Error Bounded
Piecewise Linear Approximation
of
Freeform Surfaces.

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March 16, 2008

Abstract
We present two methods for piecewise linear approximation of freeform surfaces. One scheme exploits an intermediate bilinear approximation and the other employs global curvature bounds. Both methods attempt to adaptively create piecewise linear approximations of the surfaces, employing the maximum norm.

Keywords: Adaptive Sampling, Polygonization, Bilinear Fit, Freeform Curves.

1 Introduction

The (piecewise) polynomial and rational freeform representations are frequently employed in the fields of computer graphics and computer aided geometric design. For many applications in both fields the freeform shape must be approximated using piecewise linear representations. Display devices support the drawing and rendering of only polygons in general. Fundamental tasks such as surface surface intersection and strength and heat transfer analysis are frequently addressed using lower order and mostly piecewise linear approximation of the surfaces. Hence the problem of finding an optimal in terms of both accuracy and space piecewise linear approximation of a freeform surface as a set of polygons is of significant importance.
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In [9Γ12Γ14] the norm of $\Gamma$

$$1 - \langle n_1, n_2 \rangle,$$  \hspace{1cm} (1)

is used where $n_i$ are the unit normals of surface $S(u, v)$ at locations $p_i \in S(u, v)$. Norm (1) is exploited as a curvature or divergence measure while being independent of the size of the object. Therefore a bound on the amount of deviation of the surface from its piecewise linear approximation cannot be established. Furthermore it is unclear how one can bound Equation (1) for every pair of points in an approximated region to compute a global upper bound on (1). One can only assume that Equation (1) is verified for a finite set of locations on $S(u, v)\Gamma$ probably on the boundary of the approximated region $\Gamma$ a test that is alias-prone.

In [9Γ12Γ14] other subdivision criteria are suggested as well. In [9] proximity and silhouette screen space considerations are employed for static viewing direction dependent approximation $\Gamma$ for rendering purposes. In [12] planarity and chord distance criteria are employed $\Gamma$ but for the vertices of the considered region only. In [14] the loose and alias-prone chord distance criterion of $\Gamma$

$$S \left( \frac{u_p + u_q}{2}, \frac{v_p + v_q}{2} \right) - \frac{(S(u_p, v_p) + S(u_q, v_q))}{2},$$  \hspace{1cm} (2)

is exploited. This criterion implicitly assumes a uniform parameterization.

In [8Γ10Γ13] the second partial derivatives of surface $S(u, v)$ are used to bound the distance between a $C^2$ surface $S(u, v)$ and its polygonal approximation $l(u, v)$. In [8] the bound of $\Gamma$

$$\sup_{(u, v) \in T} \| S(u, v) - l(u, v) \| < \frac{1}{8} \left( l_1^2 M_1 + 2l_1 l_2 M_2 + l_2^2 M_3 \right),$$  \hspace{1cm} (3)
is derived where

\[
M_1 = \sup_{(u, v) \in \mathcal{T}} \left\| \frac{\partial^2 S(u, v)}{\partial u^2} \right\|
\]

\[
M_2 = \sup_{(u, v) \in \mathcal{T}} \left\| \frac{\partial^2 S(u, v)}{\partial u \partial v} \right\|
\]

\[
M_3 = \sup_{(u, v) \in \mathcal{T}} \left\| \frac{\partial^2 S(u, v)}{\partial v^2} \right\|
\]  

(4)

and \( T \subset \mathbb{R}^2 \) is a right triangle with vertices \((A, B, C)\) of the form \(B = A + (l_1, 0)\) and \(C = A + (0, l_2)\) in the parametric space of \(S(u, v)\).

Unfortunately this bound can be arbitrarily loose as is demonstrated by the following example. Consider the regular parametric surface (Figure 1) \(\Gamma\)

\[
F(u, v) = \left( (au)^2 (av)^2, (au)^2, 0 \right), \quad u, v \in (0, 1/a], \quad a > 0.
\]  

(5)

The normal of \(F\) is non zero and is collinear with the \(z\) axis. Hence \(F\) is a regular surface that is also planar. Nevertheless none of the second partial derivatives of \(F\) vanish in \(u, v \in (0, 1/a]\). Moreover the second partial derivatives of \(F\) can assume arbitrarily large values by reparameterization or modifying the value of \(a\). Properly noted in [8] \(\Gamma\)
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Equation (4)) are invariant under rigid motion transformation. They are however parameterization dependent. That is, the bound established in Equation (3) depends on the surface parameterization (i.e. \( a \) in Equation (5)) and is not an intrinsic surface property. Even more elusive is the fact that while a surface can be planar, its second partial derivatives can assume arbitrary large values.

One should seek bounds that are not only invariant under rigid motion transformation but are also parameterization independent. Clearly, the second partial derivatives express not only intrinsic surface geometry but also differential properties of the isoparametric curve considered. These are manifested as the geodesic curvature \([5]\) of the isoparametric curves in the tangent plane as well as changes in the speed of the parameterization.

Let \( S(u,v) \) be a \( C^2 \) continuous regular parametric surface. Throughout this paper, without loss of generality and unless otherwise stated, we will assume a parametric domain of \( u, v \in [0,1] \). Let \( R_s(u,v) \) be an approximation of \( S(u,v) \). Then

\[
\epsilon = \max_{u,v} \| S(u,v) - R_s(u,v) \|, \tag{6}
\]

is an upper bound on the maximal deviation between the region of the surface and its approximation. In this work, we will attempt to globally estimate \( \epsilon \) when \( R_s \) is a piecewise linear approximation of \( S \).

We will restrict our discussion to an approximation that is adaptively derived using a recursive surface subdivision based approach. The divide and conquer approach is greedy and cannot ensure a globally optimal approximation. Yet, it is very simple and convenient to use. Different flatness criteria can be employed and comparisons can be easily performed. We will consider two approximation criteria: one that is based on
an intermediate bilinear surface fit and one that is based on global normal curvature bounds.

This paper is organized as follows. In Section 2 we explore the approach that is based on an intermediate bilinear surface fit. In Section 3 we apply global curvature bounds to answer the question of global bound on the error introduced by the piecewise linear approximation of a freeform surface. Possible extensions that exploits both techniques are presented in Section 4. Finally in Section 5 conclusion are drawn and some possible extension are discussed.

2 Bilinear Based Polygonal Approximation

Let \( R(u, v) \) be the bilinear surface

\[
R(u, v) = (1 - u)(1 - v)P_{00} + (1 - u)vP_{01} + u(1 - v)P_{10} + uvP_{11}, \quad u, v \in [0, 1]. \tag{7}
\]

Approximate \( R(u, v) \) using two triangles \( T_1 = \Delta(P_{00}P_{01}P_{10}) \) and \( T_2 = \Delta(P_{01}P_{11}P_{10}) \). \( T_1 \) and \( T_2 \) cover\( \Gamma \)in the parametric domain of \( R(u, v) \) the closed triangular domains of \( u + v \leq 1 \) and \( u + v \geq 1 \) respectively.

**Lemma 1** The distance (Equation (6)) between \( T_1 = \Delta(P_{00}P_{01}P_{10}) \) and \( R(u, v), u + v \leq 1 \) is less than or equal to \( \frac{1}{4} \) of the distance from \( P_{11} \) to the plane containing \( T_1 \).

**Proof:** Rotate and translate \( T_1 \) and \( R(u, v) \) into \( \hat{T}_1 \) and \( \hat{R}(u, v) \) so that \( \hat{T}_1 \) is contained in the \( Z = 0 \) plane. Because \( R(u, v) \) is rigid motion invariant this transformation does not affect the distance from the \( R(u, v) \) to the triangles \( T_1 \) and \( T_2 \) approximating it. Then the \( Z \) component of \( \hat{R}(u, v) \) is equal to \( \hat{R}_z(u, v) = uv\hat{P}_{11} \) because \( \hat{P}_{00} = \hat{P}_{01} = \hat{P}_{10} = 0 \) where \( \hat{P}_z \) denotes the \( Z \) component of the transformed point.
\( \hat{R}(u,v) \) is exactly the distance from \( \hat{R}(u,v) \) to the \( Z = 0 \) plane. Hence \( \sup_{u + v \leq 1, u, v \in [0,1]} (uv) \) provides the necessary bound. Clearly \( uv \) is an increasing monotone function in the prescribed domain. The function \( uv \) achieves its upper bound for \( T_1 \) along the boundary \( u + v = 1 \). Then the single extremum (a maximum) of \( uv = u(1 - u) \) is at \( u = \frac{1}{2} \) and \( uv = \frac{1}{4} \).

The same holds for \( T_2 \). A bilinear \( R(u,v) \) can therefore be approximated using two triangles \( T_i \), \( i = 1,2 \) and the maximal deviation from the bilinear can be easily estimated.

Let \( S(u,v) \) be a piecewise polynomial parametric surface represented as a Bspline surface. Let \( R_s(u,v) \) be a bilinear surface approximation to \( S(u,v) \) defined over the four corner points of \( S(u,v) \). \( R_s(u,v) \) can be degree raised \([3]\) and refined \([11,2]\) to the same function space of \( S(u,v) \). Call the new refined and degree raised bilinear \( R_s^r(u,v) \).

Then (see Equation (6) and also \([6,11]\) for more) \( \Gamma \)

\[
\max_{u,v} \| S(u,v) - R_s(u,v) \|
\]

\[
= \max_{u,v} \| S(u,v) - R_s^r(u,v) \|
\]

\[
= \max_{u,v} \left\| \sum_{i=0}^{m} \sum_{j=0}^{n} P_{ij} B_{i,r}^m(u) \right\| \left( B_{j,r}^n(v) - \sum_{i=0}^{m} \sum_{j=0}^{n} Q_{ij} B_{i,r}^m(u) B_{j,r}^n(v) \right) \right\|
\]

\[
= \max_{u,v} \left\| \sum_{i=0}^{m} \sum_{j=0}^{n} (P_{ij} - Q_{ij}) B_{i,r}^m(u) \right\| \left( B_{j,r}^n(v) \right) \right\|
\]

\[
\leq \max (\| P_{ij} - Q_{ij} \|, \forall i,j), \quad (8)
\]

employing the partition of unity property of the Bspline basis functions \( B_{i,r}^m(u) \) and \( B_{j,r}^n(v) \).

Hence one can put a bound on the distance between two triangles \( T_i, i = 1,2 \)
approximating \( S(u,v) \) by fitting a bilinear \( R_s(u,v) \) to \( S(u,v) \) bounding the maximal distance between \( S(u,v) \) and \( R_s(u,v) \) using Equation (8) and bounding the distance between \( T_i \), \( i = 1, 2 \) and \( R_s(u,v) \) using Lemma 1. The bound of the distance from \( S(u,v) \) to \( T_i \) is the accumulative result of the bounds established from \( S(u,v) \) to \( R_s(u,v) \) and from \( R_s(u,v) \) to \( T_i \).

The established bound is sharp in the following case (see Figure 2). Let \( \Gamma \)

\[
S(u,v) = \sum_{i=0}^{2} \sum_{j=0}^{2} P_{ij} B_i^2(u) B_j^2(v), \quad P_{ij} = \begin{cases} (i,j,1) & i = j = 1, \\ (i,j,0) & \text{otherwise}, \end{cases}
\]

where \( B_i^2(t) \) are the linear Bspline basis functions. The bilinear surface \( R_s(u,v) \) fitted to the four corners of \( S(u,v) \) is clearly planar \( \Gamma \)

\[
R_s(u,v) = \sum_{i=0}^{1} \sum_{j=0}^{1} Q_{ij} B_i^2(u) B_j^2(v), \quad Q_{ij} = (2i, 2j, 0).
\]

and the refined bilinear \( \Gamma \) elevated to the same space of \( S(u,v) \) \( \Gamma \)

\[
R_r(u,v) = \sum_{i=0}^{2} \sum_{j=0}^{2} Q_{ij} B_i^2(u) B_j^2(v), \quad Q_{ij} = (i,j,0).
\]

Because \( T_i, \ i = 1, 2 \) can exactly represent the planar bilinear \( \Gamma \) the magnitude of \( \| P_{11} - Q_{11}^r \| \) is indeed equal to the extreme distance between \( S(u,v) \) and \( T_i, \ i = 1, 2 \).

For higher degree surfaces \( \Gamma \) this bound can be further refined by establishing the maximal value a basis function can assume in the interior of the parametric domain.

In Figure 3 \( \Gamma \) several examples of polygonal approximations of freeform NURBs surfaces computed using the bilinear intermediate representation are shown. This approach is also summarized in Algorithm 1. A greedy approach in the subdivision process selects the subdivision direction that minimizes the approximation error from the two subdivision direction possibilities \( \Gamma u \) or \( \Gamma v \).
Algorithm 1

Input:
$S(u,v)$, freeform $C^1$ continuous surface;
$\epsilon$, maximal deviation from $S(u,v)$ to its polygonal approximation;

Output:
$\mathcal{P}$, set of polygons approximating $S(u,v)$ to within $\epsilon$.

Algorithm:

$\text{SubdivSrfUsingBilines}(S(u,v), \epsilon)$;
begin
    $R_s(u,v) \Leftarrow$ a bilinear surface from the four corners of $S(u,v)$;
    $R_s(u,v) \Leftarrow R_s(u,v)$ degree raised and refined to same function space of $S(u,v)$;
    $T_1, T_2 \Leftarrow$ two triangles approximating $R_s(u,v)$;
    $\epsilon_1 \Leftarrow \max_{u,v} \| S(u,v) - R_s(u,v) \|;$
    $\epsilon_2 \Leftarrow$ Maximal distance from $R_s(u,v)$ to $T_1$, $T_2$;
    if ($\epsilon_1 + \epsilon_2 < \epsilon$) then
        return \{ $T_1$, $T_2$ \};
    else begin
        (1) $S_1(u,v), S_2(u,v) \Leftarrow S(u,v)$ subdivided into two;
        return $\text{SubdivSrfUsingBilines}(S_1(u,v), \epsilon) \cup$
        $\text{SubdivSrfUsingBilines}(S_2(u,v), \epsilon)$;
    end;
end;

Figure 2: The bilinear bound established in Lemma 1 is sharp. The bound of $\| Q_{11} - P_{11} \|$ is indeed the maximal deviation of the approximation from the original surface $\Gamma S(u,v) \Gamma$ in solid lines. The (planar) bilinear $\Gamma R_s(u,v) \Gamma$ is shown in dashed lines.

\[ \begin{align*}
S(u,v) \\
P_{00} &= Q_{00}^r \\
P_{10} &= Q_{10}^r \\
P_{20} &= Q_{20}^r \\
P_{01} &= Q_{01}^r \\
P_{11} &= Q_{11}^r \\
P_{21} &= Q_{21}^r \\
P_{02} &= Q_{02}^r \\
P_{12} &= Q_{12}^r \\
P_{22} &= Q_{22}^r \\
\end{align*} \]
The teapot in Figure 3 is shown with two triangles per approximated patch. The rest of the examples in Figure 3 show a polygonization of a sufficiently flat patch into four triangles by sampling the center of the patch as a fifth interior point \( P_c \), \( u = v = \frac{1}{4} \).

Let \( R(u,v) \) be a bilinear surface as in Equation (7). Then

**Lemma 2** The distance (Equation (6)) between \( T_1 = \Delta(P_{10}P_{01}P_c) \) and \( R(u,v) \), \( u + v \leq 1, u > v \) is less than or equal to \( \frac{1}{8} \) of the maximal distance from \( P_{11} \) or \( P_{10} \) to the plane containing \( T_1 \).

**Proof:** Following the proof of Lemma 1, recognizing that the newly introduced constrain of \( P_c^z = 0 \) necessitates \( P_{11}^z = -P_{10}^z \Gamma \) and maximizing the resulting bilinear function of \( (1 - 2u)v \) at \( u = v = \frac{1}{4} \).

\[ \Box \]

Table 1 compares the number of polygons and the computation time of each approximation using both bilinear fit and uniform subdivision for the examples of Figure 3. All approximations were tuned so that the number of polygons in the uniform sampling method is larger than the bilinear fitting method. Computation time was measured on an SGI R4400 150Mhz Indy.
Figure 3: Few examples of a polygonal approximation of NURBs freeform surfaces using bilinear fit. In (a) uniform subdivision is shown while in (b) the bilinear fit method is used. In all examples the total number of polygons in (a) is larger than in (b). See also Table 1.
3 Curvature Based Polygonal Approximation

In Section 1 the second order derivatives were shown to be insufficient as curvature bounds due to their dependency on the parametrization as well as the introduction of Geodesic curvature into the expressions of the second order derivative. In this section we examine the use of the intrinsic and parametrization independent normal curvature $\kappa_n$ to bound the error of the polygonal approximation.

The projection of the second partial derivatives of $S(u, v)$ onto $n$ yields

\[
\begin{align*}
    l_{11} &= \left< \frac{\partial^2 S}{\partial u^2}, n \right>, \\
    l_{12} &= \left< \frac{\partial^2 S}{\partial u \partial v}, n \right>, \\
    l_{22} &= \left< \frac{\partial^2 S}{\partial v^2}, n \right>,
\end{align*}
\]

where $l_{ij}$ are known as the coefficients of the matrix of the second fundamental form $L$ [5]. Computing $l_{ij}$ for the surface in Figure 1 reveals that indeed all $l_{ij}$ have vanished. Nonetheless the coefficients of the second fundamental form are parameterization dependent in general. Consider for example the unit sphere

\[
S^2(\theta, \phi) = (\cos(a\theta)\cos(\phi), \cos(a\theta)\sin(\phi), \sin(a\theta)).
\]

Clearly the unit normal of $S^2$ equal $S^2$. The second partials of $S^2$ with respect to $\theta$ equals $-a^2 S^2$ and therefore $l_{11} = -a^2$ depending on $a$ which is a reparametrization of $\theta$.

Let $p \in S(u, v) \Gamma S(u, v)$ is sufficiently differentiable and regular. Let $\Gamma \in T_p \Gamma$ where $T_p$ is the tangent plane of $S(u, v)$ at $p$ be represented as the linear combination of the
two first partial derivatives of \( S(u, v) \) that spans \( T_p \). Then

\[
\Gamma = \left\langle (\gamma_u, \gamma_v), \left( \frac{\partial S}{\partial u}, \frac{\partial S}{\partial v} \right) \right\rangle = \gamma_u \frac{\partial S}{\partial u} + \gamma_v \frac{\partial S}{\partial v},
\]

(10)

where \( \gamma = (\gamma_u, \gamma_v) \) is a vector in the parametric space of \( S(u, v) \). The normal curvature \( \kappa_n \) of surface \( S(u, v) \) at \( p \) in direction \( \Gamma \) is equal to [5]

\[
\kappa_n = \frac{\gamma^T L \gamma}{\gamma^T G \gamma},
\]

(11)

where \( G \) and \( L \) are the matrices of the first and second fundamental forms

\[
G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix},
\]

(12)

and \( g_{ij} = \left\langle \frac{\partial S}{\partial u^i}, \frac{\partial S}{\partial u^j} \right\rangle \). \( \kappa_n \) is invariant under rigid motion and reparameterization.

In [7] an upper bound on the normal curvature of a surface \( S(u, v) \) is computed as the sum of the squares of the principal normal curvatures \( \kappa_n^1(u, v) \) and \( \kappa_n^2(u, v) \) and represented as a piecewise rational scalar field

\[
\xi(u, v) = \left( \kappa_n^1(u, v) \right)^2 + \left( \kappa_n^2(u, v) \right)^2.
\]

(13)

One can symbolically compute \( \xi(u, v) \) [7] and employ the convex hull property of the Bézier and NURBs representations to derive an upper bound on \( \sqrt{\xi(u, v)} \). Denote by \( \overline{\xi} \) an upper bound on \( \sqrt{\xi(u, v)} \):

\[
\overline{\xi} = \sqrt{2} \kappa_n^i \Gamma i = 1, 2 \text{ if } \kappa_n^1 = \kappa_n^2 \text{ and } \overline{\xi} = \max(\kappa_n^1, \kappa_n^2) \text{ if either } \kappa_n^1 = 0 \text{ or } \kappa_n^2 = 0.
\]

Let \( \mathcal{C}_i = C(s_i)C(s_{i+1}) \) be a linear approximation of curve \( C(s) \), \( s_i \leq s \leq s_{i+1} \) where \( s \) is the arclength parameter of \( C \) (See Figure 4). Assume \( C(s) \) is curvature continuous and let \( \kappa(s) > 0 \), \( s_i \leq s \leq s_{i+1} \) be the curvature field of \( C(s) \). Denote by \( \kappa_{\max} = \max_{s_i \leq s \leq s_{i+1}} \kappa(s) \) the maximal curvature of \( C(s) \), \( s_i \leq s \leq s_{i+1} \). Assume

\[
\|C(s_{i+1}) - C(s_i)\| < \frac{1}{\kappa_{\max}}.
\]

Then

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Figure 4: A bound on $\epsilon$ is established as a function of the maximal curvature $\kappa_{\text{max}}$ of $C(s)$ and the arc length $s_{i+1} - s_i$.

Lemma 3

$$
\epsilon = \max_{s_i \leq s \leq s_{i+1}} \| C(s) - L_i \| \leq \frac{(c(s_{i+1} - s_i))^2}{8} \kappa_{\text{max}}, \quad \epsilon = \frac{\pi}{2}.
$$

Proof: Let $C_i$ be the circle of radius $\frac{1}{\kappa_{\text{max}}}$ through $C(s_i)$ and $C(s_{i+1})$ (See Figure 4). $C(s), s_i \leq s \leq s_{i+1}$ is bounded between $C_i$ and $L_i$ simply because $\kappa_{\text{max}} \geq \kappa(s) \geq 0$ where $\kappa_{\text{max}}$ and 0 are the constant curvatures of $C_i$ and $L_i$, respectively. Hence the maximal deviation between $C(s)$ and $L_i$ is bounded from above by the maximal deviation between $L_i$ and $C_i$.

$$
\epsilon = \max_{s_i \leq s \leq s_{i+1}} \| C(s) - L_i \|
< \| C_i - L_i \|
= \frac{1}{\kappa_{\text{max}}} - \frac{1}{\kappa_{\text{max}}} \cos \theta
= \frac{1}{\kappa_{\text{max}}} - \frac{1}{\kappa_{\text{max}}} \left( 1 - \frac{\theta^2}{2} + O(\theta^4) \right)
< \frac{1}{\kappa_{\text{max}}} \frac{\theta^2}{2}
= \frac{1}{\kappa_{\text{max}}} \frac{(c(s_{i+1} - s_i))^2}{8}, \quad 1 \leq \exists \epsilon \leq \frac{\pi}{2}
$$
Figure 5: An $\Omega$ shaped cubic and a $B$ shaped quadratic B-spline curves approximated using uniform sampling (a) and optimal sampling based on Lemma 3 (b). Both approximations carry the same number of samples. Original curve is shown in gray, piecewise linear approximation in black.

$$= \frac{(c(s_{i+1} - s_i))^2 \kappa_{max}}{8}, \quad (14)$$

because $O(\theta^4) = \cos \theta - 1 + \frac{\theta^2}{2}$ is non negative throughout.

Lemma 3 can be easily extended to the symmetric case for which $\kappa(s) < 0$. If $\kappa(s)$ can assume both positive and negative values in the domain in question, two bounding circles may be used to bound both the convex and concave regions of $C(s)$.

Lemma 3 can be employed in the generation of a more optimal piecewise linear approximation of freeform parametric curves. Given $C(t)$, $t$ is an arbitrary regular parametrization. One can estimate the arc length at the neighborhood of $C(t_0)$ using $\frac{ds}{dt}$. Figure 5 shows an example of using Lemma 3 compared to the uniform in parametric space sampling case.

We are now ready to establish a practical upper bound on the maximal possible deviation from $S(u, v)$ of every line segments connecting two points on $S(u, v)$. Let $D$ be the largest diagonal of the bounding box of $S(u, v)$. Assume $S(u, v)$ is fairly flat that is $\frac{1}{\kappa_{max}} > s_{i+1} - s_i$. Then and because $s_{i+1} - s_i$ is difficult to compute, $D$ can be employed to approximate $s_{i+1} - s_i$. Further since $\xi$ bounds $\kappa_{max}$ and following
Lemma 3Γ \( \frac{\partial^2 \Gamma}{\partial a^2} \) can be employed to provide an estimate on the allowed length of the piecewise linear approximation in the local neighborhood of the surface.

Finally it is interesting to compare the result of Lemma 3 to the known result from approximation theory [4] of Γ

\[
sup_{a \leq t \leq b} \| f(t) - l(t) \| \leq \frac{(b - a)^2}{8} sup_{a \leq t \leq b} \| f''(t) \|, \tag{15}
\]

where \( l(t) \) is a linear approximation of \( f(t) \) and recalling that \( f''(t) = \kappa(t) \) for arc length parameterization.

Assume surface \( S(u, v) \) is a cylinder and hence parabolic. Let the lines of curvature [5] be the isoparametric directions such that \( \kappa_n^u = 0 \) \( \kappa_n^v = \kappa \). Clearly there is a need to subdivide \( S(u, v) \) only in \( v \). \( \xi(u, v) = (k_n^u(u, v))^2 = \kappa^2 \Gamma \) and is constant throughout. On the other hand the diagonal \( D \) is reducing in size by subdividing either in \( u \) or in \( v \) providing little cues as to the obviously preferred subdivision direction. It is unfortunate that there is no simple way to derive a preferred subdivision direction either \( u \) or \( v \) using only the information that is provided by \( \xi(u, v) \). One can consider symbolically computing and exploiting the normal curvatures in the isoparametric directions \( (\kappa_n^u(u, v))^2 \) and \( (\kappa_n^v(u, v))^2 \) as well as the normal curvature along the diagonal \( u = v \). Not only that all this symbolic manipulation is computationally intensive but even with the aid of \( (\kappa_n^u(u, v))^2 \) and \( (\kappa_n^v(u, v))^2 \) it is insufficient. Consider the case where the surface is planar but with a non rectilinear boundary. In [9] it is properly noted that the boundary of the surface should also be considered by applying a flatness or linearity tests to all four boundaries. A given surface \( S(u, v) \) can be flat and hence \( \kappa_i^l \equiv 0, \ i = 1, 2 \) yet \( S(u, v) \) must be subdivided due to its freeform shaped boundary. Thus an additional test should examine the curves of the boundary of the approximated
4 Extensions

In Algorithm 1 line (1) surface \( S(u, v) \) is subdivided at the middle of the parametric domain. If \( S(u, v) \) is not \( C^1 \) continuous a preprocessing subdivision stage must be applied to ensure \( C^1 \) continuity. The motivation for such a preprocessing stage is obvious. The discontinuities of \( S(u, v) \) must be reflected in the polygonal approximation. This streamline also motivates an attempt to subdivide \( S(u, v) \) at locations of extreme curvature.

Clearly a subdivision of \( S(u, v) \) in \( v \) at the \( v \) location with the highest normal curvature in the tangent plane direction perpendicular to iso-\( v \) parametric direction can yield a more optimal polygonal approximation. The perpendicular to an iso-\( v \) parametric direction can be approximated using the iso-\( u \) parametric direction.

Herein the symbolic computation of \( (\kappa^u_n(u, v))^2 \) and \( (\kappa^v_n(u, v))^2 \) can be employed to convey the locations of the extreme values of the normal curvatures in the isoparametric directions and hence provide better subdivision locations. Figure 6 shows a surface \( S(u, v) \) with its \( (\kappa^u_n(u, v))^2 \) and \( (\kappa^v_n(u, v))^2 \) scalar curvature fields. As a biquadratic surface \( S(u, v) \) is curvature discontinuous at interior knots as can be seen from the shape of \( (\kappa^u_n(u, v))^2 \). Nonetheless the three highly curved regions of \( S(u, v) \) along the triangular cross section clearly show up in \( (\kappa^u_n(u, v))^2 \). The use of these scalar curvature fields can improve the subdivision process creating a more optimal polygonal approximation.

One can attempt to subdivide a surface at the parameter values of its interior knot instead of at the center of the parametric domain. The existence of knots suggests the existence of high resolution information in the surface and hence provides cues on the
\[(\kappa_n^u(u, v))^2 \times 0.001\]

\[
S(u, v)
\]

Figure 6: A biquadratic Bspline surface \(S(u, v)\) with its scalar curvature fields \((\kappa_n^u(u, v))^2\) and \((\kappa_n^v(u, v))^2\). \((\kappa_n^u(u, v))^2\) is scaled by a factor of 0.001.

complexity of the shape. This simple heuristic yields surprisingly good results without curvature estimation computation. In Figure 6 the maximal normal curvature in the isoparametric direction is found to be close to the interior knots’ parameter values, manifested as the discontinuities in the figure. In Figure 3 the heuristic of subdividing at interior knots was employed yielding good results throughout.

5 Conclusion

We presented two approaches to bound the maximal deviation of piecewise linear approximations of freeform surfaces. We found that the use of curvature based polygonal approximation is difficult due to its significant order of \(\xi(u, v)\). Lack of ability to derive the preferred subdivision direction and insufficiency in determining termination conditions
of subdivisions.

The use of a bilinear fit is not only sufficient but is also found more efficient computationally and with an ability to determine a preferred subdivision direction. Moreover, the bilinear fit can exploit $(\kappa_n^u(u, v))^2$ and $(\kappa_n^v(u, v))^2$ to derive a more optimal subdivision direction either $u$ or $v$. The bilinear fit method can be combined with other methods presented in the literature. For example, the use of proximity and silhouette screen space consideration [9] can be embedded into the bilinear tolerance directly or alternatively incorporated as a parallel test.

One can attempt to globally approach the problem of piecewise linear approximation of freeform surfaces. Given a freeform surface $S(u, v)$ one would like to compute the “best” approximation of $S(u, v)$ using $n$ triangles. The greedy subdivision methods discussed herein consider the distribution of the triangles within a uniform isoparametric grid. As a result, a different surface parameterization will result in a different polygonal approximation. This artificial constraint should be eliminated. Moreover, current schemes including the ones exploited herein are local and greedy in their nature. By computing the entire global solution at a single stage more optimal solutions can be expected. This open question can also be formulated as follows: given $n$ vertices position them in the parametric space of surface $S(u, v)$ so that a triangulation over these vertices will be globally optimal under some norm. This global approach was not investigated to the best knowledge of the author and yet is of great interest.

6 Acknowledgments

The author is grateful to Craig Gotsman and the reviewers of this paper for their detailed valuable remarks on this paper.
References


