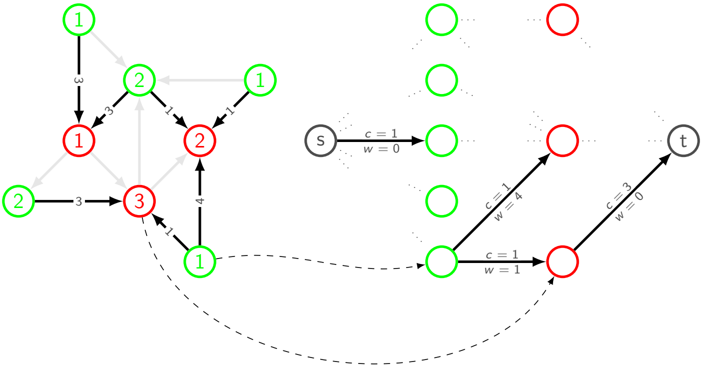

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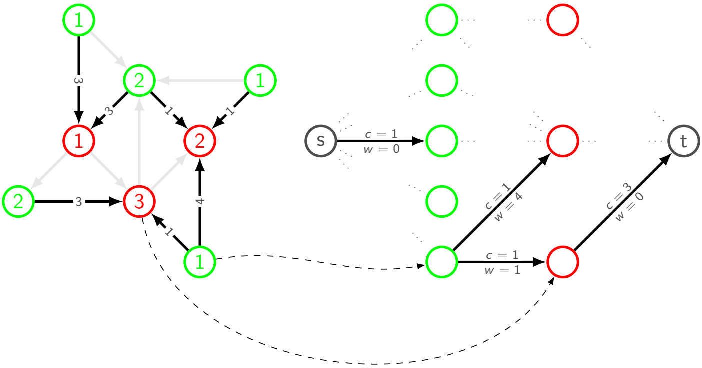

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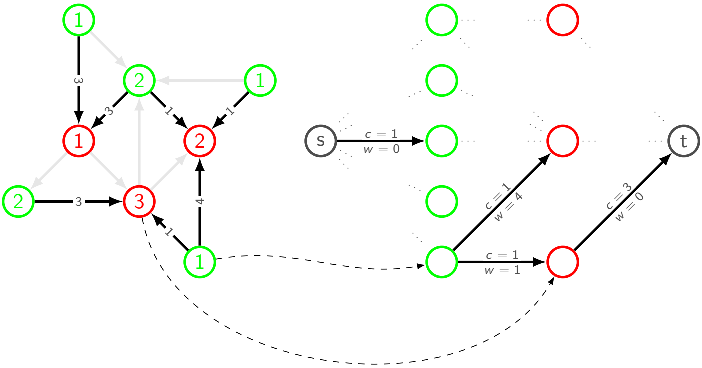

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Group Carpool

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Open Question

Better than $1/2$-approximation?

Thank You !