The Bisector Surface of Rational Space Curves *

Gershon Elber
Department of Computer Science
Technion, Israel Institute of Technology
Haifa 32000, Israel
Email: gershon@cs.technion.ac.il

and

Myung-Soo Kim
Department of Computer Science
POSTECH
Pohang 790-784, South Korea
Email: mskim@postech.ac.kr

Abstract

Given a point and a rational curve in the plane, their bisector curve is rational [2]. However, in general, the bisector of two rational curves in the plane is not rational [3]. Given a point and a rational *space* curve, this paper shows that the bisector surface is a rational ruled surface. Moreover, given two rational space curves, we show that the bisector surface is rational (except for the degenerate case in which the two curves are coplanar).

Key Words: Bisector, Voronoi surface, medial surface, skeleton.

*This research was supported in part by the Korean Ministry of Science and Technology under Grants 96-NS-01-05-A-02-A and 96-NS-01-05-A-02-B of STEP 2000, by Korea Science and Engineering Foundation (KOSEF) under Grant 96-0100-01-01-2, and by the Electronics and Telecommunications Research Institute (ETRI) of Korea.

1 Introduction

Given two objects in the plane or space, their bisector curve or surface is defined as the set of points which are equidistant from the two objects. The medial axis and medial surface are also closely related to the bisector curve and surface; that is, given an object in the plane or space, the medial axis or surface is defined as the set of interior points of the object which have minimum distance from at least two different points on the boundary of the object.

Bisector surfaces and medial surfaces have many important applications in engineering [1, 5, 6, 8, 9]. (See Sherbrooke et al. [9] for a detailed survey.) However, their construction is non-trivial except for some special cases. Dutta and Hoffmann [1] considered the bisectors for simple surfaces such as natural quadrics and toroidal surfaces. For these special types of surfaces, the bisector surfaces have simple closed-form representations. For general algebraic surfaces, Hoffmann et al. [5, 6] formulated the bisector surface (called the Voronoi surface) using a simultaneous system of non-linear polynomial equations. The solution scheme is based on the *dimensionality paradigm* [5], which requires a preprocessing step that determines the global topological structure of the solution space.

Sheehy et al. [8] determine the topology of medial vertices, edges, and faces of a solid by computing the domain Delaunay triangulation of a relatively sparse distribution of object boundary points. Each cell of the triangulation connects different parts of the object boundary. A cell with four corners, each from a different boundary surface patch, is supposed to contain a medial vertex. The cells for medial edges and faces can also be classified in a similar way. More precise approximation for vertex, edge, and face can be computed by using the dimensionality paradigm [5].

Approximation of bisectors is computationally quite expensive. Therefore, it is desirable to classify special cases in which the bisectors have exact representations or simple approximations. Farouki and Johnstone [2, 3] investigated the bisector problem for planar rational curves. Given a point and a rational curve in the plane, the bisector curve is rational [2]. However, in general, the bisector of two polynomial or rational curves in the plane is algebraic, but not rational [3]. In practice, numerical tracing techniques are required to approximate the bisector curve [4]. This paper considers the bisector problem for space curves and shows that the added extra dimension of the 3D space actually alleviates the level of difficulty! That is, given a point and a rational space curve, their bisector is a rational ruled surface. Moreover, given two rational space curves, their bisector surface is rational (except for the two–dimensional degenerate case in which the two curves are coplanar).

The basic idea of our approach is as follows. Given two C^1-continuous space curves, $C_1(s)$ and $C_2(t)$, we consider necessary conditions for two points $C_1(s_0)$ and $C_2(t_0)$ to generate a bisector point P:

1. Point P must be located in the normal plane $L_1(s_0)$ of $C_1(s)$ at $C_1(s_0)$
2. Similarly, point P must also be located in the normal plane $L_2(t_0)$ of $C_2(t)$ at $C_2(t_0)$.
3. Moreover, point P is located in the bisector plane $L_{12}(s_0, t_0)$ which is the set of equidistant points from $C_1(s_0)$ and $C_2(t_0)$.

When the three planes: $L_1(s_0)$, $L_2(t_0)$, and $L_{12}(s_0, t_0)$, are in general position (i.e., no two of them are parallel to each other), there exists a unique intersection point P of the three planes.

When the curves $C_1(s)$ and $C_2(t)$ are rational space curves, the planes $L_1(s)$, $L_2(t)$, and $L_{12}(s, t)$ can be represented as implicit equations (of x, y, z) with rational coefficients in s and t:

$$\begin{aligned} L_1(s): &\quad a_1(s)x + b_1(s)y + c_1(s)z = d_1(s) \\ L_2(t): &\quad a_2(t)x + b_2(t)y + c_2(t)z = d_2(t) \\ L_{12}(s,t): &\quad a_{12}(s,t)x + b_{12}(s,t)y + c_{12}(s,t)z = d_{12}(s,t), \end{aligned}$$

where all the coefficients are rational functions of s and t. Based on Cramer's rule, it is quite straightforward to show that x, y, z are rational functions of these coefficients; therefore, they are rational functions of s and t. As a result, we can represent the bisector surface $P(s, t)$ as a rational surface of s and t.

When one of the two curves (say, $C_1(s)$) degenerates to a single point Q, the orthogonal plane $L_1(s)$ is not defined. Let $L_{12}(t)$ denote the bisector plane between Q and $C_2(t)$, then we have the following plane equations of $L_2(t)$ and $L_{12}(t)$:

$$\begin{aligned} L_2(t): &\quad a_2(t)x + b_2(t)y + c_2(t)z = d_2(t) \\ L_{12}(t): &\quad a_{12}(t)x + b_{12}(t)y + c_{12}(t)z = d_{12}(t), \end{aligned}$$

where all the coefficients are rational functions of t. The common solution of the above two equations is the intersection line of two planes: $L_2(t)$ and $L_{12}(t)$. That is, each parameter t contributes a line to the bisector surface; thus the bisector surface becomes a ruled surface. The direction $N(t)$ of each ruling line is given by the cross product of the normals of $L_2(t)$ and $L_{12}(t)$:

$$N(t) = (a_2(t), b_2(t), c_2(t)) \times (a_{12}(t), b_{12}(t), c_{12}(t)),$$

which is rational. To represent the ruled bisector surface as a rational surface, we need to construct a rational directrix curve on the surface. Let $L_1(t)$ be the plane which passes through the given point Q and is orthogonal to the ruled direction $N(t)$ at t:

$$L_1(t): \qquad a_1(t)x + b_1(t)y + c_1(t)z = d_1(t),$$

where $N(t) = (a_1(t), b_1(t), c_1(t))$ and $d_1(t) = \langle N(t), Q \rangle$. All the coefficients of $L_1(t)$, $L_2(t)$, and $L_{12}(t)$ are rational; therefore, their common intersection point is also rational in t. The trace of these intersection points generates a rational directrix curve on the ruled bisector surface. Since the ruling direction curve $N(t)$ is also rational, the bisector surface is a rational ruled surface.

Given two rational space curves, the existence of a rational bisector surface has great potential in surface design as well as in conventional engineering applications of medial surfaces. It is easy to control the geometric shape of a (bisector) surface by changing the shape and orientation of the two base curves or a ruled (bisector) surface with the base curve and base point. In a sense, the design of a bisector surface requires a similar amount of work for the design of sweep surfaces such as surface of revolution or surface of extrusion. Further research is required to investigate the representational power of the bisector surface of two space curves. All the figures in this paper were created using tools implemented as part of the IRIT [7] solid modeling system, developed at the Technion, Israel.

This paper is organized as follows. In Section 2, given two rational space curves, we construct the bisector surface as a rational surface. In Section 3, given a point and a rational space curve, we construct the bisector surface as a rational ruled surface. Finally, in Section 4, we conclude the paper.

2 Bisector Surface of Two Space Curves

In this section, given two rational space curves, we construct their bisector as a rational surface. Section 2.1 derives the rational representation of the bisector surface. Section 2.2 provides some examples of rational bisector surfaces of two space curves. In Section 2.3, we classify the degenerate cases in which the bisector surface may not have rational representation.

2.1 Rational Representation of Bisector Surface

Let $C_1(s) = (x_1(s), y_1(s), z_1(s))$ and $C_2(t) = (x_2(t), y_2(t), z_2(t))$ be two regular parametric C^1-continuous space curves. The regularity and C^1-continuity conditions imply that the tangent vectors $T_1(s) = (x_1'(s), y_1'(s), z_1'(s))$ and $T_2(t) = (x_2'(t), y_2'(t), z_2'(t))$ are non-zero continuous vector fields. (Note that $T_1(s)$ and $T_2(t)$ are non-unit vectors, in general.) Let $L_1(s_0)$ denote the normal plane of $C_1(s)$ at $C_1(s_0)$, which contains $C_1(s_0)$ and is orthogonal to $T_1(s_0)$; the normal plane $L_2(t_0)$ is defined in a similar way. Then, due to the continuity of $T_1(s)$ and $T_2(t)$, the normal planes $L_1(s)$ and $L_2(t)$ are well-defined and move continuously along the curves $C_1(s)$ and $C_2(t)$, respectively.

When a point P is on the bisector of two curves, there exist (at least) two points $C_1(s)$ and $C_2(t)$ such that point P is simultaneously contained in the normal planes $L_1(s)$ and $L_2(t)$. As a result, the point P satisfies the following two linear equations:

$$L_1(s): \quad \langle P - C_1(s), T_1(s) \rangle = 0, \tag{1}$$

$$L_2(t): \quad \langle P - C_2(t), T_2(t) \rangle = 0. \tag{2}$$

Moreover, point P is also contained in the bisector plane $L_{12}(s,t)$ between the two points $C_1(s)$ and $C_2(t)$. The plane $L_{12}(s,t)$ is orthogonal to the vector $C_1(s) - C_2(t)$ and passes through the mid point $\frac{C_1(s)+C_2(t)}{2}$ of $C_1(s)$ and $C_2(t)$. Therefore, the bisector plane $L_{12}(s,t)$ is defined by the following linear equation:

$$L_{12}(s,t): \quad \left\langle P - \frac{C_1(s) + C_2(t)}{2}, C_1(s) - C_2(t) \right\rangle = 0. \tag{3}$$

Any orthogonal bisector point $P \in B_o(C_1, C_2)$ must be a common intersection point of the three planes of $L_1(s)$, $L_2(t)$, and $L_{12}(s,t)$, for some s and t. Therefore, the point P can be computed by solving the following simultaneous linear equations in P:

$$\begin{aligned} \langle P, T_1(s) \rangle &= \langle C_1(s), T_1(s) \rangle, \\ \langle P, T_2(t) \rangle &= \langle C_2(t), T_2(t) \rangle, \\ \langle P, C_1(s) - C_2(t) \rangle &= \frac{\|C_1(s)\|^2 - \|C_2(t)\|^2}{2}. \end{aligned} \tag{4}$$

Then, we have the following matrix equation,

$$\begin{bmatrix} x_1'(s) & y_1'(s) & z_1'(s) \\ x_2'(t) & y_2'(t) & z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} d_1(s) \\ d_2(t) \\ m(s,t) \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned}
P &= (p_x, p_y, p_z), \\
d_1(s) &= \langle C_1(s), T_1(s) \rangle, \\
&= x_1(s)x_1'(s) + y_1(s)y_1'(s) + z_1(s)z_1'(s), \\
d_2(t) &= \langle C_2(t), T_2(t) \rangle, \\
&= x_2(t)x_2'(t) + y_2(t)y_2'(t) + z_2(t)z_2'(t), \\
m(s,t) &= \frac{\|C_1(s)\|^2 - \|C_2(t)\|^2}{2}, \\
(x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) &= (x_1(s) - x_2(t), y_1(s) - y_2(t), z_1(s) - z_2(t)).
\end{aligned}$$

By Cramer's rule, Equation (5) can be solved as follows:

$$\begin{aligned}
p_x &= \frac{\begin{vmatrix} d_1(s) & y_1'(s) & z_1'(s) \\ d_2(t) & y_2'(t) & z_2'(t) \\ m(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & z_1'(s) \\ x_2'(t) & y_2'(t) & z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{vmatrix}}, \\
p_y &= \frac{\begin{vmatrix} x_1'(s) & d_1(s) & z_1'(s) \\ x_2'(t) & d_2(t) & z_2'(t) \\ x_{12}(s,t) & m(s,t) & z_{12}(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & z_1'(s) \\ x_2'(t) & y_2'(t) & z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{vmatrix}}, \\
p_z &= \frac{\begin{vmatrix} x_1'(s) & y_1'(s) & d_1(s) \\ x_2'(t) & y_2'(t) & d_2(t) \\ x_{12}(s,t) & y_{12}(s,t) & m(s,t) \end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & z_1'(s) \\ x_2'(t) & y_2'(t) & z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{vmatrix}}.
\end{aligned} \tag{6}$$

The bisector surface $P(s,t) = (p_x(s,t), p_y(s,t), p_z(s,t))$ has a simple rational representation as long as the common denominator of p_x, p_y, and p_z in the above equation does not vanish. That is, the tangent vectors $T_1(s)$, $T_2(t)$, and the difference vector $C_1(s) - C_2(t)$ must be linearly independent so that the coordinate functions $p_x(s,t)$, $p_y(s,t)$, and $p_z(s,t)$ are well-defined and rational. In Section 2.3, we show that the degenerate cases essentially reduce to the special case in which the two curves $C_1(s)$ and $C_2(t)$ are coplanar.

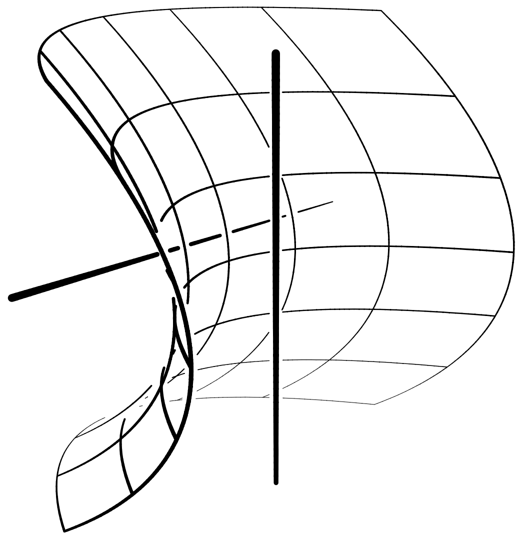


Figure 1: The bisector surface of two orthogonal lines in the 3D space.

2.2 Examples of Rational Bisector Surfaces

Figure 1 shows an example of the bisector surface of two non-intersecting orthogonal straight lines in the 3D space. In this example, the two base curves $C_1(s)$ and $C_2(t)$ have simple curve representations:

$$C_1(s) = (1, s, 0) \quad \text{and} \quad C_2(t) = (0, 0, t). \tag{7}$$

Therefore, we have

$$\begin{aligned} d_1(s) &= s, \\ d_2(t) &= t, \\ m(s,t) &= \frac{1 + s^2 - t^2}{2}, \\ (x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) &= (1, s, -t). \end{aligned}$$

Consequently, the bisector surface $P(s,t)$ has the following simple rational (in fact polynomial) representation:

$$P(s,t) = \left(\frac{1 - s^2 + t^2}{2}, s, t \right).$$

In general, the bisector surface of any two non-intersecting skew lines in the 3D space is a hyperbolic paraboloid [1]. Using the bisector surface of two lines, we can also compute the bisector surface of two cylinders of the same radius.

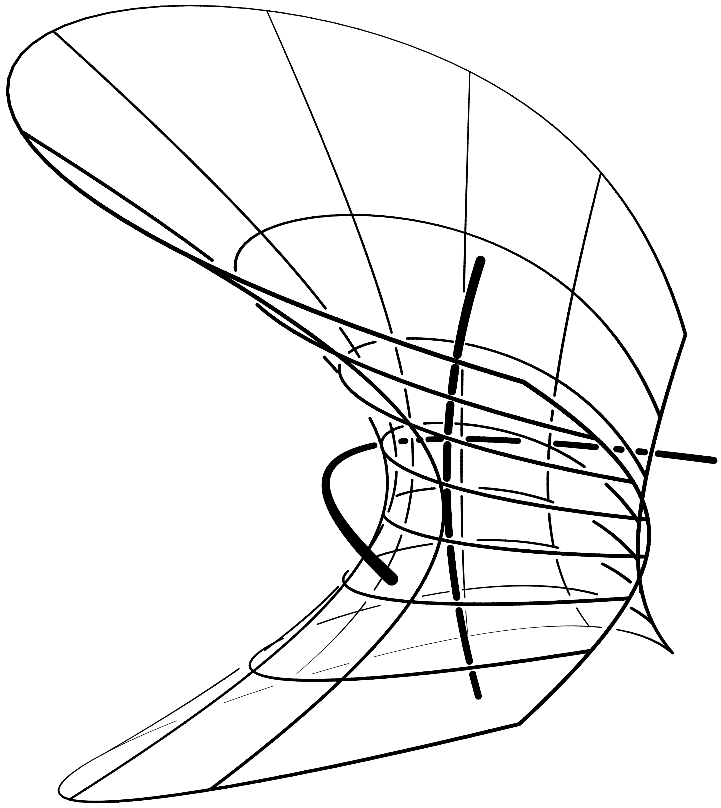


Figure 2: A bisector surface of two quadratic Bézier curves in the 3D space.

In Figure 2, two quadratic Bézier curves are given in the 3D space. Each curve is a planar curve (since every quadratic curve must be planar). However, they are not coplanar. The resulting bisector is a rational surface of degree (6,6).

Figure 3 (a) shows the bisector surface of the unit circle $C_1(s)$ in the xy-plane and the line $C_2(t)$ on the z-axis:

$$C_1(s) = \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}, 0\right), \tag{8}$$

$$C_2(t) = (0, 0, t). \tag{9}$$

In this example, we have

$$\begin{aligned} d_1(s) &= 0, \\ d_2(t) &= t, \\ m(s,t) &= \frac{1-t^2}{2}, \\ (x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) &= \left(\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}, -t\right). \end{aligned}$$

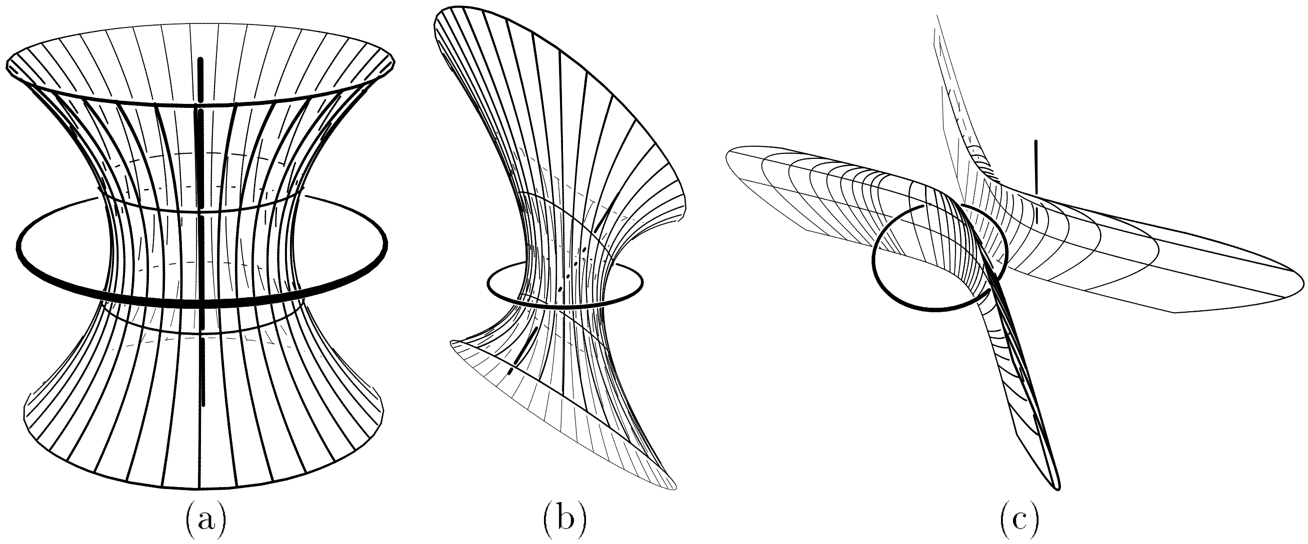


Figure 3: Three examples of the bisector surface of a line and a circle.

Consequently, the bisector surface $P(s,t)$ has the following rational representation:

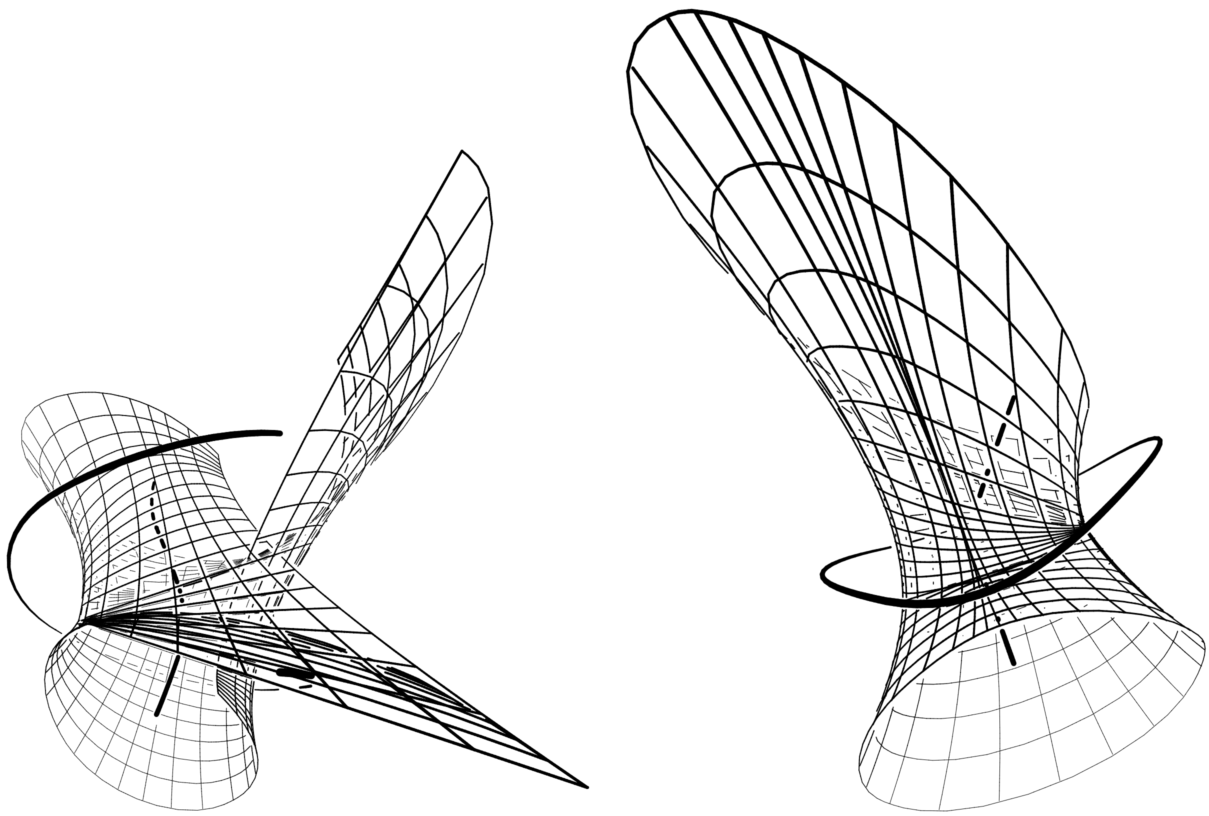
$$P(s,t) \;=\; \left(\frac{(1-s^2)(1+t^2)}{2(1+s^2)}, \frac{s(1+t^2)}{(1+s^2)}, t \right).$$

Figures 3(b) and 3(c) also generate similar rational surfaces of degree (3,9). Using the bisector surface of a line and a circle, we can compute the bisector surface of a cylinder and a torus with the same minor radius. In general, the bisector surface of two space curves can be used to compute the bisector surface of two canal surfaces which are generated by sweeping two spheres of the same fixed radii along the two space curves. The bisector surface in Figure 3 (c) has *poles*. In other words, the denominator of the rational bisector surface vanishes to zero for some values of s and t, creating *multiple sheets* that meet at infinity. The bisector surface presented in Figure 3 (c) is hence only a finite subset of the real infinite bisector.

Figure 4 shows two more examples of the bisector surface of two non-planar curves. Figure 5 shows another set of two examples of bisector surfaces, this time rendered as (transparent) shaded surfaces.

2.3 Degenerate Cases

When the two curves $C_1(s)$ and $C_2(t)$ are in the same plane, the three vectors $T_1(s)$, $T_2(t)$, and $C_1(s) - C_2(t)$ are always linearly dependent; therefore, Equation (6) has no solution since the



(a) (b)

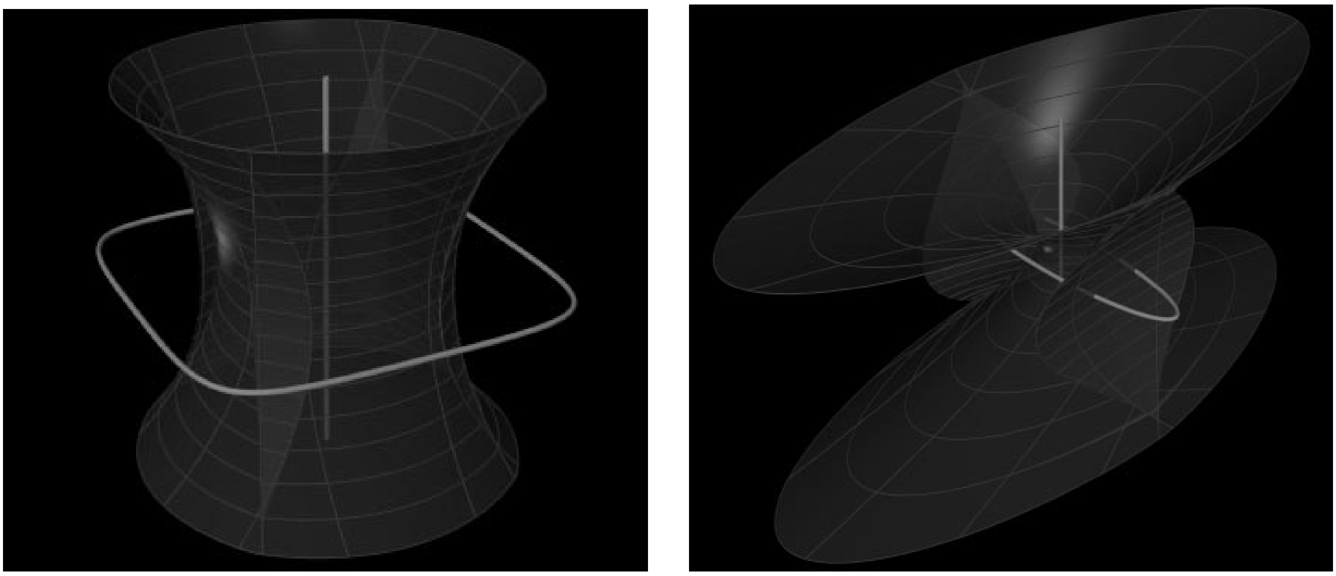
Figure 4: Two examples of the bisector surface of non-planar curves. In (a), the bisector surface of two 360° arcs of an approximation of a helix is presented. In (b), the bisector of an open quadratic and a periodic B-spline space curve is presented.

common denominator vanishes identically:

$$D(s,t) = \begin{vmatrix} x_1'(s) & y_1'(s) & z_1'(s) \\ x_2'(t) & y_2'(t) & z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & z_{12}(s,t) \end{vmatrix} \equiv 0. \tag{10}$$

Given two rational space curves $C_1(s)$ and $C_2(t)$, the determinant $D(s,t)$ is a rational function of s and t. Thus the common denominator of Equation (6) vanishes for all (s,t) on the algebraic curve $D(s,t) = 0$ in the st-parameter space. As discussed above, when the two curves are coplanar, the solution space of the algebraic curve $D(s,t) = 0$ degenerates to the whole st-parameter space. In the following discussion, we show that this coplanar case is the only case in which the determinant vanishes identically: $D(s,t) \equiv 0$.

For a fixed s_0, we first consider the cases in which $D(s_0,t) \equiv 0$. Since the curve $C_2(t)$ is rational, the determinant $D(s_0,t)$ is a rational function of t. Therefore, the determinant is identically zero or has at most finitely many solutions of t. We classify the degenerate cases in which the determinant



(a) (b)

Figure 5: Two examples of shaded (transparent) bisector surfaces. In (a), the bisector surface is that of a straight line and a rounded square. In (b), the bisector surface is that of a straight line and an ellipse.

vanishes identically for all t. After that, by making the parameter s_0 change arbitrarily (so that we have $D(s,t) \equiv 0$ for all s and t), we show how each degenerate case reduces to a case in which the two curves $C_1(s)$ and $C_2(t)$ are coplanar.

Case I: $T_1(s_0)$ and $T_2(t)$ are Parallel

For a fixed tangent vector $T_1(s_0)$, we consider the case in which the tangent vector $T_2(t)$ is linearly dependent on $T_1(s_0)$ for all t. The regularity of $C_1(s)$ and $C_2(t)$ implies that $T_1(s_0) \neq 0$ and $T_2(t) \neq 0$. Therefore, the two vectors $T_1(s_0)$ and $T_2(t)$ are parallel to each other for all t, and the curve $C_2(t)$ must be a straight line which is parallel to the vector $T_1(s_0)$. If the same condition holds for all other values of s, the tangent vector field $T_1(s)$ is parallel to the line $C_2(t)$; therefore, the curve $C_1(s)$ must be a straight line which is parallel to the line $C_2(t)$. The two parallel lines determine a unique plane. Consequently, the two curves $C_1(s)$ and $C_2(t)$ are coplanar.

Case II: $C_1(s_0) - C_2(t)$ **and** $T_2(t)$ **are parallel**

We assume that $T_1(s_0)$ and $T_2(t)$ are linearly independent, and consider the case in which the three vectors $T_1(s_0)$, $T_2(t)$, and $C_1(s_0) - C_2(t)$ are linearly dependent. We assume that $C_1(s_0) - C_2(t) \neq 0$, for all t, and consider the case in which the two vectors $C_1(s_0) - C_2(t)$ and $T_2(t)$ are linearly dependent. (Case III will consider the case in which $C_1(s_0) - C_2(t)$ and $T_2(t)$ are linearly independent.) Since the regularity of $C_2(t)$ implies $T_2(t) \neq 0$, we have

$$C_1(s_0) - C_2(t) = \alpha(t)T_2(t), \tag{11}$$

for some $\alpha(t) \in I\!R$. That is,

$$C_1(s_0) = C_2(t) + \alpha(t)T_2(t),$$

which means that the point $C_1(s_0)$ is in the tangent line of $C_2(t)$. If the relation holds for all t, the curve $C_2(t)$ must be a radial straight line emanating from the point $C_1(s_0)$. Moreover, if this relation holds for all other values of s, the curve $C_1(s)$ must be contained in the same straight line as $C_2(t)$. Therefore, the two curves are collinear.

Case III: $(C_1(s_0) - C_2(t)) \times T_2(t)$ **is orthogonal to** $T_1(s_0)$

We consider the case in which $C_1(s_0) - C_2(t)$ and $T_2(t)$ are linearly independent; that is, the case in which the two vectors $C_1(s_0) - C_2(t)$ and $T_2(t)$ span a plane, and equivalently, we have $(C_1(s_0) - C_2(t)) \times T_2(t) \neq 0$. When the three vectors $T_1(s_0)$, $T_2(t)$, and $C_1(s_0) - C_2(t)$ are linearly dependent, the vector $T_1(s_0)$ must be contained in the plane spanned by the two vectors $C_1(s_0) - C_2(t)$ and $T_2(t)$, and equivalently, we have the following relation:

$$\langle T_1(s_0), (C_1(s_0) - C_2(t)) \times T_2(t) \rangle = 0. \tag{12}$$

Consequently, the vector field $(C_1(s_0) - C_2(t)) \times T_2(t)$ is always orthogonal to the fixed vector $T_1(s_0)$. Note that the vector field $(C_1(s_0) - C_2(t)) \times T_2(t)$ is parallel (though in the opposite direction) to the normal vector of the conical surface $S(r,t)$ which is generated by connecting the origin and each point of the curve $C_1(s_0) - C_2(t)$ by a straight line:

$$S(r,t) = \{\ r(C_1(s_0) - C_2(t)) \mid r, t \in I\!R\ \}.$$

The surface normal of $S(r,t)$ is given by

$$\begin{aligned} \frac{\partial S}{\partial r}(r,t) \times \frac{\partial S}{\partial t}(r,t) &= (C_1(s_0) - C_2(t)) \times (-rT_2(t)) \\ &= -r(C_1(s_0) - C_2(t)) \times T_2(t). \end{aligned}$$

Therefore, the constraint: $\langle T_1(s_0), (C_1(s_0) - C_2(t)) \times T_2(t)\rangle = 0$ implies that the conical surface $S(r, t)$ must be a cylindrical surface or a plane (whose surface normal is always orthogonal to the vector $T_1(s_0)$). We consider the two special cases as follows:

1. In the cylindrical case, the apex at the origin is on each ruling of the cylindrical surface. Thus the surface $S(r, t)$ degenerates to a straight line which emanates from the origin. By taking $r = -1$, we realize that the curve $C_2(t) - C_1(s_0)$ is on the straight line which emanates from the origin. This means that the curve $C_2(t)$ is a straight line which contains the point $C_1(s_0)$. If this relation holds for all other values of s, the two curves $C_1(s)$ and $C_2(t)$ must be collinear lines.

2. The other possible case is when the curve $C_1(s_0) - C_2(t)$ is contained in a plane which passes through the origin. That is, the curve $C_2(t)$ is a planar curve and the point $C_1(s_0)$ is contained in the same plane. If this relation holds for all other values of s, the two curves $C_1(s)$ and $C_2(t)$ must be coplanar.

3 Bisector Surface of a Point and a Space Curve

3.1 Rational Representation of a Ruled Bisector Surface

In this section, given a point and a space curve, we construct their bisector as a rational ruled surface. Let $Q = (q_x, q_y, q_z)$ be a fixed point and $C(t) = (x(t), y(t), z(t))$ be a regular parametric C^1-continuous space curve. Then, $T(t) = (x'(t), y'(t), z'(t))$ is a non-zero continuous tangent vector field along $C(t)$.

Let P be a bisector point of $C(t)$ and Q with its foot points at $C(t_0)$ and Q, respectively. Then, the point P is contained in the normal plane $L(t_0)$ of $C(t)$ at $C(t_0)$ and also in the bisector plane $L_b(t_0)$ of the two points $C(t_0)$ and Q. The two planes $L(t_0)$ and $L_b(t_0)$ intersect in a line $l(t_0)$. Every point in the intersection line $l(t_0)$ has the same orthogonal foot point $C(t_0)$ on the curve $C(t)$. Since each point $C(t_0)$ contributes a line on the bisector surface, the overall bisector becomes a ruled surface. We now show that this surface is also rational.

Let $N(t)$ be the direction vector of the intersection line $l(t)$ between two planes $L(t)$ and $L_b(t)$. Since vector $N(t)$ is contained in both $L(t)$ and $L_b(t)$, it is orthogonal to the normal vectors of $L(t)$ and $L_b(t)$. Therefore, we can construct $N(t)$ by computing the cross product of the two normal vectors:

$$N(t) = T(t) \times (C(t) - Q),$$

which is a rational vector field.

To represent the bisector surface as a rational ruled surface, we also need to construct a rational directrix curve $P(t)$ on the bisector surface. A natural choice of $P(t)$ is the closest point of $l(t)$ to Q. Toward this goal, we construct an auxiliary plane $L_n(t)$ which passes through the fixed point Q and is orthogonal to the line $l(t)$ (i.e., the direction vector $N(t)$). The points P in the plane $L_n(t)$ satisfy the following linear equation:

$$L_n(t): \quad \langle P - Q, N(t) \rangle = 0,$$

which is rational in t. The common intersection point P of three planes: $L(t)$, $L_n(t)$, and $L_b(t)$ can be computed by solving the following simultaneous linear equations in P (see also Equation (4)):

$$\begin{aligned} \langle P, T(t) \rangle &= \langle C(t), T(t) \rangle, \\ \langle P, N(t) \rangle &= \langle Q, N(t) \rangle, \\ \langle P, C(t) - Q \rangle &= \frac{\|C(t)\|^2 - \|Q\|^2}{2}. \end{aligned} \tag{13}$$

Then, we have the following matrix equation,

$$\begin{bmatrix} x'(t) & y'(t) & z'(t) \\ x_n(t) & y_n(t) & z_n(t) \\ x_b(t) & y_b(t) & z_b(t) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} d_1(t) \\ d_2(t) \\ m(t) \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} P &= (p_x, p_y, p_z), \\ N(t) &= (x_n(t), y_n(t), z_n(t)), \\ d_1(t) &= \langle C(t), T(t) \rangle \;=\; x(t)x'(t) + y(t)y'(t) + z(t)z'(t), \\ d_2(t) &= \langle Q, N(t) \rangle \;=\; q_x x_n(t) + q_y y_n(t) + q_z z_n(t), \\ m(t) &= \frac{\|C(t)\|^2 - \|Q\|^2}{2}, \\ (x_b(t), y_b(t), z_b(t)) &= (x(t) - q_x, y(t) - q_y, z(t) - q_z). \end{aligned}$$

By Cramer's rule, Equation (14) can be solved as follows:

$$p_x = \frac{\begin{vmatrix} d_1(t) & y'(t) & z'(t) \\ d_2(t) & y_n(t) & z_n(t) \\ m(t) & y_b(t) & z_b(t) \end{vmatrix}}{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x_n(t) & y_n(t) & z_n(t) \\ x_b(t) & y_b(t) & z_b(t) \end{vmatrix}},$$

$$p_y = \frac{\begin{vmatrix} x'(t) & d_1(t) & z'(t) \\ x_n(t) & d_2(t) & z_n(t) \\ x_b(t) & m(t) & z_b(t) \end{vmatrix}}{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x_n(t) & y_n(t) & z_n(t) \\ x_b(t) & y_b(t) & z_b(t) \end{vmatrix}},$$

$$p_z = \frac{\begin{vmatrix} x'(t) & y'(t) & d_1(t) \\ x_n(t) & y_n(t) & d_2(t) \\ x_b(t) & y_b(t) & m(t) \end{vmatrix}}{\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x_n(t) & y_n(t) & z_n(t) \\ x_b(t) & y_b(t) & z_b(t) \end{vmatrix}}. \qquad (15)$$

Once the rational directrix curve $P(t)$ and the rational vector field $N(t)$ are obtained, the rational ruled bisector surface $P(s,t)$ can be constructed as follows:

$$P(s,t) = P(t) + sN(t), \quad \text{for } s,t \in I\!R.$$

The directrix curve $P(t) = (p_x(t), p_y(t), p_z(t))$ has a rational representation as long as the common denominator of p_x, p_y, and p_z in the above equation does not vanish. That is, the three vectors $T(t)$, $N(t)$, and $C(t) - Q$ must be linearly independent so that the coordinate functions $p_x(t)$, $p_y(t)$, and $p_z(t)$ are well-defined and rational. Since the vector $N(t)$ is the cross product of $T(t)$ and $C(t) - Q$, the three vectors are linearly independent as long as the two vectors $T(t)$ and $C(t) - Q$ are linearly independent.

We assume that $C(t) - Q \neq 0$ and consider the degenerate case in which the two vectors $T(t)$ and $C(t) - Q$ are linearly dependent. The regularity of $C(t)$ implies that $T(t) \neq 0$; therefore, the two vectors $Q - C(t)$ and $T(t)$ must be parallel to each other, and we have

$$Q - C(t) = \alpha(t)T(t),$$

for some $\alpha(t) \in I\!R$. Then, the condition: $Q = C(t) + \alpha(t)T(t)$ implies that the point Q is on the tangent line of $C(t)$ for all t. This means that the curve $C(t)$ is a straight line which emanates from Q. Therefore, the point Q and the curve $C(t)$ must be collinear.

3.2 Examples of Ruled Bisector Surfaces

Figures 6 and 7 show two examples of the bisector surface of a curve and a point. In Figure 6, a cubic planar Bézier curve is shown that is also coplanar with the given point, resulting in a ruled surface that degenerates into a cylindrical surface. (See Section 3.3 for more details.) In contrast, Figure 7 shows the bisector ruled surface of a general space curve and a point.

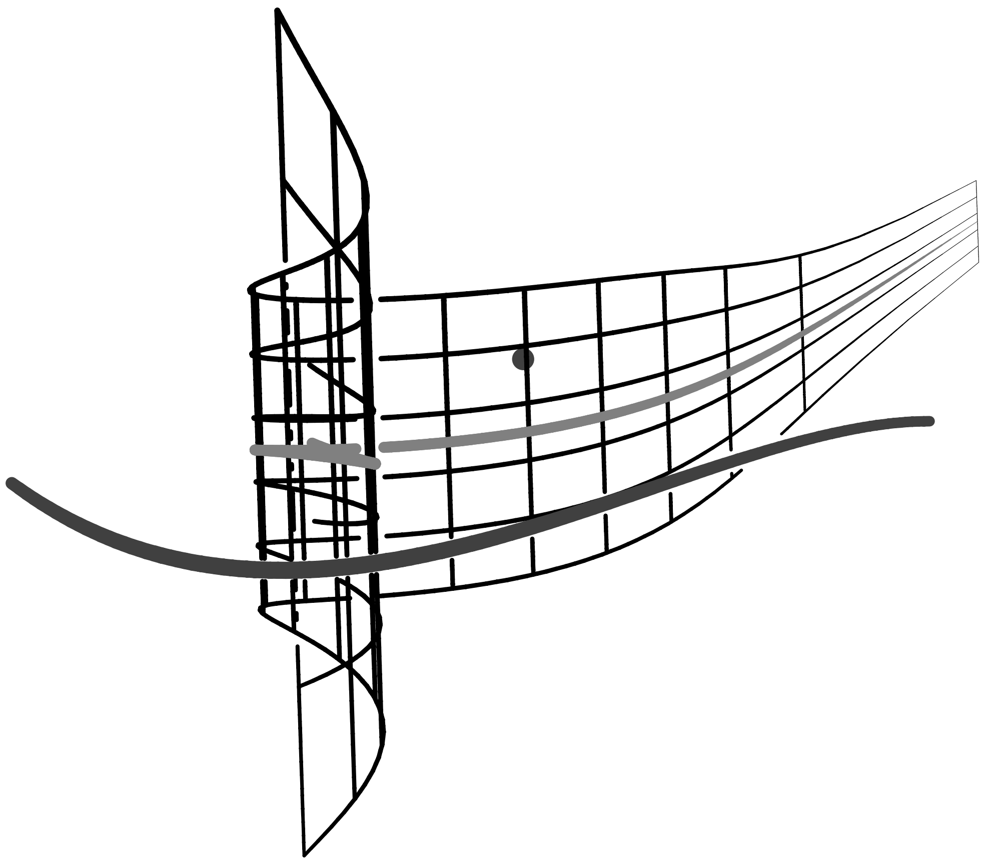


Figure 6: The bisector surface of a cubic Bézier planar curve and a coplanar point (both shown in gray) is a cylindrical surface. In light gray, the coplanar isoparametric curve of the bisector surface (which is contained in the same plane) is shown. This isoparametric curve is, in fact, the planar bisector curve of the planar curve and the point.

3.3 Bisector Curve of a Point and a Planar Curve

Given a point and a rational curve in the plane, Farouki and Johnstone [2] showed that their bisector curve is rational. When we consider the bisector surface of a point Q and a rational curve $C(t)$ in the xy-plane, the two planes $L(t)$ and $L_b(t)$ are always orthogonal to the xy-plane and the plane $L_n(t)$ is the same as the xy-plane. Therefore, the directrix curve $P(t)$ is the same as the rational bisector curve generated by Farouki and Johnstone [2]. Moreover, the ruling line $l(t)$ (which is the intersection of two planes: $l(t) = L(t) \cap L_b(t)$) is always orthogonal to the xy-plane. Therefore, the bisector surface $P(s,t)$ is a cylindrical surface in which $P(t)$ is its directrix curve and all the ruling lines are parallel to the z-axis. (See Figure 6 for such an example.) The cylindrical surface is a special case of developable surface. An interesting question is: “what are the special cases of the point-curve bisector problem in which the bisector surface is a developable surface ?” We leave this

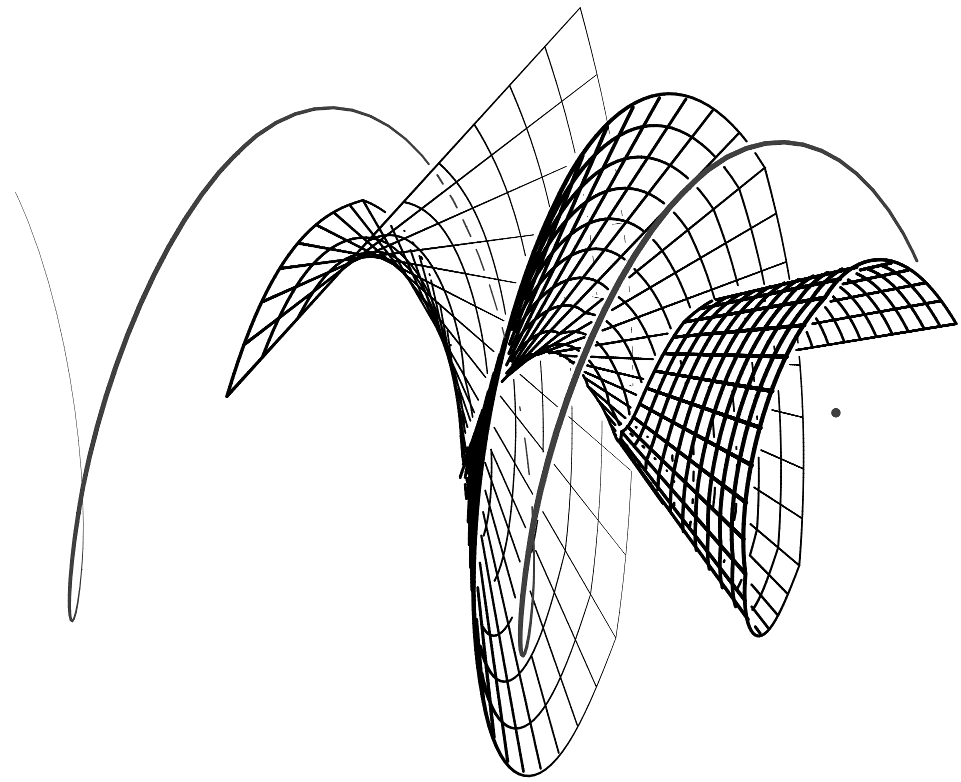
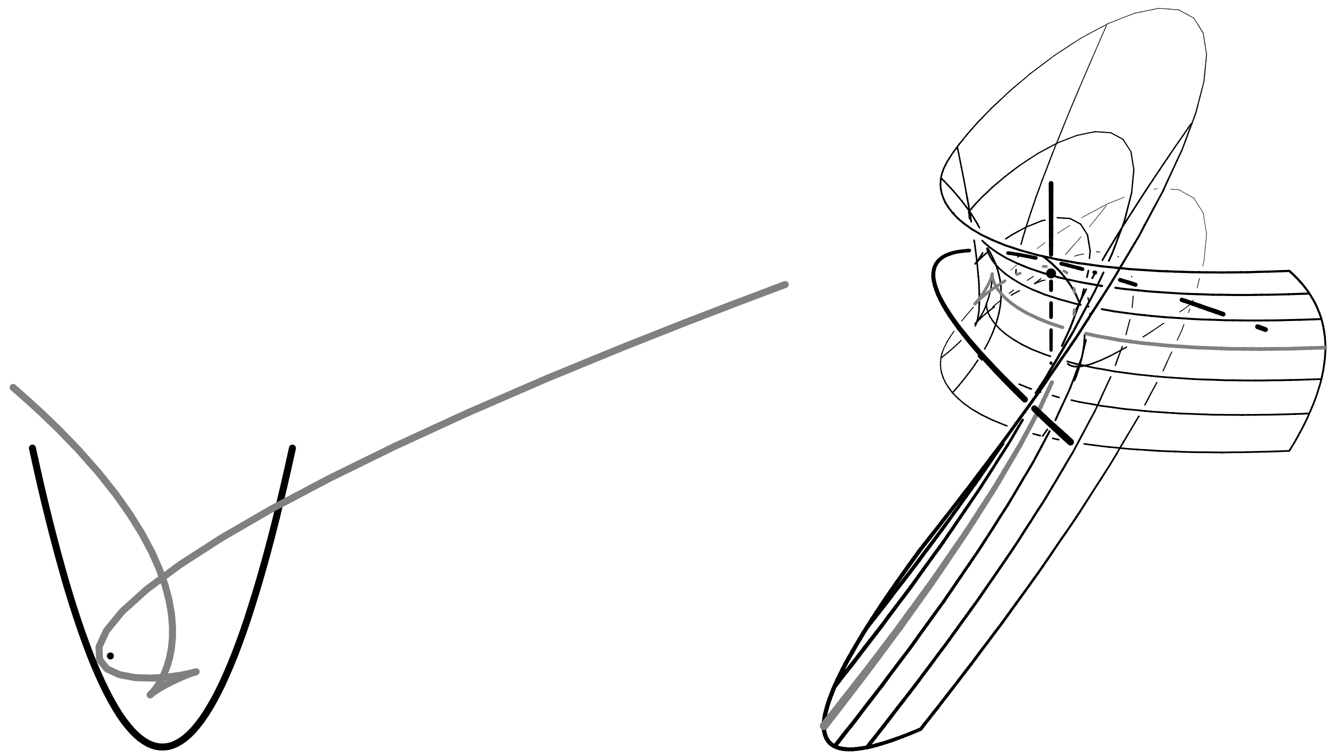


Figure 7: The bisector surface of a point and a helical curve (both shown in gray) is, in fact, a ruled surface.

question as an open problem that may hopefully be answered in the future.

Moreover, this case of a planar point-curve bisector is also a special case of the bisector surface of two space curves, in which one curve is given as a planar curve and the other curve is given an orthogonal line to the plane. Consider a planar curve $C_1(s)$ that is given in the xy-plane, and the line $C_2(t)$ that is parallel to the z-axis and also passes through a given point Q in the xy-plane. Let t_0 be the parameter value such that $Q = C_2(t_0)$. Then, on the bisector surface $P(s,t)$, the isoparametric curve $P(s,t_0)$ is the same as the bisector curve of $C_1(s)$ and Q in the xy-plane. Figure 8 shows such an example (see also Reference [2]).

The bisector of two planar rational curves is not rational, in general. However, we can approximate the bisector by rotating one planar curve by a certain small angle ϵ out of the plane (or both curves in the opposite directions), and computing the intersection of the bisector surface with the plane containing the two curves. Figure 9 shows such an example of this approach, where a rotation of degree $1°$ is applied to one planar curve. Another approach is to translate one planar curve along the normal direction of the plane by a small distance ϵ, compute the bisector surface, and finally



(a) A planar point-curve bisector.

(b) The bisector surface of the z-axis and a planar parabolic curve.

Figure 8: The special case of bisector for a planar curve vs. a point (shown in (a)) can be derived from the bisector surface of the planar curve and a line orthogonal to the plane, and considering only the proper isoparametric curve in the plane (in gray).

intersect the bisector surface with the given plane. The numerical stability of these approaches is yet to be evaluated.

4 Conclusions

In this paper, we have shown that the bisector of two rational space curves is a rational surface (except for the degenerate case in which the two curves are coplanar). Given a point and a rational space curve, the bisector surface is shown to be a rational ruled surface. Moreover, given a point and a rational planar curve in the same plane, the bisector is a rational cylindrical surface which is orthogonal to the plane.

All the examples in this paper were computed in a fraction of second on a 150MHz R4000 SGI Indy system, while using tools implemented on the IRIT [7] modeling system that is developed at the Technion, Israel. Hence, the capability of efficiently constructing a rational surface (respectively, a

Figure 9: The bisector (bold curves) of two planar curves (light curves) can be approximated by computing the intersection of the plane containing the two curves with the bisector surface of the two curves rotated out of the plane by ϵ degrees. In this example, we take $\epsilon =$ degree $1°$.

rational ruled surface), given a prescription of curves and/or points, has great potential in geometric modeling. Further work is required to investigate the representational power of rational bisectors (of points, curves, and surfaces) and the possibility of exploiting them in surface modeling. The geometric intuition in using two control curves may suggest the bisector surface as a useful surface construction scheme much like sweeping or extrusion operations. Based on the result of this paper, this new direction of surface design paradigm seems plausible not only for geometric modeling but also for geometric analysis, in general.

We are currently investigating the extendibility of this methodology to other rational varieties (including those in high-dimensional spaces) and the usage of rational bisectors in formulating, solving, and/or simplifying various geometric problems.

References

[1] Dutta, D., and Hoffmann, C., (1993), "On the Skeleton of Simple CSG Objects," *ASME J. of Mechanical Design*, Vol. 115, pp. 87–94.

[2] Farouki, R., and Johnstone, J., (1994), "The Bisector of a Point and a Plane Parametric Curve," *Computer Aided Geometric Design*, Vol. 11, No. 2, pp. 117–151.

[3] Farouki, R., and Johnstone, J., (1994), "Computing Point/Curve and Curve/Curve Bisectors," *Design and Application of Curves and Surfaces: Mathematics of Surfaces V*, R.B. Fisher (Ed.), Oxford University Press, pp. 327–354.

[4] Farouki, R., and Ramamurthy, R. "Specified-Precision Computation of Curve/Curve Bisectors," To appear in *International Journal of Computational Geometry & Applications*, (1997).

[5] Hoffmann, C., (1990), "A Dimensionality Paradigm for Surface Interrogations," *Computer Aided Geometric Design*, Vol. 7, No. 6, pp. 517–532.

[6] Hoffmann, C., and Vermeer, P., (1991), "Eliminating Extraneous Solutions in Curve and Surface Operations," *Int'l J. of Computational Geometry and Applications*, Vol. 1, No. 1, pp. 47–66.

[7] IRIT 6.0 User's Manual, February 1996, Technion. http://www.cs.technion.ac.il/~irit.

[8] Sheehy, D., Armstrong, C., and Robinson, D., (1996), "Shape Description by Medial Surface Construction," *IEEE Tans. Visualization and Computer Graphics*, Vol. 2, No. 1, pp. 42–72.

[9] Sherbrooke, E., Patrikalakis, N., and Brisson, E., (1996), "An Algorithm for the Medial Axis Transform of 3D Polyhedral Solids," *IEEE Tans. Visualization and Computer Graphics*, Vol. 2, No. 1, pp. 44–61.