$\boldsymbol{\alpha}$-Sectors of Rational Varieties*

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Abstract

In this paper we present the α-sector of two varieties in $\mathbb{R}^2$ or $\mathbb{R}^3$, which consists of all the points whose ratio of distances from the varieties is $\alpha : 1 - \alpha$ (for $0 \leq \alpha \leq 1$). This is a generalization of the bisector of varieties, which is their 0.5-sector. The α-sectors of rational varieties are nonrational in general. We therefore propose the pseudo α-sector which approximates the α-sector with a rational variety. Both the exact and the pseudo α-sectors reduce to the bisector when $\alpha = 1/2$.

1 Introduction

Given m different objects $O_1, \ldots, O_m$, the Voronoi region of an object O_i ($1 \leq i \leq m$) is defined as the set of points that are closer to the object O_i than to any other object O_j (for $j \neq i$). The boundary of each Voronoi region is composed of portions of *bisectors*, i.e., the set of points that are equidistant from two different objects O_i and O_j ($i \neq j$).

In this paper we extend the bisector to a more general concept, so-called the *α-sector*. Instead of taking an equal distance from two varieties, the α-sector allows different relative distances from the two varieties. Even in the simple case of the α-sector of a point and a line, we can generate all different types of conics, depending on the value of $0 < \alpha < 1$. Exact α-sectors of rational varieties are nonrational even in the special cases where the bisectors are rational. We thus propose the pseudo α-sectors—a rational approximation of the exact α-sectors. Both the exact and the pseudo α-sectors reduce to bisectors for $\alpha = \frac{1}{2}$.

Let V_1 and V_2 be two varieties, and let $\mathcal{B}_\alpha$ (hereafter denoted simply as $\mathcal{B}$) be the α-sector of V_1 and V_2.

The distance to the two varieties is measured along the normal spaces of the two varieties and hence two varieties of dimensions n and m will impose $n + m$ orthogonality constraints

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on their α-sector. For example, for a curve $C(t)$ $(n=1)$ and a surface $S(u,v)$ $(m=2)$, the following three constraints are imposed,

$$0 = \left\langle \frac{\partial S}{\partial u}, S - \mathcal{B} \right\rangle$$
$$0 = \left\langle \frac{\partial S}{\partial v}, S - \mathcal{B} \right\rangle$$
$$0 = \left\langle \frac{\partial C}{\partial t}, C - \mathcal{B} \right\rangle.$$

In addition, the α-sector must satisfy the distance ratios of,

$$\alpha \|\mathcal{B} - \mathcal{V}_1\| = (1-\alpha)\|\mathcal{B} - \mathcal{V}_2\|, \tag{1}$$

where $0 \leq \alpha \leq 1$. Unfortunately, the square of Equation (1) is linear in $\mathcal{B}$ only for $\alpha = \frac{1}{2}$, where the α-sector reduces to the bisector. Nevertheless, there is a nice property that the two special α-sectors, for $\alpha = 0$ and $\alpha = 1$, identify with the original varieties. The ability to continuously change α is a useful tool in a variety of applications, e.g., for metamorphosis between two different freeform shapes. In the next sections we consider a few simple examples of α-sectors of two varieties.

While Equation (1) is quadratic, we later "linearize" it and introduce the *pseudo α-sector* which is simple to represent as a rational function.

The rest of this paper is organized as follows. In Section 2 we give solve the simple point-point case (in any dimension). In Section 3 we compute the α-section of a point and a line in $\mathbb{R}^2$. Them, in Section 4 we characterize all conic sections as such α-sectors. In Sections 5, 6, and 7 we analyze the 3-dimensional point-plane, line-line, and curve-curve cases, respectively. In Section 8 we present the pseudo α-sector, the "linearized" form of the α-sector, which is always rational if the two original varieties are rational. We terminate with some concluding remarks in Section 9.

2 Point-Point α-Sector in $\mathbb{R}^d$

It is well known that the α-sector of two points in $\mathbb{R}^d$ is a d-dimensional sphere. Let $\mathcal{B} = (b_{x_1}, ..., b_{x_d})$ be the α-sector of $P = (p_1, ..., p_d)$ and $Q = (q_1, ..., q_d)$. Then,

$$\alpha \sqrt{\sum_{i=1}^{d} (b_{x_i} - p_i)^2} = (1-\alpha)\sqrt{\sum_{i=1}^{d} (b_{x_i} - q_i)^2}.$$

The α-sector thus equals

$$0 \;\; = \;\; \sum_{i=1}^{d} (2\alpha - 1) b_{x_i}^2 + 2((1-\alpha)^2 q_i - \alpha^2 p_i) b_{x_i} + (\alpha^2 p_i^2 - (1-\alpha)^2 q_i^2).$$

When $\alpha = 0.5$ the α-sector of P and Q is the bisector hyper-plane defined between the two points,

$$0 = \sum_{i=1}^{d} 2(q_i - p_i)b_{x_i} + (p_i^2 - q_i^2) = \sum_{i=1}^{d}(q_i - p_i)(2b_{x_i} + (p_i + q_i)).$$

When $\alpha \neq 0.5$ the α-sector of P and Q is the hypersphere,

$$0 \;=\; \sum_{i=1}^{d} b_{x_i}^2 + \frac{2((1-\alpha)^2 q_i - \alpha^2 p_i)}{2\alpha - 1} b_{x_i} + \frac{\alpha^2 p_i^2 - (1-\alpha)^2 q_i^2}{2\alpha - 1}$$

or

$$R^2 \;=\; \sum_{i=1}^{d} (b_{x_i} - \beta_i)^2,$$

where

$$\beta_i = \frac{\alpha^2 p_i - (1-\alpha)^2 q_i}{2\alpha - 1}, R^2 = \sum_{i=1}^{n} \beta_i^2 - \frac{\alpha^2 p_i^2 - (1-\alpha)^2 q_i^2}{2\alpha - 1}$$

centered at $\beta = (\beta_1, ..., \beta_d)$ and of radius R.

3 Point-Line α-Sector in $\mathbb{R}^2$

We assume without loss of generality that the line is the Y-axis, that is, the parametric line $C(t) = (0, t)$, and that the point is $P = (1, 0)$. We set $\alpha = 0$ for the line and $\alpha = 1$ for the point.

The α-sector $\mathcal{B} = (b_x, b_y)$ of the Y-axis and the point P satisfies the line-orthogonality constraint

$$0 = \left\langle \mathcal{B} - C(t), \frac{dC(t)}{dt} \right\rangle = \langle (b_x, b_y) - (0, t), (0, 1) \rangle = b_y - t, \tag{2}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + b_y^2\right) = (1-\alpha)^2 (b_x^2 + (b_y - t)^2). \tag{3}$$

The solution of Equations (2)–(3) is the quadratic curve

$$\left(\frac{2\alpha - 1}{\alpha^2}\right) x^2 + y^2 - 2x + 1 = 0, \tag{4}$$

where x, y are used instead of b_x, b_y.

Figure 1 shows the α-sectors of the line $(0, t)$ and the point $(1, 0)$ for various values of α. When $\alpha < \frac{1}{2}$, the coefficients of x^2 and y^2 have opposite signs, and so the α-sector is a hyperbola. When $\alpha = \frac{1}{2}$, the coefficient of x^2 vanishes, and so the bisector is a parabola. When $\alpha > \frac{1}{2}$, the coefficients of x^2 and y^2 have the same sign, and so the α-sector is an ellipse.

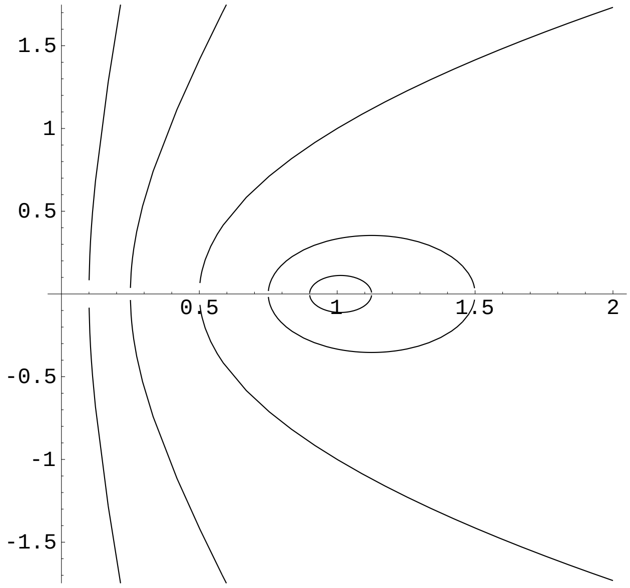


Figure 1: The α-sectors of the point $(1, 0)$ and the line $(0, t)$ for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$.

4 Characterization of Planar Conic Sections

We have shown above that the α-sector of a point and a line is a conic section—a hyperbola, a parabola, or an ellipse—depending on the value of α. We now show that every conic section is an α-sector (for some value of $0 < \alpha < 1$) of some point and some line in the plane. We do that constructively and show how to compute the point, line, and α from the quadratic curve.

4.1 Reduction of Quadratic Forms to a Canonical Representation

We start with a reduction of the general quadratic form to a canonical representation. Given the general quadratic equation

$$0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F,$$

we rotate the plane about the origin by θ. That is, we substitute $x = \overline{x}\cos(\theta) - \overline{y}\sin(\theta)$ and $y = \overline{x}\sin(\theta) + \overline{y}\cos(\theta)$. Then,

$$\begin{aligned} 0 &= A(\overline{x}\cos(\theta) - \overline{y}\sin(\theta))^2 + B(\overline{x}\cos(\theta) - \overline{y}\sin(\theta))(\overline{x}\sin(\theta) + \overline{y}\cos(\theta)) \\ &\quad + C(\overline{x}\sin(\theta) + \overline{y}\cos(\theta))^2 + D(\overline{x}\cos(\theta) - \overline{y}\sin(\theta)) + E(\overline{x}\sin(\theta) + \overline{y}\cos(\theta)) + F. \end{aligned}$$

In the new coordinate system, the coefficient, $\overline{B}$, of $\overline{xy}$ is

$$\begin{aligned} \overline{B} &= -2A\cos(\theta)\sin(\theta) + B(\cos^2(\theta) - \sin^2(\theta)) + 2C\cos(\theta)\sin(\theta) \\ &= B\cos(2\theta) - (A - C)\sin(2\theta). \end{aligned}$$

For $\overline{B}$ to vanish we must require

$$\frac{B}{A-C} = \tan(2\theta).$$

In other words, by rotating the general conic section by $\theta = \frac{1}{2}\arctan\left(\frac{B}{A-C}\right)$, the general conic reduces to

$$0 = \overline{A}\overline{x}^2 + \overline{C}\overline{y}^2 + \overline{D}\overline{x} + \overline{E}\overline{y} + \overline{F}.$$

Assume first that neither $\overline{A}$ nor $\overline{C}$ are zero, and translate the plane by $(-x_0, -y_0)$. That is, substitute $\overline{x} = \hat{x} - x_0$ and $\overline{y} = \hat{y} - y_0$, and obtain

$$\begin{aligned} 0 &= \overline{A}(\hat{x}-x_0)^2 + \overline{C}(\hat{y}-y_0)^2 + \overline{D}(\hat{x}-x_0) + \overline{E}(\hat{y}-y_0) + \overline{F} \\ &= \overline{A}\hat{x}^2 + \overline{C}\hat{y}^2 + (\overline{D} - 2\overline{A}x_0)\hat{x} + (\overline{E} - 2\overline{C}y_0)\hat{y} \\ &\quad +\overline{A}x_0^2 + \overline{C}y_0^2 - \overline{D}x_0 - \overline{E}y_0 + \overline{F}. \end{aligned}$$

By setting $x_0 = \overline{D}/(2\overline{A})$ and $y_0 = \overline{E}/(2\overline{C})$, the general conic is further reduced to

$$0 = \hat{A}\hat{x}^2 + \hat{C}\hat{y}^2 + \hat{F}. \tag{5}$$

Assume now that either $\overline{A}$ or $\overline{C}$ is zero, that is, that the conic is a parabola. Without loss of generality we may assume $\overline{A} = 0$. Translate the plane by $(0, -y_0)$, that is, substitute $\overline{y} = \hat{y} - y_0$, and obtain

$$\begin{aligned} 0 &= \overline{C}\overline{y}^2 + \overline{D}\overline{x} + \overline{E}\overline{y} + \overline{F} \\ &= \overline{C}(\hat{y}-y_0)^2 + \overline{D}\overline{x} + \overline{E}(\hat{y}-y_0) + \overline{F} \\ &= \overline{C}\hat{y}^2 + \overline{D}\overline{x} + (\overline{E} - 2\overline{C}y_0)\hat{y} + \overline{C}y_0^2 - \overline{E}y_0 + \overline{F}. \end{aligned}$$

By setting $y_0 = \overline{E}/(2\overline{C})$, the general parabola is further reduced to

$$0 = \hat{C}\hat{y}^2 + \hat{D}\hat{x} + \hat{F}. \tag{6}$$

The last case, in which both $\overline{A}$ and $\overline{C}$ are zero, is a line and is thus of no interest.

We have applied two rigid transformations (motions) on the conic section. Having the conic as a bilinear form, $\mathbf{R}$ a rotation matrix and $\mathbf{T}$ a translation matrix as described above,

$$\begin{aligned} 0 &= \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \overline{x} & \overline{y} & 1 \end{bmatrix} \mathbf{RMR} \begin{bmatrix} \overline{x} \\ \overline{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{x} & \overline{y} & 1 \end{bmatrix} \begin{bmatrix} \overline{A} & 0 & \frac{\overline{D}}{2} \\ 0 & \overline{C} & \frac{\overline{E}}{2} \\ \frac{\overline{D}}{2} & \frac{\overline{E}}{2} & \overline{F} \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{x} & \hat{y} & 1 \end{bmatrix} \mathbf{R}^{-1}\mathbf{T}^{-1}\mathbf{MT}\,\mathbf{R} \begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & 1 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & 0 \\ 0 & \hat{C} & 0 \\ 0 & 0 & \hat{F} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix}. \end{aligned} \tag{7}$$

In other words, the matrix M is diagonalized.

It is possible [2] to parameterize this canonical representation of the conic $0 = \hat{A}\hat{x}^2 + \hat{C}\hat{y}^2 + \hat{F}$ (for $\hat{F} \neq 0$). By substituting

$$\hat{x} = \frac{r^2 - t^2}{r^2 + t^2}\sqrt{\frac{-\hat{F}}{\hat{A}}}, \qquad \hat{y} = \frac{2rt}{r^2 + t^2}\sqrt{\frac{-\hat{F}}{\hat{C}}}$$

in

$$\frac{-\hat{A}}{\hat{F}}\hat{x}^2 + \frac{-\hat{C}}{\hat{F}}\hat{y}^2 = 1,$$

the canonical conic is satisfied by every r and t. One can choose any parameterization for r and t. For example, in order to keep the degree of the parameterization as low as possible one can select $r = 1 - t$, getting a rational quadratic representation for $\hat{x}$ and $\hat{y}$.

With the determined parametric form of $\hat{C}(t) = (\hat{x}(t), \hat{y}(t))$, the original conic can be restored by applying the inverse rigid motion:

$$C(t) = (\ \mathbf{R}^{-1}\mathbf{T}^{-1})\hat{C}(t).$$

4.2 Point-Line α-Sectors from Conic Sections

To allow enough freedom in choosing the α-sector, we introduce a scaling factor on the distance between the point and the line. Specifically, we compute the α-sector of the line $C(t) = (\beta, t)$ and the point $P = (\gamma, 0)$. As before, we set $\alpha = 0$ for the line and $\alpha = 1$ for the point. A similar computation to that given above shows that the α-sector of $C(t)$ and P is

$$\left(\frac{2\alpha - 1}{\alpha^2}\right) x^2 + y^2 - 2\left[\gamma - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta\right] x + \gamma^2 - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta^2 = 0. \tag{8}$$

Given the quadratic curve, we first transform it to a canonic form by applying a rigid transformation (rotation and translation). Specifically, we change the coordinate system so that the quadratic curve has either the form

1. $y^2 + Ax + B = 0$ with $A \neq 0$ (Equation (6), a parabola whose axis is the x-axis); or
2. $Ax^2 + y^2 + B = 0$, with $A < 1$, $A \neq 0$, and $AB \leq 0$ (Equation (5), an ellipse whose major axis is the x-axis if $0 < A < 1$, or a hyperbola if $A < 0$).[1]

A proper choice of α, β, and γ yields each of these two equations:

[1] The case $A, B > 0$ is impossible. The case $A, B < 0$ is handled by dividing the equation by A and by exchanging the roles of x and y.

1. A parabola. We solve the system

$$\begin{aligned} \frac{2\alpha - 1}{\alpha^2} &= 0, & (9) \\ -2\left[\gamma - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta\right] &= A, & (10) \\ \gamma^2 - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta^2 &= B. & (11) \end{aligned}$$

Setting

$$\alpha = 0.5,\ \beta = \frac{A}{4},\ \text{and}\ \gamma = -\frac{A}{4} - \frac{B}{A}$$

satisfies the system and yields the desired result: The bisector of the line $(A/4, t)$ and the point $(-A/4 - B/A, 0)$ is the parabola $y^2 + Ax + B = 0$.

2. An ellipse or a hyperbola. We solve the system

$$\begin{aligned} \frac{2\alpha - 1}{\alpha^2} &= A, & (12) \\ -2\left[\gamma - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta\right] &= 0, & (13) \\ \gamma^2 - \left(\frac{1-\alpha}{\alpha}\right)^2 \beta^2 &= B. & (14) \end{aligned}$$

Setting

$$\alpha = \frac{1}{1+\sqrt{1-A}},\ \beta = \sqrt{\frac{B}{A(A-1)}},\ \text{and}\ \gamma = \sqrt{\frac{B(A-1)}{A}}$$

satisfies the system and yields the desired result: The $(1/(1+\sqrt{1-A}))$-sector of the line $(\sqrt{B/(A(A-1))}, t)$ and the point $(\sqrt{B(A-1)/A}, 0)$ is the curve $Ax^2+y^2+B = 0$.

Note that α, β, and γ are always defined, since $A < 1$ and because A and B always have opposite signs (unless $B = 0$).

Also note that when A approaches 1, the ellipse is asymptotically a circle. The circle is the 1-sector of the vertical line at $-\infty$ and the origin.

4.3 Reduction of Quadric Forms to Parametric Representations

The same diagonalization process used in Section 4.1 can be applied to quadrics. Let us represent the general quadric

$$0 = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J$$

in a bilinear form in $\mathbb{R}^3$:

$$0 = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} A & \frac{D}{2} & \frac{E}{2} & \frac{G}{2} \\ \frac{D}{2} & B & \frac{F}{2} & \frac{H}{2} \\ \frac{E}{2} & \frac{E}{2} & C & \frac{D}{2} \\ \frac{G}{2} & \frac{F}{2} & \frac{H}{2} & J \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} & 1 \end{bmatrix} \mathbf{U}^{-1}\mathbf{M}\mathbf{U} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} & 1 \end{bmatrix} \begin{bmatrix} \hat{A} & 0 & 0 & 0 \\ 0 & \hat{B} & 0 & 0 \\ 0 & 0 & \hat{C} & 0 \\ 0 & 0 & 0 & \hat{J} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ 1 \end{bmatrix}.$$

Hence, given a general quadric surface, we can reduce it (by diagonalization—a rigid motion in $\mathbb{R}^3$) to

$$0 = \hat{A}\hat{x}^2 + \hat{B}\hat{y}^2 + \hat{C}z^2 + \hat{J}.$$

A procedure for parameterizing a quadric, similar to that given in Section 4.1, is described in [2]. This reference examines the various combinations of the signs of $\hat{A}$, $\hat{B}$, and $\hat{C}$, yielding nonsingular cases of ellipsoids (all positive), hyperboloids of one sheet (one is negative), and hyperboloids of two sheets (two are negative).

With the determined parametric form of the canonical quadric $\hat{S}(u,v) = (\hat{x}(u,v), \hat{y}(u,v), \hat{z}(u,v))$, the original position and orientation of the quadric could be restored by applying the inverse rigid motion:

$$S(u,v) = \mathbf{U}^{-1}\hat{S}(u,v).$$

5 Point-Plane α-Sector in $\mathbb{R}^3$

The α-sector of a point and a plane in $\mathbb{R}^3$ is similar to that of a point and a line in $\mathbb{R}^2$. We assume without loss of generality that the plane is the YZ-plane, that is, it is the parametric plane $S(u,v) = (0,u,v)$, and that the point is $P = (1,0,0)$. We set $\alpha = 0$ for the plane and $\alpha = 1$ for the point.

Let $\mathcal{B} = (b_x, b_y, b_z)$ be the α-sector of $S(u,v)$ and P. As in the two-dimensional case we have the two plane-orthogonality constraints

$$0 = \left\langle \mathcal{B} - S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle = \langle (b_x, b_y, b_z) - (0,u,v), (0,1,0) \rangle = b_y - u, \quad (15)$$

$$0 = \left\langle \mathcal{B} - S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle = \langle (b_x, b_y, b_z) - (0,u,v), (0,0,1) \rangle = b_z - v, \quad (16)$$

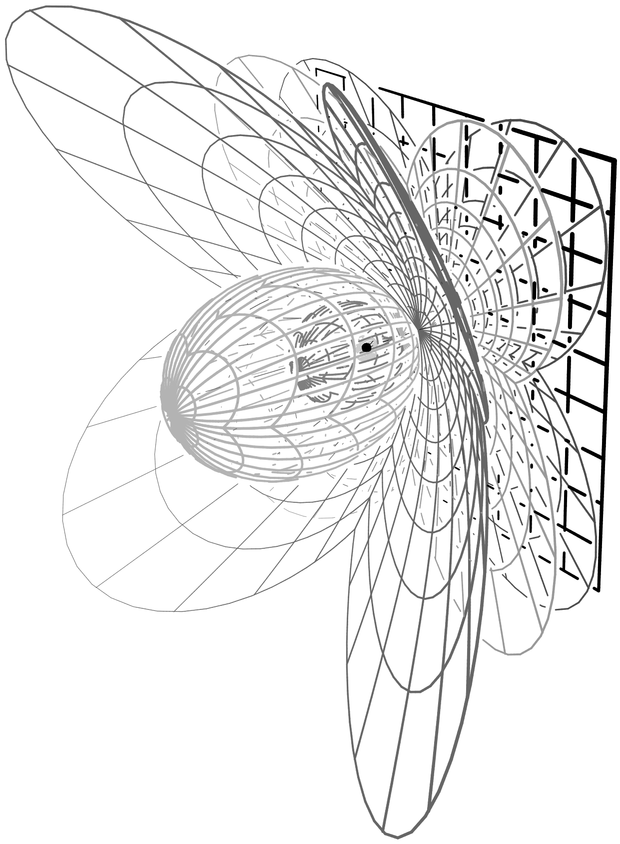


Figure 2: The α-sectors of the point $(1,0,0)$ and the plane $(0,u,v)$.

and the distance constraint

$$\alpha^2\left((b_x-1)^2+b_y^2+b_z^2\right)=(1-\alpha)^2(b_x^2+(b_y-u)^2+(b_z-v)^2). \tag{17}$$

The solution of Equations (15)–(17) is the quadratic surface

$$\left(\frac{2\alpha-1}{\alpha^2}\right)x^2+y^2+z^2-2x+1=0, \tag{18}$$

where x,y,z are used instead of b_x,b_y,b_z. This is a hyperboloid for $0<\alpha<\frac{1}{2}$, an elliptic paraboloid for $\alpha=\frac{1}{2}$, and an ellipsoid for $\frac{1}{2}<\alpha<1$.

Figure 2 shows the parametric forms of α-sectors of the point P and the plane $S(u,v)$ defined above, for various values of α, computed using the reduction presented in Section 4.3.

6 Line-Line α-Sector in $\mathbb{R}^3$

Yet another simple example is the α-sector of the two lines $C_1(u)=(1,u,0)$ and $C_2(v)=(0,0,v)$. We set $\alpha=0$ for $C_2(v)$ and $\alpha=1$ for $C_1(u)$. Let $\mathcal{B}=(b_x,b_y,b_z)$ be the α-sector of

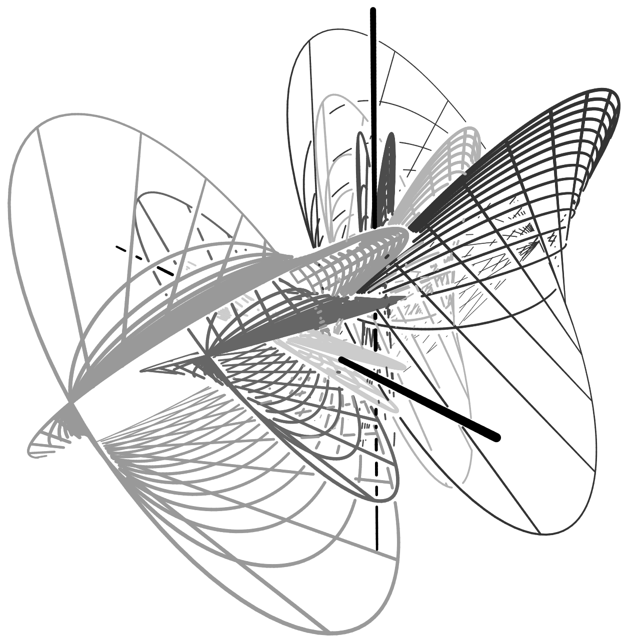


Figure 3: The α-sectors of the lines $(1, u, 0)$ and $(0, 0, v)$.

$C_1(u)$ and $C_2(v)$. Here we have the two line-orthogonality constraints

$$0 = \left\langle \mathcal{B} - C_1(u), \frac{dC_1(u)}{du} \right\rangle = \langle (b_x, b_y, b_z) - (1, u, 0), (0, 1, 0) \rangle = b_y - u, \tag{19}$$

$$0 = \left\langle \mathcal{B} - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = \langle (b_x, b_y, b_z) - (0, 0, v), (0, 0, 1) \rangle = b_z - v, \tag{20}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + (b_y - u)^2 + b_z^2 \right) = (1 - \alpha)^2 (b_x^2 + b_y^2 + (b_z - v)^2). \tag{21}$$

The solution of Equations (19)–(21) is the quadratic surface

$$\left(\frac{2\alpha - 1}{\alpha^2} \right) x^2 - \left(\frac{1 - \alpha}{\alpha} \right)^2 y^2 + z^2 - 2x + 1 = 0. \tag{22}$$

The $\frac{1}{2}$-sector (bisector) of $C_1(u)$ and $C_2(v)$ is thus the surface

$$y^2 - z^2 + 2x - 1 = 0,$$

whose parametric form is given as $(\frac{1-u^2+v^2}{2}, u, v)$. This repeats the result of [1].

Figure 3 shows the parametric forms of the α-sectors of the lines $C_1(u)$ and $C_2(v)$ defined above, for various values of α computed using the reduction presented in Section 4.3. For $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$, Equation (22) yields a hyperboloid of two sheets, and for $\alpha = \frac{1}{2}$ it yields a hyperbolic paraboloid. For $\alpha = 0$ and $\alpha = 1$ the hyperboloid of two sheets reduces to a 0-width cylinder (one of the original lines).

7 Curve-Curve α-Sector in $\mathbb{R}^3$

In general, the α-sector of two general parametric rational curves in $\mathbb{R}^3$ is a nonrational surface. Let $C_1(s) = (x_1(s), y_1(s), z_1(s))$ and $C_2(t) = (x_2(t), y_2(t), z_2(t))$ be two regular C^1-continuous spatial curves. The regularity and C^1-continuity conditions imply that the tangent vectors $T_1(s) = (x_1'(s), y_1'(s), z_1'(s))$ and $T_2(t) = (x_2'(t), y_2'(t), z_2'(t))$ are nonzero continuous vector fields. (Note that $T_1(s)$ and $T_2(t)$ are nonunit vectors in general.) Let $L_1(s_0)$ (resp., $L_2(t_0)$) denote the plane normal to $C_1(s)$ (resp., $C_2(t)$) at $s = s_0$ (resp., $t = t_0$). Due to the continuity of $T_1(s)$ and $T_2(t)$, the normal planes $L_1(s)$ and $L_2(t)$ are well-defined and move continuously along the curves $C_1(s)$ and $C_2(t)$, respectively.

When a point $Q = (x, y, z)$ is on the α-sector of the two curves, there exists (at least) one pair of points $C_1(s)$ and $C_2(t)$ such that Q is simultaneously contained in the normal planes $L_1(s)$ and $L_2(t)$. Hence the point Q satisfies the two linear equations

$$\begin{aligned} \langle C_1'(s), C_1(s) - Q \rangle &= 0, \\ \langle C_2'(t), C_2(t) - Q \rangle &= 0. \end{aligned}$$

In addition, the point Q is contained in the α-sector surface of the two points $C_1(s)$ and $C_2(t)$. This surface is defined by the quadratic equation

$$\alpha^2 \langle Q - C_1(s), Q - C_1(s) \rangle = (1-\alpha)^2 \langle Q - C_2(t), Q - C_2(t) \rangle .$$

Thus Q is a solution (if exists) of the system

$$\begin{array}{ll} \text{(a)} & \langle C_1'(s), Q \rangle = \langle C_1'(s), C_1(s) \rangle \\ \text{(b)} & \langle C_2'(t), Q \rangle = \langle C_2'(t), C_1(t) \rangle \\ \text{(c)} & (2\alpha - 1) \langle Q, Q \rangle + 2 \langle (1-\alpha)^2 C_2(t) - \alpha^2 C_1(s), Q \rangle \\ & -(1-\alpha)^2 \langle C_2(t), C_2(t) \rangle + \alpha^2 \langle C_1(s), C_1(s) \rangle = 0. \end{array} \tag{23}$$

Equations (23)(a,b) are planes in $\mathbb{R}^3$, whereas Equation (23)(c) is a sphere. Our strategy is to compute the intersection line of the two planes, and then to verify whether this line pinches the sphere in 0, 1, or 2 points.

We expand Equations (23)(a,b):

$$\begin{array}{ll} \text{(a)} & \langle (x_1'(s),\ y_1'(s),\ z_1'(s)), (x,\ y,\ z) \rangle = \langle C_1'(s), C_1(s) \rangle , \\ \text{(b)} & \langle (x_2'(t),\ y_2'(t),\ z_2'(t)), (x,\ y,\ z) \rangle = \langle C_2'(t), C_2(t) \rangle , \end{array}$$

or, alternatively,

$$\begin{bmatrix} x_1'(s) & y_1'(s) \\ x_2'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \langle C_1'(s), C_1(s) \rangle - z_1'(s) z \\ \langle C_2'(t), C_2(t) \rangle - z_2'(t) z \end{bmatrix} .$$

We can now use Cramer's rule to solve x and y as functions of z:

$$
\begin{aligned}
x &= \frac{\begin{bmatrix} \langle C_1'(s), C_1(s)\rangle - z_1'(s)z & y_1'(s) \\ \langle C_2'(t), C_2(t)\rangle - z_2'(t)z & y_2'(t) \end{bmatrix}}{\begin{bmatrix} x_1'(s) & y_1'(s) \\ x_2'(t) & y_2'(t) \end{bmatrix}} = \psi_x(s,t)z + \phi_x(s,t), \\
y &= \frac{\begin{bmatrix} x_1'(s) & \langle C_1'(s), C_1(s)\rangle - z_1'(s)z \\ x_2'(t) & \langle C_2'(t), C_2(t)\rangle - z_2'(t)z \end{bmatrix}}{\begin{bmatrix} x_1'(s) & y_1'(s) \\ x_2'(t) & y_2'(t) \end{bmatrix}} = \psi_y(s,t)z + \phi_y(s,t),
\end{aligned}
$$

where $\psi_x(s,t)$, $\phi_x(s,t)$, $\psi_y(s,t)$, $\phi_y(s,t)$ are rational functions of s and t. (In the sequel we omit the dependency on (s,t) for ease of notation of ψ and ϕ.) Substituting these values of x and y in Equation (23)(c) yields

$$
\begin{aligned}
&(2\alpha - 1)\left((1 + \psi_x^2 + \psi_y^2)z^2 + 2(\psi_x\phi_x + \psi_y\phi_y)z + \phi_x^2 + \phi_y^2\right) \\
&\quad + 2\left(((1-\alpha)^2 x_2(t) - \alpha^2 x_1(s))(\psi_x z + \phi_x)\right. \\
&\qquad \left. + ((1-\alpha)^2 y_2(t) - \alpha^2 y_1(s))(\psi_y z + \phi_y) + ((1-\alpha)^2 z_2(t) - \alpha^2 z_1(s))z\right) \\
&\quad - (1-\alpha)^2 \langle C_2(t), C_2(t)\rangle + \alpha^2 \langle C_1(s), C_1(s)\rangle = 0.
\end{aligned} \tag{24}
$$

Let $\Psi = (\psi_x, \psi_y, 1)$ and $\Phi = (\phi_x, \phi_y, 0)$. Further, denote

$$
\begin{aligned}
A &= (2\alpha - 1)\langle \Psi, \Psi \rangle, \\
B &= 2\left(\langle \Psi, \Phi\rangle + \left\langle (1-\alpha)^2 C_2(t) - \alpha^2 C_1(s), \Psi \right\rangle\right), \\
C &= (2\alpha - 1)\langle \Phi, \Phi\rangle + 2\left\langle (1-\alpha)^2 C_2(t) - \alpha^2 C_1(s), \Phi \right\rangle \\
&\qquad - (1-\alpha)^2 \langle C_2(t), C_2(t)\rangle + \alpha^2 \langle C_1(s), C_1(s)\rangle.
\end{aligned}
$$

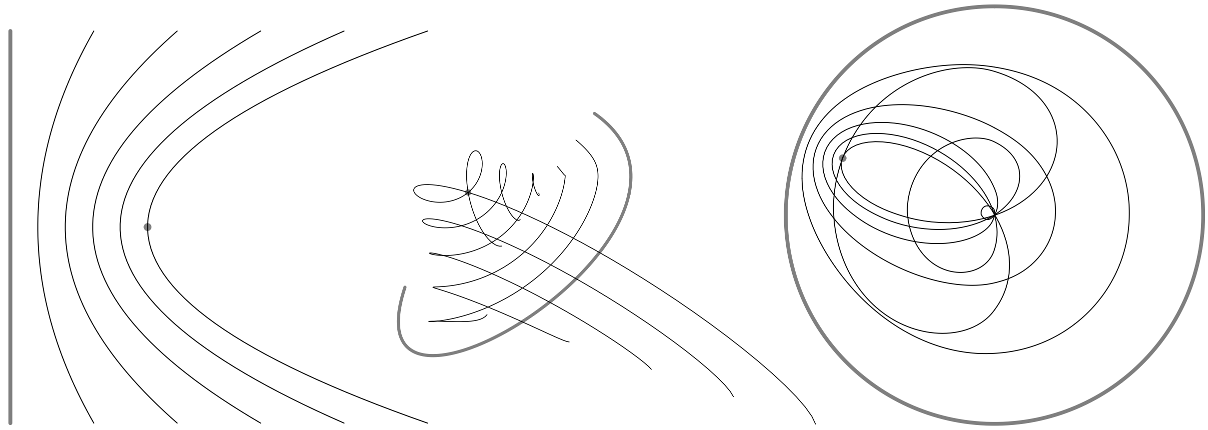
With these notations, Equation (24) becomes

$$
Az^2 + Bz + C = 0.
$$

The last equation has 0, 1, or 2 solution when its discriminant is negative, zero, or positive, respectively. Thus, the α-sector of two rational curves in $\mathbb{R}^3$ is a partially-defined nonrational function of s and t.

8 Pseudo α-Sectors

Unfortunately, Equation (1) is quadratic in $\mathcal{B}(t)$; thus, the α-sector of two rational varieties is nonrational in general. We seek a linear constraint that would replace this equation and yet would yield similar properties to the α-sector. We look for a linear constraint that can measure the relative distances to the two given varieties. While the L_2 norm is quadratic,



(a) (b) (c)

Figure 4: (a) The pseudo α-sectors of the point $(0,1)$ and the line $(0,t)$ in $\mathbb{R}^2$ for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$. (Compare with Figure 1.) (b) The pseudo α-sectors of the point $(-0.5, 0.5)$ and the parametric cubic curve $(-0.9-1.2t+9t^2-6.6t^3, -0.1-3.6t+8.4t^2-3.7t^3)$ in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. (c) The pseudo α-sectors of a point and a circle in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. The original curves and points are shown in gray.

we propose as a linear constraint the plane that is at $\alpha : (1-\alpha)$ relative distances from the two examined points on the two varieties.

For example, for the pseudo α-sector of a curve $C(t)$ and a point P in $\mathbb{R}^2$, we will impose the two linear constraints

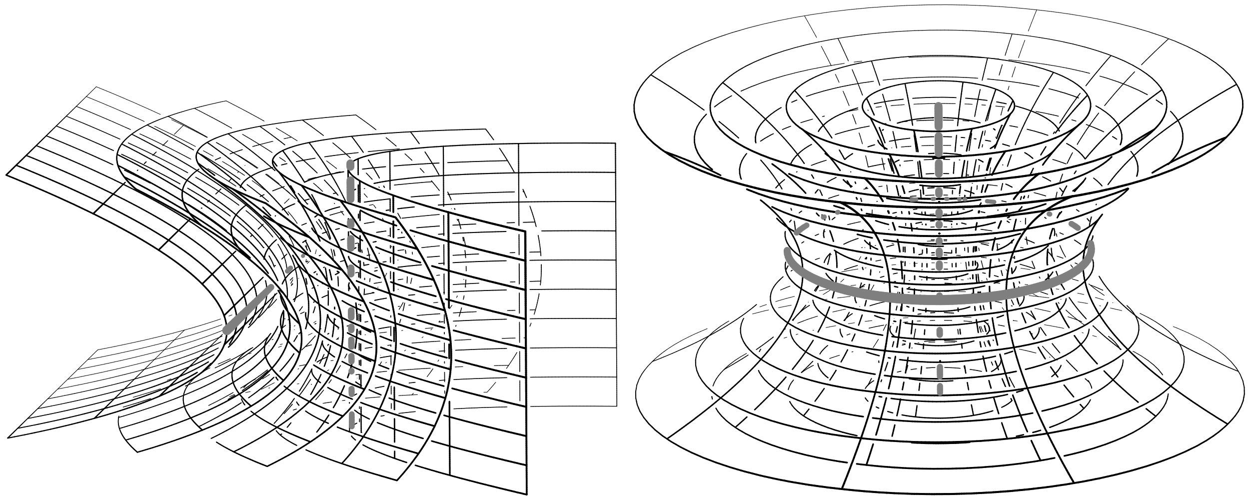
$$\left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt} \right\rangle = 0, \tag{25}$$

$$\langle \mathcal{B}(t) - (\alpha P + (1-\alpha)C(t)), C(t) - P \rangle = 0. \tag{26}$$

While Equation (25) is the regular orthogonality constraint, the constraint of Equation (26) ensures that the pseudo α-sector point is on the plane containing the point $\alpha P + (1-\alpha)C(t)$ and orthogonal to the vector $C(t) - P$. If $C(t)$ has a rational representation, we can easily use Cramer's rule to obtain a rational representation for $\mathcal{B}(t) = (b_x(t), b_y(t))$.

Figure 4 shows three examples of planar pseudo α-sectors of: (i) a point and a line (Figure 4(a)); (ii) a point and a cubic curve (Figure 4(b)); and (iii) a point and a circle (Figure 4(c)).

The extension to more complex shapes and/or to higher dimensions is straightforward. For example, the pseudo α-sector of two curves $C_1(u)$ and $C_2(v)$ in $\mathbb{R}^3$ is the solution of the three linear constraints



(a) (b)

Figure 5: (a) The pseudo α-sectors of two lines in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. (b) The α-sectors of a line and a circle in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. The original curves are shown in gray.

$$\left\langle \mathcal{B}(t) - C_1(u), \frac{dC_1(u)}{du} \right\rangle = 0, \tag{27}$$

$$\left\langle \mathcal{B}(t) - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = 0, \tag{28}$$

$$\langle \mathcal{B}(t) - (\alpha C_1(u) + (1-\alpha)C_2(v)), C_1(u) - C_2(v) \rangle = 0. \tag{29}$$

Again, if $C_1(u)$ and $C_2(v)$ have rational representations, we can use Cramer's rule to obtain a rational representation for $\mathcal{B}(t)$. Figure 5 shows two such three-space pseudo α-sectors of: (i) two lines (Figure 5(a)) and (ii) a line and a circle (Figure 5(b)).

The pseudo α-sector is clearly different from the α-sector. They are identical only for $\alpha = \frac{1}{2}$, where they are also equivalent to the regular bisector. Note that unlike the regular α-sectors, the pseudo 0- and 1-sectors are different from the original varieties. This is because of the weaker distance constraint: points on the pseudo bisector do not satisfy the $\alpha : (1-\alpha)$ distance ratio; this property is imposed only on their projections on the lines joining the respective points on the varieties.

8.1 Properties of the Point-Curve Pseudo 0-Sector

We next show two properties of the pseudo 0-sector of a point and a curve. (Here $\alpha = 0$ and $\alpha = 1$ correspond to the point and to the curve, respectively). Unlike the 0-sector of a point and a curve, the pseudo 0-sector $\mathcal{B}$ does not identify with the point, but rather passes through the point.

The first theorem refers to the multiplicity of the pseudo 0-sector at the point:

Theorem 1 *Let P and $C(t)$ be a point and a rational parametric curve in the plane, respectively. Let also k be the number of points of $C(t)$ at which the line orthogonal to $C(t)$ passes through P. Then the pseudo 0-sector of P and $C(t)$ passes k times through P.*

Note that the theorem refers to a number of points of $C(t)$ and not to a number of lines. This is because of multiplicity of lines: a line can be perpendicular to $C(t)$ at more than one point.

Proof: Let $C(t_0)$ be a point on the curve C, and let Q be its corresponding point on the pseudo 0-sector of P and C. We need to show that $Q = P$ if and only if the normal to C at $C(t_0)$ passes through P.

Recall the definition of the pseudo α-sector: A point Q is on the pseudo α-sector of P and $C(t)$ if there exists some value t_0 for which (a) Q is on the normal to C at $C(t_0)$; and (b) the projection of Q onto $\overline{P, C(t_0)}$ is at relative distances α and $1 - \alpha$ from P and $C(t_0)$, respectively.

The first direction is trivial: if $Q = P$, then P must lie on the normal to C at $C(t_0)$ due to condition (a).

The opposite direction is also straightforward. Let $t = t_0$ be a parameter for which the normal to C at $C(t_0)$ passes through P. Thus we have: (a) P is on the normal to C at $C(t_0)$; and (b) P is at relative distances 0 and 1 from P and $C(t_0)$, respectively. Therefore P itself is the pseudo 0-sector point that corresponds to P.

Hence the pseudo 0-sector of P and $C(t)$ passes through P as many times as different values of t_0 with the above property. $\square$

Refer again to Figure 4. In Figure 4(a), the pseudo 0-sector of a point and a vertical line passes once through the point. Indeed, only one perpendicular line to the vertical line passes through the point. In Figure 4(b), the pseudo 0-sector of a point and a cubic curve passes three times through P_2. This is because of the three perpendicular lines to the cubic curve that pass through the point. Finally, in Figure 4(c), the pseudo 0-sector of a point and the circle passes twice through the point. This is a multiplicity case: a line perpendicular to the circle and passing through the point is perpendicular to the circle at two points.

The second theorem refers to the slope of the pseudo 0-sector when it passes through the point:

Theorem 2 *Let P and $C(t)$ be a point and a rational parametric curve in the plane, respectively. Assume that P is the pseudo 0-sector point that corresponds to $C(t_0)$. Then the slope of the pseudo 0-sector at P and the slope of C at $C(t_0)$ are identical.*

Proof: As stated above, the distance constraint at $\mathcal{B}(t)$ and $C(t)$ (for any $0 \le \alpha \le 1$) is

$$\langle \mathcal{B}(t) - (\alpha P + (1-\alpha)C(t)), P - C(t)\rangle = 0.$$

The left side of the equation is a function of t which is identically 0. Hence, its derivative according to t is also identically 0:

$$\langle \mathcal{B}'(t) - (1-\alpha)C'(t), P - C(t)\rangle + \langle \mathcal{B}(t) - (\alpha P + (1-\alpha)C(t)), -C'(t)\rangle = 0.$$

Now we substitute $\alpha = 0$ and $t = t_0$, and use the assumption that $\mathcal{B}(t_0) = P$, to get

$$\langle \mathcal{B}'(t_0) - C'(t_0), P - C(t_0)\rangle + \langle P - C(t_0), -C'(t_0)\rangle = 0. \tag{30}$$

However, the orthogonality constraint at t_0 says that

$$\langle P - C(t_0), C'(t_0)\rangle = 0. \tag{31}$$

By combining Equations (30) and (31) we obtain

$$\langle \mathcal{B}'(t_0), P - C(t_0)\rangle = 0,$$

and the claim follows. $\square$

8.2 Cusps in Pseudo α-Sectors

The α-sector could be singular. That is, even if the original varieties are continuous and regular, the α-sector can have singular points due to the fact that the parametric form of the α-sector curve can fail its regularity condition and reduce its speed to zero. Such points are known as cusps and can be identified via the simultanous vanishing of the partial derivatives of the shape. Herein, $\frac{db_x}{dt} = \frac{db_y}{dt} = 0$ can identify the cusp locations of the planar α-sector curve.

Consider, for example, the point $P = (0, 2)$ and the parabola $C(t) = (t, t^2)$ (See Figure 6). Equations (25) and (26) yield the system

$$\begin{aligned} (x - t) + 2t(y - t^2) &= 0, \\ t(x - t(1-\alpha)) + (t^2 - 2)(y - (2\alpha + t^2(1-\alpha))) &= 0. \end{aligned}$$

The solution of the above system is the pseudo α-sector of P and $C(t)$:

$$\mathcal{B}(x, y) = (b_x, b_y) = \left(\frac{2(1-4\alpha)t + (1+6\alpha)t^3 - 2\alpha t^5}{2+t^2}, \frac{4\alpha + (2-3\alpha)t^2 + (1+\alpha)t^4}{2+t^2} \right).$$

In order to compute the cusp points we solve the system

$$\frac{d\,b_x}{d\,t} = \frac{d\,b_y}{d\,t} = 0,$$

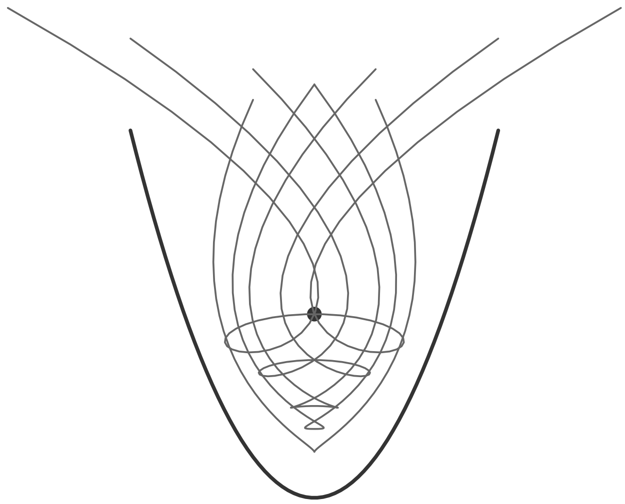


Figure 6: Cusp points of the pseudo α-sector of the point $(0, 2)$ and the parabola (t, t^2) for $\alpha = 0.25,\ 0.375,\ 0.50,\ 0.75$.

namely,

$$\frac{2 - 8\alpha + 3(1+6\alpha)t^2 - 10\alpha t^4}{2+t^2} + \frac{2t((8\alpha-2)t - (1+6\alpha)t^3 + 2\alpha t^5)}{(2+t^2)^2} = 0, \qquad (32)$$

$$\frac{2(2-3\alpha)t + 4(1+\alpha)t^3}{2+t^2} + \frac{2t(-4\alpha + (3\alpha-2)t^2 - (1+\alpha)t^4}{(2+t^2)^2} = 0. \qquad (33)$$

This system has two complex solutions ($\alpha = 0.5$, $t = \pm\sqrt{\sqrt{42}/3 + 2}\cdot i$) and five real solutions:

1. $\alpha = 0.25$, $t = 0$.

2-3. $\alpha = 0.5$, $t = \pm\sqrt{\sqrt{42}/3 - 2}$.

4-5. $\alpha = 7$, $t = \pm\sqrt{1.5}$.

Figure 6 shows P and C, and their respective pseudo α-sectors for several values of α. The first three cusp points are seen clearly in the figure.

9 Conclusion

In this paper we have defined the α-sector of two varieties, and computed it for the pairs point-point (in $\mathbb{R}^d$), point-line (in $\mathbb{R}^2$), point-plane (in $\mathbb{R}^3$), line-line (in $\mathbb{R}^3$), curve-curve (in $\mathbb{R}^3$). We have also shown a characterization of all conic sections: every conic section is an α-sector of a point and a line in $\mathbb{R}^2$, for some line, point, and value of α. The α-sector may be useful in various applications, such as morphing between two freeform geometric shapes.

We have also described its "linearized" version, the pseudo α-sector, in which we replace a nonlinear constraint by a linear approximation of it.

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Figures 1 and 6 were created by Mathematica [4]. Figures 2, 3, 4(a,b,c), and 5(a,b) were created with the aid of the IRIT [3] solid-modeling environment.

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