# Constrained Multiresolution Geometric Modeling

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## 1 Introduction

Freeform curves, surface and solids are generally represented in B-spline basis. Various geometric quantities, such as control points, knots and weights have to be specified. Controlling the shape of an object under complex deformations by manipulating the control points directly is often difficult. The movement of control points gives an indication of the resulting deformation, but being extraneous to the object, the control points do not allow for precise control of the shape. In addition, large deformations of complex objects with many details to be preserved become nearly impossible without any "higher level" control mechanisms. User-friendly shape-control tools, therefore, generally make use of modeling techniques that integrate constraints. The present paper surveys the state-of-the-art of geometric modeling techniques that integrate constraints, including direct shape manipulation, physics-based modeling, solid modeling and freeform deformations as well as implicit surface modeling. In particular, we will focus on recent advances of multiresolution modeling of shapes under constraints. Going beyond the limits of traditional modeling techniques, they allow for editing of complex objects while automatically preserving the details.

## 2 Interactive freeform techniques

Controlling the shape of an object under complex deformations is often difficult. The traditional approach to interacting with deformable objects is to manipulate control points since they allow precise control over models. CAGD textbooks by Farin, Hoschek and Lasser, Bartels et al., and Cohen et al. [Far96], [HL93], [BB87], [CRE01] cover the complete theory of parametric freeform curve and surface representations such as NURBS curves and tensor product surfaces, triangular Bézier patches, n-sided patches, but also Coons and Gregory surfaces. Limited by the expertise and patience of the user, the direct use control points as the manipulation handles necessitates an explicit specification of the deformation. Therefore, large deformations can be extremely difficult to achieve because they mandate moving a huge number of individual control points by hand, and the precise modification of the freeform object can be laborious. Deformation tools based on geometric constraints offer more direct control over the shape. In this section high-level interactive freeform curve and surface manipulation techniques are presented. These use either geometric constraints as direct deformation handles (sect. 2.1) or as definitions of functional behavior via geometric properties (sect. 2.2). Finally, geometry-driven (freeform) solid modeling techniques are described (sect. 2.3).

### 2.1 Direct curve and surface manipulation

Rather than manipulating control points, [BB89] show how to pick any point on a B-spline curve and change its location, i.e. the curve is constrained to pass through a user-specified location. The new curve shape is computed by minimizing the control point's offset. [FB93] control the shape of a B-spline curve by enforcing prescribed geometric constraints, such as the position of a curve point, tangent direction and magnitude, or curvature magnitude. An extension to tensor product B-spline surfaces is given in [Fow92]. This satisfies the user-defined position of surface points, normal direction, tangent plane rotation (twisting effect), and the first partial derivative's magnitude (tension effect). [BR94] deform B-spline surfaces by determining the displacement and radius of influence for each constrained surface point. [HHK92] proposes points picking for freeform deformations. Curve constraints, i.e. enforcing the surface to contain a given curve or to model a character line, have been considered by [CW92], [GL96], [PGL02]. Direct shape manipulation techniques are closely related to Variational Design, where the objective of obtaining fair and graceful shapes is achieved by minimizing some energy, see sect. 4.1. In general, a freeform shape has much more degrees of freedom than constraints to satisfy. In order to compute a new shape the remaining degrees of freedom are prescribed by minimizing some energy functional, such as bending. For example, [WW92] maintains the imposed constraints while calculating a surface that is as smooth as possible. [CW92] derives interactive sculpting techniques for B-spline surfaces based on energy minimization, keeping some linear geometric surface-constrained features unchanged. [CG91] enforces linear geometric constraints for shape design of finite elements governed by some surface energy. While energy minimization affects the surface globally, finite element methods allow for local control. [FB88] later used the technique of hierarchical B-splines attempt to overcome this drawback for B-spline surfaces.

### 2.2 Feature Modeling

Constrained geometric modeling also occurs in feature modeling – a quite different context. Geometric modeling tools are commonly used in various phases of product development, for example, to generate product images or NC-code. Many applications, however, require functional information that is not contained in geometric models. A feature in a product model combines geometric information with functional information, such as information about its function for the user in a design application, or its manufacturing process in a manufacturing application. Features are higher level entities compared to the underlying geometry and as such are easier to maintain and manipulate at the user level.

The concept of features has been investigated mainly in mechanical environments [DFG94], [Ros90]. This is due to the fact that classical mechanical parts are defined by canonical geometry shapes, which can easily be classified. Constraints occur at different stages in feature modeling. In [BB00], a semantic feature modeling approach is presented. All properties of features including their geometric parameters, their boundaries, their interactions, and their dependencies, are declared by means of constraints. Another issue in feature modeling is feature validation, which concerns the meaning of a feature, given by its information content [DKB96]. A feature modeling system should ensure that product modifications by a user are in accordance with the meaning of the features. Herein constraints are used to specify such feature validity conditions; constraint satisfaction techniques are applied to maintain feature validity under product modifications from multiple views.

More recently, freeform feature modeling approaches have been developed [CM92], [Vos99], [VVB03]. In contrast to the feature-based approach adopted by CAD systems for classical mechanical design, freeform features are strongly related to aesthetic or styling aspects when modeling with freeform surfaces. The Brite-Euram Project called FIORES (Formalization and Integration of an Optimized Reverse Engineering Styling Workflow) focused on the development of modeling tools for direct shape modifications closer to the stylist's way of thinking [DP98]. Here again, properties of aesthetic features are expressed in terms of constraints, including convexity, shape preserving deformations, eliminations and cuts, and continuity conditions [FGM99].

### 2.3 Solid modeling

The history of solid modeling goes back to the 1980s when the term "solid modeling" was introduced; see survey papers [RV82], [RR92]. This was also the period when early advances were motivated primarily by the mechanical engineering industry. Traditional solid modeling approaches include implicit functions (CSG and blubby models), boundary representations and cell decompositions. The use of constraints has mainly been developed from interaction with freeform solids.

Sederberg and Parry [SP86] developed free-form deformation (FFD), a technique for globally deforming three-dimensional objects. The object to be deformed is embedded in a three-dimensional parametric space, usually defined by a Bézier or B-spline solid (called the control lattice). The vertices of the object are assigned parametric values that depend on their positions inside the parametric solid. Local control can be achieved by patching FFD lattices along their boundaries. As the user moves the control points of the solid, the vertices of the embedded object move in response. The geometric structure and definition of the embedded object are independent of the FFD process. Polygon-based, parametric, implicit and other types of objects can be embedded and deformed using the same FFD interface. Hsu et al. [HHK92] improved upon traditional FFD with their technique that permits users to manipulate the embedded object directly. In this manner, the system computes how the Bézier (or B-spline) control points must move in order to produce the desired deformation. Shi-Min et al. [SHCJ01] proposed a similar scheme in which an FFD function is computed based on the manipulation and translation of a single point. Complex deformations are then achieved via the composition of several such single-point FFDs. MacCracken and Joy [MJ96] generalized FFD by incorporating arbitrary-topology subdivision-based lattices.

Rappoport et al. [RSB95] derived a technique for *preserving the volumes* of tri-variate Bézier solids. They were the first to publish a way to use free-form solids for representing a sculpted object. In their method, different solids are patched together at their boundaries to create more complex objects. Their algorithm uses an energy minimization function whose purpose is to preserve the volume during sculpting. In addition to the volume-preserving constraint, their system can satisfy interpatch continuity constraints, positional constraints, attachment constraints, and inter-point constraints. All of these are formulated using a Lagrange multiplier method.

Hirota et al. [HML99] presented an algorithm for *preserving the global volume* of a solid undergoing a free-form deformation. Unfortunately, their algorithm works only for Brep solids. During initialization, each triangle in the surface is projected onto the x-y plane, and the volume under the triangle is stored. During the deformation process, this volume is constantly re-computed and compared to the original. By taking the difference between the volumes of the original and deformed volume elements, the total change in volume is computed. A simple energy functional is minimized, subject to the volume preservation constraint using the augmented Lagrangian method.

Self-intersection could clearly occur in the FFD function. Local self-intersection can be identified via the vanishing Jacobian of the FFD, an approach proposed in [GD01].

## 3 Implicit Surfaces

Implicit surfaces have sparked great interest in the computer graphics and animation community [WMW86], [BS91], [DG95], [PAS95], [GWG98], with applications for geometric modeling and scientific visualization [LC87]. Deformations of implicit surfaces can be obtained intuitively by articulating the skeleton or by changing the parameters of implicit primitives that hierarchically define the surface [Can98], [CD97]. Another, more intricate way to deform implicit models is to change the iso-surface progressively by modifying the sample field function defining it [WB98], [DC98].

Two kinds of constraints are particularly easy to integrate. First, collision detection can be accelerated, since in-out functions are provided. Second, implicit surfaces provide a good tool for physics-based animation; see section 4.

Volume of an implicit object is another constraint that it is important to preserve during deformation [DG95], [CD97], [DC98]. For example, volume constant deformations in a morphing process can make virtual objects look like real ones. A more complete overview on implicit surface modeling can be found in [B97], [Can99].

## 4 Physics-based modeling

Physics-based modeling attaches physical properties to geometric structures in order to achieve better or more fair shapes for design purposes, or in order to increment realism in computer animations. The constraints are formulated in terms of energy functionals or kinetic and mass laws that are, in many cases, non-linear.

### 4.1 Variational shape design

Although it is difficult to exactly define, in mathematic terms, what *fairness* of a curve or surface is, it is commonly accepted that smooth and graceful shapes are obtained by minimizing the amount of energy stored in the surface. The energy functionals originating from elasticity theory are in general non-linear, such as the bending energy for curves $\int \kappa^2(t)dt$ or the thin-plate energy for surfaces $\int \kappa_1^2 + \kappa_2^2 dA$. These and other higher order non-linear energy functionals have been used in [MS92], [Gre94].

In order to accelerate computations, linearized versions of these energy functionals are generally used; see, for example, [CG91], [CW92], [WW92], [GC95]

$$\mathcal{E} = \int_\sigma (\alpha \text{ stretch} + \beta \text{ bend}) d\sigma$$

where $\alpha$ and $\beta$ are weights on stretching and bending. This produces a surface which tends to minimize its area to avoid folding and to distribute curvature over large regions in order to result in fair shapes. The stretch-and-bend functionals are typically approximated via the following quadratic terms: $\alpha_{11}X_u^2 + \alpha_{12}X_uX_v + \alpha_{22}X_v^2$ and $\beta_{11}X_{uu}^2 + \beta_{12}X_{uv}^2 + \beta_{22}X_{vv}^2$, respectively, only to be linearized in the optimization process.

Historically, use of such energy functionals goes back to early spline and CAGD literature [Meh74], [Rei67] and has led to a research area, called Variational Design of smooth curves and surfaces, today [FRS87], [HS87], [HS92], [BH94], [BHS93], [Hah98], [Had95].

### 4.2 Dynamic modeling

Deformations of objects are obtained by externally applying forces. The dynamic approach based on well-established laws of physics aims to produce smooth and natural motions in order to create realistic-looking computer animation. Traditional animation techniques [Las87] have to be considered as well. To synthesize convincing motions, the animator must specify the variables at each instant in time, while also satisfying kinematic constraints.

[TPB87] introduced freeform deformable models to computer graphics, pioneering the development of dynamic parametric curves, surfaces and solids. Animation of implicit surfaces goes back to [WMW86]. Gravitational, spring, viscous and collision forces applied to the geometric model act as constraints when deforming objects. Non-linear dynamic behavior [TF88] results from simulating inelastic deformation. Different dynamic behavior of deformable objects has been developed by many varying the imposed constraints, the numerical solution method or by applying these to different geometric models, including modal dynamics [PW89], animation of non-rigid articulated objects [WW90], FEM-based methods [CG91], D-Nurbs [TQ94], implicit surfaces [DC95], deformable voxel methods [CZK98] and dynamic subdivision surfaces [QMV98]. In [PLH02], dynamic parameters are directly evaluated over B-spline curves, while parameterization of the curve is ignored.

## 5 Multiresolution Editing

Multiresolution analysis has received considerable attention in recent years in many fields of computer graphics, geometric modeling and visualization [SDS96], [WW01]. It provides a powerful tool for efficiently representing functions at multiple levels-of-detail with many inherent advantages, including compression, LOD display, progressive transmission and LOD editing.

In the literature the term multiresolution (MR) is employed in different contexts, including wavelets, subdivision and hierarchies or multigrids. Multiresolution representations based on wavelets have been developed for parametric curves [CQ92], [LM92], [FS94], and can be generalized to tensor-product surfaces, to surfaces of arbitrary topological type [LDW97], to spherical data [SS95], and to volume data [CMP97]. Wavelets provide a rigorous unified framework. Herein, a complex function is decomposed into a "coarser" low resolution part, together with a collection of detail coefficients, necessary to recover the original function. Other multiresolution representations exist for data defined on irregular meshes [BHN96], [Bon98], for arbitrary meshes [ZSS97], [KCV98], [EDD95], [H96], for tensor product surfaces, known as hierarchical B-splines [FB88], and for volumetric data sets represented using tri-variate functions [RE99].

In the context of geometric modeling, LOD editing is an attractive MR application because it allows the modification of the overall shape of a geometric model at any scale while automatically preserving all fine details. In contrast to classical control-point-based editing methods where complex detail-preserving deformations need to manipulate a lot of control points (see sect. 2), MR methods can achieve the same effect by manipulating only a few control points of some low resolution representation; see [FS94], [DSS95]. However, there are application areas, including CAGD and computer animation, where deformations under constraints are needed. As stated in the introduction, it is obvious that constraints offer an additional and finer control of the deformation applied to curves and surfaces.

Continuing the previous sections, the present section reports on constrained modeling methods using MR representations. Section 5.1 presents an LOD editing method for B-spline curves and surfaces that allows the integration of linear and non-linear geometric constraints, including fixed position, symmetry and constant area. Section 5.2 presents wavelet-based MR curve editing methods preserving area and length of curves. Section 5.3 is about variational MR methods, where minimum energy is the constraint to be satisfied. Finally, section 5.4 describes MR subdivision methods.

## 5.1 Constrained Multiresolution control

### Multiresolution editing of freeform curves

In [FS94, GC95], a wavelet decomposition for uniform cubic B-splines is presented toward interactive and intuitive manipulation of freeform shape. In [KE97], results from [LM92] are similarly employed toward the support of non uniform knot sequences. While local support is considered the major advantage of the B-spline representation, it is also its achilles heel. Global changes are fundamentally difficult to apply to a highly refined shape and a pain-staking laborius manual effort is required to move one control point at a time. The ability to decompose a given freeform B-spline curve or a surface as offered by [FS94, GC95, KE97] is a large step in the direction that alleviates these difficulties. The user can now modify the shape locally or globbally as he/she see fits. In Figure 1, a single select-and-drag operation is applied to a non uniform quadratic B-spline curve at six different resolutions. The outcome clearly shows the power of multiresolution editing, allowing for both local and global control.

**Fig. 1.** Multiresolution manipulation of a non uniform quadratic B-spline curve with 138 control points. In all six images (a)-(f), a single select-and-grad operation was applied to the top of the 's' letter in the upward direction.

Let $C(t) = \sum_{i=0}^{n-1} P_i B_{i,\boldsymbol{\tau},k}(t)$ be a planar non uniform B-spline curve of order $k$ and $n$ control points. Let the knot sequence of $C(t)$ be

$$\boldsymbol{\tau} = \{t_0, t_1, \cdots, t_{k-1}, \cdots, t_n, \cdots t_{n+k-1}\}.$$

$C(t)$ is defined for the domain $[t_{k-1}, t_n)$. The knots from $t_k$ to $t_{n-1}$ are denoted the *interior knots* and their removal does not affect the domain of $C(t)$.

The knot sequence of $\boldsymbol{\tau}$, together with the order $k$, define a subspace $\boldsymbol{\Phi}$ of piecewise polynomial functions. This subspace contains the all polynomial functions but also piecewise polynomials with potential discontinuities at each of the interior knots, depending on the multiplicity of the knot. Let $\boldsymbol{\tau}_0 = \boldsymbol{\tau}$ and further let $\boldsymbol{\tau_{i+1}} \subset \boldsymbol{\tau_i}$ by removing only interior knots from $\boldsymbol{\tau_i}$. Then:

- The domain spaned by all the $\boldsymbol{\tau}_i$ is the same and equal to $[t_{k-1}, t_n)$, $\forall i$.
- The subspace $\boldsymbol{\Phi_{i+1}}$ induced by $\boldsymbol{\tau_{i+1}}$ and $k$ is a strict subspace of $\boldsymbol{\Phi_i}$. That is $\boldsymbol{\Phi_{i+1}} \subset \boldsymbol{\Phi_i}$.

Clearly $C(t) \in \boldsymbol{\Phi_0}$. Denote $C(t)$ as $C_0(t)$. Let $\boldsymbol{\Phi_1}$ be a new subspace formed out of $\boldsymbol{\Phi_0}$, by removing a single knot $t_j = \boldsymbol{\tau_0}/\boldsymbol{\tau_1}$. We seek to find the orthogonal projection, under the $L_2$ norm, of $C_0(t)$ onto $\boldsymbol{\Phi_1}$. Denote this projection by $C_1(t) \in \boldsymbol{\Phi}_1$ and let the difference be $D_1(t) = C_0(t) - C_1(t)$. We call $C_1(t) = \sum_{i=0}^{n-2} Q_i B_{i,\boldsymbol{\tau}_1,k}(t)$ a low resolution version of $C_0(t)$ and $D_1(t)$ the details. $D_1(t)$ is in a new subspace $\boldsymbol{\Psi}_0 \subset \boldsymbol{\Phi_0}$ which means we can express $D_1(t)$ in terms of the basis functions of $\Psi_0$ as

$$D_1(t) = \sum_{i=0}^{n-1} d_i B_{i,\boldsymbol{\tau}_0,k}(t).$$

$D_1(t) \in \boldsymbol{\Psi}_1$ is orthogonal to the space of $\boldsymbol{\Phi}_1$. Hence, the following must hold,

$$0 = \langle D_1(t), B_{m,\boldsymbol{\tau}_1,k} \rangle = \sum_{i=0}^{n-1} d_i \langle B_{i,\boldsymbol{\tau}_0,k}, B_{m,\boldsymbol{\tau}_1,k} \rangle . \quad (1)$$

Turns out Equations (1) completely prescribes the coefficients of $D_1(t)$ upto uniform scaling of the function. This $D_1(t)$ is also known as the *B-wavelet* function of knot $t_j$ in subspace $\Psi_0$. Figure 2 presents the orthogonal projection of our "multiresolution" curve onto several subspaces, all the way to a single Bezier curve.

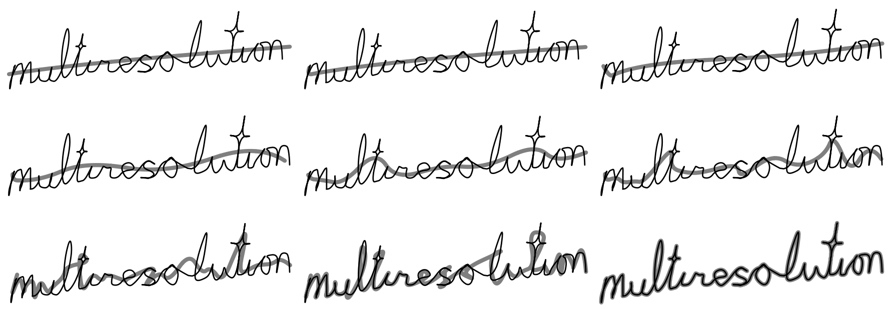


**Fig. 2.** Projections (in thick gray) of the original "multiresolution" curve from Figure 1 (in thin line) over different spline subspaces is presented. The top left is the smallest space (single quadratic polynomials) all the way to the bottom right which is the orignal space.

Unfortunately, the computation of the coefficients of $D_1(t)$, following Equations (1) is expensive, as it necessitates the resolution of products and integrals of B-spline basis functions,

$$\langle B_{i,\boldsymbol{\tau},k}(t), B_{j,\boldsymbol{\tau},k}(t) \rangle = \int B_{i,\boldsymbol{\tau},k}(t) B_{j,\boldsymbol{\tau},k}(t) dt.$$

One option is to limit these computations to uniform knot sequences only, removing half the knots each time, effectively doubling the knot spacing. This approach was taken by [FS94, GC95] and it allows one to precompute the B-wavelets once for each order.

For non uniform knot sequences the B-wavelets must be computed every time and while one can reach interactive rates for curves with dozens of control points, this computation as already stated is expensive. Figure 3 presents few examples of B-wavelets. See [LM92, E92, KE97] for more on the computation of products and integrals of B-spline basis functions as well as more on this B-wavelet decomposition. A similar computation is also necessary toward the computation of $C_{i+1}(t)$ from $C_i(t)$, given the subspace $\boldsymbol{\Phi_i}$.

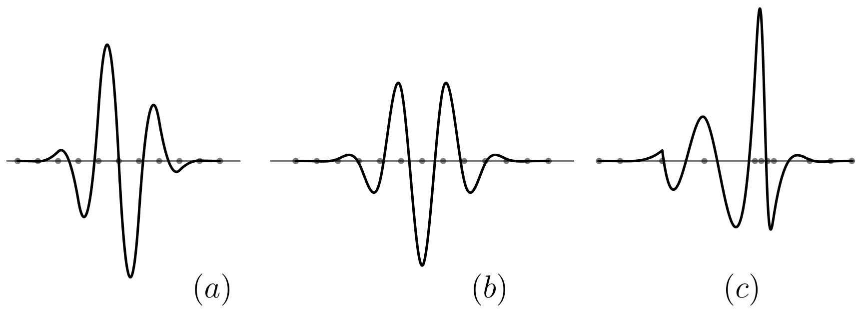



**Fig. 3.** B-Wavelets of a uniform quadratic (a), a uniform cubic (b), and a non uniform knot seqence of a cubic curve (c). The third knot from the left in (c) is a triple knot, resulting in a $C^1$ discontinuity in the B-Wavelet.

Due to the computational costs, alternatives were sought. One alternative is to approximate the low resolution projection using a simple least squares fit [EG95]. Given $C_i(t) \in \boldsymbol{\Phi_i}$, find a least squares fit $C_{i+1}(t) \in \boldsymbol{\Phi_{i+1}}$ to $C_i(t)$ by sampling $C_i(t)$ at $m$ location, $m >> n_{i+1}$, $n_{i+1}$ the number of coefficients in $C_{i+1}(t)$. Nevertheless, for the task in hand of interactive multiresolution manipulation with constraints, this B-wavelet decomposition is not really necessary! Consider curve $C_i(t) \in \boldsymbol{\Phi_i}$, $\boldsymbol{\Phi_i} \subset \boldsymbol{\Phi_0}$. Now consider a change of a single control point in $C_0(t)$ against a change of a single control point in $C_i(t)$. The later will clearly affect a larger domain of the original curve $C(t) = C_0(t) \in \boldsymbol{\Phi}_0$ compared to a change in $C_0(t)$. A single control points $P_j$ is supported along the non zero domain of its basis function $B_j(t)$. The less interior knot there are, the larger the domain of $B_j(t)$ is.

Then, a modification to the shape using a change in curve $C_i(t) \in \boldsymbol{\Phi_0}$ could be added to the original curve $C_0(t)$ using knot insertion [CLR80], refining $C_i(t)$ at all the knots of $\boldsymbol{\tau}_0/\boldsymbol{\tau}_i$. In practice, direct manipulation is prefered over control points manipulation, hiding the representation (i.e. control points) from the novice user. If point $C(t_1)$ is directly selected and dragged along the vector $\mathbf{V}$ to $C(t_1) + \mathbf{V}$, a new $\Delta_i(t) \in \boldsymbol{\Phi_i}$ curve could be constructed as

$$\Delta_i(t) = \frac{1}{\sigma} \sum_{i=0}^{n_i} B_{i,\boldsymbol{\tau}_i,k}(t_1) B_{i,\boldsymbol{\tau}_i,k}(t),$$

using the support of the different basis functions at $t_1$ as the weights and

$$\sigma = \sum_{i=0}^{n_i} \left(B_{i,\boldsymbol{\tau}_i,k}(t_1)\right)^2 = \sum_{i=J-k+1}^{J} \left(B_{i,\boldsymbol{\tau}_i,k}(t_1)\right)^2, \qquad t_J \le t_1 < t_{J+1},$$

yielding $\Delta_i(t_1) = 1$.

**Linear constraints**

Multiresolution editing has a drawback we already discussed. It can be imprecise. We now aim to add support for constraints to our multiresolution editing capabilities. To begin with, we consider the two simple linear constraints of position and tangency.

Recall curve $C(t) = \sum_{i=0}^{n} P_i B_{i,\boldsymbol{\tau},k}(t)$. A positional constaint could be prescribed as $C(t_P) = P$. Then, if the original curve satifisies the constraint or $C(t_p) = C_0(t_p) = P$, we are now required to have $\Delta_i(t_p) = 0$, an additional linear constraint that is easy to satisfy. In practice, two possible simple approaches could be employed to solve this underconstrained linear system, having $\Delta_i(t_p) = 0$ and $\Delta_i(t_1) = 1$ as constriants and achieving an $L_2$ minimizing solution elsewhere along the domain. Either the singular valued decomposition (SVD) or the QR factorization [GV96] of the linear systems of equations would do. Interestingly enough, the QR factorization is also employed by [WW92] for similar reasons.

A tangenial constraint could be supported in an almost identical way. Here, $C'(t_T) = T$ and $C'(t)$, that is expressed in term of basis functions one degree lower, is elevated back to the same function space using degree elevation, resulting again in a linear alternative constraint to satisfy of $\Delta_i'(t_T) = 0$. Second order or even higher derivatives constraints could easily be incorporated as well, in a similar fashion. Figure 4 shows one example of multiresolution editing with positional and tangential constraints.

**Fig. 4.** Multiresolution editing without (a), and with two positional (b), and three tangential (c) linear constraints.

Other linear constraints could also be support with some more effort. A planar curve, having the domain of $t \in [0, 1]$, is considered $x$-symmetric if

$x(t) = x(1-t)$ and $y(t) = -y(1-t)$. Analogically, one can define $y$-symmetry and even rotational symmetry as $x(t) = -x(1-t)$ and $y(t) = -y(1-t)$. Assuming a symmetric knot sequence, that is $\tau_{i+1} - \tau_i = \tau_{k+n-i-1} - \tau_{k+n-i-2}$, $0 \le \forall i \le n/2$,

$$c(1-t) = \sum_{i=0}^{n-1} P_i B_i(1-t) = \sum_{i=0}^{n-1} P_i B_{n-1-i}(t) = \sum_{i=0}^{n-1} P_{n-1-i} B_i(t)$$

due to the symmerty of the basis functions. But now the constraint of $x(t) = x(1-t)$ reduces to

$$\sum_{i=0}^{n-1} x_i B_i(t) = \sum_{i=0}^{n-1} x_{n-1-i} B_i(t), \qquad \text{or} \qquad \sum_{i=0}^{n-1} (x_i - x_{n-1-i}) B_i(t) = 0.$$

Hence and because of the independence of the basis functions, the symmetry constraint is now reduced to $O(n/2)$ linear constraints of the form

$$x_i = x_{n-1-i}, \quad i = 0, \cdots, \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

and

$$y_i = -y_{n-1-i}, \quad i = 0, \cdots, \left\lceil \frac{n}{2} \right\rceil - 1.$$

Finally, we consider area constraints. The area of a closed curve equals,

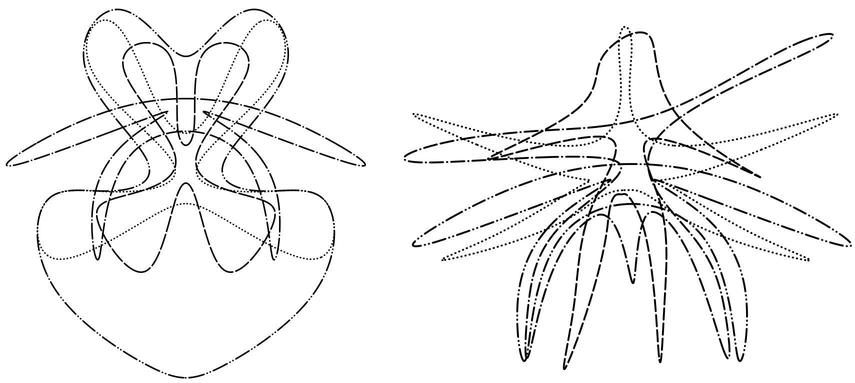
$$\begin{aligned}
\mathcal{A} &= \frac{1}{2} \oint -x'(t)y(t) + x(t)y'(t)dt, \\
&= \oint -\sum_i x_i B'_{i,k}(t) \sum_j y_j B_{j,k}(t) + \sum_i x_i B_{i,k}(t) \sum_j y_j B'_{j,k}(t)dt \\
&= \sum_i x_i \sum_j y_j \oint -B'_{i,k}(t)B_{j,k}(t) + B_{i,k}(t)B'_{j,k}(t)dt. \\
&= \begin{bmatrix} x_0, x_1, \cdots, x_{n-1} \end{bmatrix} \begin{bmatrix} \xi_{0,0} & \xi_{0,1} & \cdots & \xi_{0,n-1} \\ \xi_{1,0} & \xi_{1,1} & \cdots & \xi_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1,0} & \xi_{n-1,1} & \cdots & \xi_{n-1,n-1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\xi_{i,j} &= \oint -B'_{i,k}(t)B_{j,k}(t) + B_{i,k}(t)B'_{j,k}(t)dt \\
&= \oint -(k-1)\left( \frac{B_{i,k-1}(t)}{t_{i+k-1} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k} - t_{i+1}} \right) B_{j,k}(t) \\
&\quad + (k-1)\left( \frac{B_{j,k-1}(t)}{t_{j+k-1} - t_j} - \frac{B_{j+1,k-1}(t)}{t_{j+k} - t_{j+1}} \right) B_{i,k}(t)dt.
\end{aligned}$$

The area constraint is not linear. Nonetheless, it is a bilinear constraint so one could fix the $x_i$ coefficients, resulting in a linear constraint in $y_i$ and then reverse the role of $x_i$ and $y_i$, in the next iterations. During an interactive session when the user select-and-drag curve's locations we are in need of solving these constraints every mouse event or almost every pixel. This interleaving process becomes fully transparent to the end user at such low granularity.

Figure 5 shows two examples of direct manipulation of freeform curves under symmetry and area constraints. The curves were directly manipulated in real time while the symmetry and/or area constraints are fully preserved. More on the symmetry and area constraints in multiresolution editing as well as the special case of linear curves and the extension to freeform surfaces could be found in [E01].



(a) (b)

**Fig. 5.** $Y$-symmetry constraint (a) and area constraint (b) are employed in multiresolution context. These two examples were created in few seconds using direct curve manipulation under constraints.

### 5.2 Area and length preserving MR curve editing

In a wavelet based multiresolution setting complex objects can be edited at a chosen scale with mainly two effects: First, modifying some low-resolution control points and add back the details modifies the overall shape of the object. Second, modifying a set of fine detail coefficients modifies the character of the object without affecting its overall shape. In this section a wavelet based multiresolution editing method is presented, that entirely integrateis the constant area constraint completely into the multiresolution formulation of the deformation.

*Wavelet based MR curve*

Let us briefly sketch the notation of the wavelet based multiresolution analysis that will be used in this section. For more detais see [Mal89], [FS94], and [SDS96]. Suppose we have a certain functional space $E$ and some nested linear *approximation spaces* $V^j \subset E$ with $V^0 \subset V^1 \subset \cdots \subset V^n$. Since we are dealing with closed curves, these spaces have finite dimension. Let $V^j$ be spanned by a set of basis functions $\varphi^j = [\varphi_1^j, \ldots, \varphi_m^j]^T$, called *scaling functions*. A space $W^j$ being the complement of $V^j$ in $V^{j+1}$ is called the *detail space*. Its basis functions $\psi^j = [\psi_1^j, \ldots, \psi_{N-m}^j]^T$ are such that together with $\varphi^j$ they form a basis of $V^{j+1}$. The functions $\psi_i^j$ are called *wavelets*. The space $V^n$ can therefore be decomposed as follows:

$$V_n = V_{n-1} \bigoplus W_{n-1} = V_{n-2} \bigoplus_{j=n-2}^{n-1} W_j = \cdots = V_0 \bigoplus_{j=0}^{n-1} W_j. \quad (2)$$

A **multiresolution curve** is then defined as a planar parametric curve $c(t) = (x^n)^T(\varphi^n)$, element of $V^n$, where $x^n$ is a column of control points $x_0^n, \ldots, x_{D2^n}^n \in I\!R^2$. Due to property (2) the same curve can be expressed in terms of the basis functions of the different decompoditions of $V^n$, each of it corresponding to a certain resolution of the curve. The multiresolution curve at any **level of resolution** $L \in [0, n]$, i.e. element of $V_L \bigoplus_{j=L}^{n-1} W_j$ is then given by some coarse control points $x^L$ that form approximations of the initial control polygon and by the detail coefficients $d^L, \ldots, d^{n-1}$ as follows:

$$\mathbf{c}(t) = (\mathbf{x}^L)^T(\varphi^L) + (\mathbf{d}^L)^T(\psi^L) + \cdots + (\mathbf{d}^{n-1})^T(\psi^{n-1}), \quad L = 0, \ldots, n.$$

The *filter bank* algorithm [Mal89], [FS94] is used to compute the coefficients of all levels of resolutions from the initial coefficients $x^n$ and vise versa.

*Area preserving deformation of a MR curve*

An advantage of a MR representation of the curve is that LOD editing consists of simply applying deformations on the coarse control points at some level $L$, the overall shape of the curve is therefore modified and the fine details are preserved, see fig. 6 (a,b).

However the enclosed area of a (closed) modified curve is generally not preserved. In [HBS02] is has been shown that the constant area constraint can be integrated completely into the MR editing process. To this end a MR formula of the area constraint has been developed that allows to compute the area of a curve in terms of the coefficients at any resolution level $L$.

The area (see sect. 5.1) of a multiresolution curve can now be evaluated at any level of resolution $L$ in terms of the bilinear equation

$$2\mathcal{A} = (\mathbf{X}^L) \left[ \; M^L \; \right] (\mathbf{Y}^{\,L})^T \,, \quad \forall L \in \{0, \ldots, n\},$$

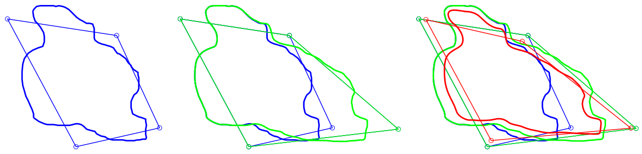


**Fig. 6.** area preserving multiresolution deformation. (a) original curve and coarse control polygon, $n = 7, L = 2$. (b) deformed curve without area constraint (green). (c) deformed area preserving curve (red).

where $X^L$ and $Y^L$ are the line vectors of the x- and y-coordinates resp. of all coarse and wavelet coefficients of the MR representation of the curve, i.e.

$$\begin{pmatrix} \mathbf{X}^L \\ \mathbf{Y}^L \end{pmatrix} = (\mathbf{x}^L, \mathbf{d}^L, \mathbf{d}^{L+1}, \ldots, \mathbf{d}^{n-1}),$$

and

$$M^L = \begin{bmatrix} I(\varphi^L, \varphi^L) & I(\varphi^L, \psi^l)_{l=L}^{n-1} \\ I(\psi^k, \varphi^L)_{k=L}^{n-1} & I(\psi^k, \psi^l)_{k,l=L}^{n-1} \end{bmatrix}$$

Note that $\varphi^L$ and $\psi^k$ are vector notations. Therefore the elements of the previous area-matrix are in fact bloc matrices whose elements are of type $I(\varphi_i, \psi_j) = \oint \varphi_i(t)\psi_j'(t) - \varphi_i'(t)\psi_j(t)dt$ and whose sizes vary in function of the resolution level $L$. It has been shown in [HBS02] that the area matrices $M^L$ can be computed efficiently by recursively applying the refinement equations.

The *area preserving editing process* now works as follows: Let $\mathcal{A}_{ref}$ be the refernce area to be preserved. After choosing the **decomposition** level $L$, the user modifies one or more coarse control points (fig. 6(b)), defining the desired **deformation**. Let $(X_0, Y_0)^T$ denote the coefficient vectors of the deformaed MR curve at level $L$. The algorithm then computes new positions, denoted by $(X, Y)^T$, of the coarse control points (and possibly the detail coefficients) such that they are as close as possible to the user defined deformation while **preserving the area** $\mathcal{A}_{ref}$, see fig. 6(c). The last step remains to solve the following min-max problem:

$$\max_{\lambda} \min_{X,Y} \quad (|X - X_0|^2 + |Y - Y_0|^2) + \lambda(XMY^T - 2\mathcal{A}_{ref})$$

If only local area preserving deformations are desired, the degrees of freedom in $(X, Y)^T$ can be reduced to a user-defined subset of control points. Figure 7 shows an example, where the upper left coarse control points has been kept fixed during deformation and area preservation.

*Length preserving deformation of a MR curve.*

Deformation of curves with constant length is needed typically if one wants to create wiggles or folding of a curve. Sauvage et al. [SHB03] developed a multiresolution approach of length preserving curve deformation for the

**Fig. 7.** Local area preserving multiresolution deformation. $n = 7$, $L = 2$. original curve (blue), deformed curve at level 2 (green), area preserving deformed curve at level 2 (red).

particular case of piecewise linear curves using the Lazy wavelets [Swe97]. Let $c(t)$ be a polyline of control points $c_i^n$. Coarse coefficients and wavelet coefficients are then computed by

$$\begin{cases} x_i^j = x_{2i}^{j+1} \\ d_i^j = x_{2i+1}^{j+1} - \frac{1}{2}(x_{2i}^{j+1} + x_{2i+2}^{j+1}) \ . \end{cases}$$

In the case of polylines the legth is given by $L = \sum_{i=0}^{N-2} ||c_{i+1}^n - c_i^n||_2$. One can either keep the total length constant or preserve the length of each segment. We choose the second way because of two main reasons:

- It ensures the balance between segment's length that is to say the control points don't gather in a small part of the curve.
- It allows the length constraints to be expressed in such a way that computationally inefficient square roots evaluations can be avoided.

The length constraint being a non-linear functional has no multiresolution representation as the area constraint. However in [SHB03] it is shown that length preserving MR curve editing offers a direct control of wrinkle generation. The level of resolution $L$ where the length adaptation is performed has two advantages. First, wrinkles can be generated locally on a user defind extend, and magnitude and frequency of the wrinkles can be controlled.

The algorithm works in two steps. Once the user has defined the deformation by modifying some coarse control points at an arbitrary scale, he fixes the level of resolution $L$ where he wants the length preserving being done. In other words, with $L$ he chooses the extend and frequency of wrinkle creation. Following some geometric rules, the detail coefficients of the deformed curve belonging to level $L + 1$ are then modified in order to make the control polygon at level $L + 1$ having the same length as the level $L + 1$ control polygon of the initial curve. The second step of the algorithm consists then of length preserving by smoothing via an optimization method and precisely satisfies the length constraint.

### 5.3 Variational MR curves

The variational modeling paradigm is used in order to find the "best" curve or surface amongst all solutions that meet the constraints. The constraints may result from the particular modeling technique used, for example sample point

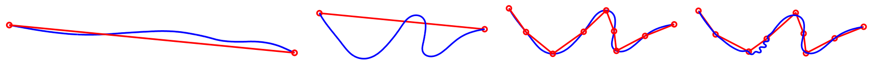


**Fig. 8.** Length preserving MR editing: Two successive deformations at different levels of decomposition are shown. The initial curve (a) is edited at the coarsest level. (b) its length is adapted at the scale $L = 1$ resulting in large wrinkles. (c), (d) 2 neighbouring control points are moved closer at the scale 3 and length preserving at scale $L = 6$ creates small wrinkles.

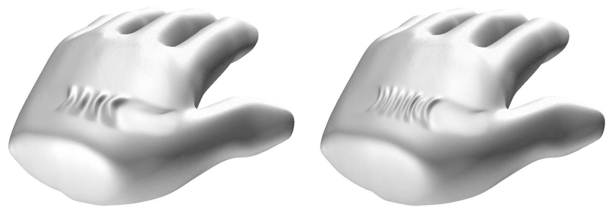


**Fig. 9.** Wrinkles on the back of the hand by length preserving MR deformation. The scheme has been applied on several lines of a triangular mesh modeling a hand. It creates wrinkles at the back of the hand automatically by pinching the skin. The skin is also stretched around the wrinkles. The model is purely geometric, no time consuming physical simulation is used.

approximation, or direct curve manipulation (see sect. 2.1. In the context of smooth curve and surface design the notion of "best" is formulated by minimizing some energy functional, see sect. 4.1.

Gortler and Cohen [GC95] show how the variational constraint, which generalizes least squares, can be solved through a MR formulation of a planar curve. A wavelet based MR curve satisfying some linear contraints and minimizing a linearized bending energy functional may be found by solving the following linear system [WW92]

$$\begin{bmatrix} \bar{H} & \bar{A}^T \\ \bar{A} & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix},$$

where $\bar{A}$ is the constraint matrix, $\bar{H}$ is the Hessian matrix of the basis functions, and $\lambda$ is the vector of Langrange multiplyers. The bars signify that the variables are wavelet coefficients. Gortler and Cohen show then how wavelets allow to accelerate the iterative conjugent-gradient-solving of the variational problem.

### 5.4 Multiresolution Subdivision methods with constraints

Subdivision has become a popular tool in computer graphics. Much literature derives and analyzes new subdivision algorithms for curve, surfaces and solids. For an overview see the SIGGRAPH 2000 course notes [ZS00] and the textbook [WW01]. Subdivision curves and surfaces are intrinsically hierarchical. Different levels of subdivision of a coarse mesh provide different levels of resolution. Constrained modeling techniques can then interact with different subdivision levels in order to obtain particular local design effects.

MacCracken and Joy [MJ96] developed an extension of Catmull-Clark subdivision surfaces to the volumetric setting, mainly for the purpose of freeform deformation in 3D space. Qin et al. introduced dynamic Catmull-Clark subdivision surfaces [QMV98]. McDonnell and Qin [MQ00] simulate volumetric subdivision objects using a mass-spring model. A generalization of McDonnell et al. [MQW01] includes haptic interaction. Capell et al. [CGC02] use the subdivision hierarchies to construct a hierarchical basis to represent displacements of a solid model for dynamic deformations. Additionally, some linear constraints, such as point displacements can be added at any level of subdivision.

Variational subdivision is another modeling technique, where constraints are combined with classical subdivision. Instead of applying explicit rules for the new vertices, Kobbelt's [Kob96] variational subdivision scheme computes the new vertices such that a fairness functional is minimized. At each step a linear system has to be solved. The resulting curves have minimal total curvature. Furthermore, in [KS98] is shown how wavelets can be constructed by using the Lifting Scheme [Swe97] which are appropriate for variational subdivision curves. Weimer and Warren [WW98a], [WW98b], [WW99] developed variational subdivision schemes that satisfy partial differential equations, for instance, fluid or thin-plate equations.

## 6 Conclusion

In this paper geometric modeling techniques have been surveyed that all make use of constraints of different nature in order to provide high-level user friendly manipulation tools of geometric objects. Basic research, developing new curve and surface representation, is going on and new deformation and editing tools have to be invented. For example, it is still a challenge to develop modeling tool for subdivision surface equivalent to those existing for NURBS surfaces.

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