Bisectors and α-Sectors of Rational Varieties

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Abstract

The bisector of two rational varieties in $\mathbb{R}^d$ is, in general, non-rational. However, there are some cases in which such bisectors are rational; we review some of them, mostly in $\mathbb{R}^2$ and $\mathbb{R}^3$. We also describe the α-sector, a generalization of the bisector, and consider a few interesting cases where α-sectors become quadratic curves or surfaces. Exact α-sectors are non-rational even in special cases and in configurations where the bisectors are rational. This suggests the pseudo α-sector which approximates the α-sector with a rational variety. Both the exact and the pseudo α-sectors identify with the bisector when $\alpha = \frac{1}{2}$.

1 Introduction

Given m different objects $O_1, \ldots, O_m$, the Voronoi region of an object O_i $(1 \leq i \leq m)$ is defined as the set of points that are closer to the object O_i than to any other object O_j $(j \neq i)$. The boundary of each Voronoi region is composed of portions of bisectors, i.e., the set of points that are equidistant from two different objects O_i and O_j $(i \neq j)$. The medial axis of an object is defined as the set of interior points for which the minimum distance to the boundary corresponds to two or more different boundary points; that is, the medial axis is the self-bisector of the boundary of an object.

The concepts of Voronoi diagram and medial axis greatly simplify the design of algorithms for various geometric computations, such as shape decomposition [1], finite-element mesh generation [19, 20], motion planning with collision avoidance [13], and NC tool-path generation [14]. When the objects involved in these applications have freeform shapes, the bisector construction for rational varieties is indispensable. Unfortunately, the bisector of two rational varieties, in general, is non-rational. Moreover, even the bisector of two simple geometric primitives (such as spheres, cylinders, cones, and tori) is not always simple.

In the first part of this paper we review some important special cases where the bisectors are known to be rational. Farouki and Johnstone [10] showed that the bisector of a point and a rational curve in the same plane is a rational curve. Elber and Kim [4] showed that in $\mathbb{R}^3$ the bisector of two rational space curves is a rational surface, whereas the bisector of a point and a rational space curve is a rational ruled surface (which is also developable [16]). Moreover, the bisector of a point and a rational surface is also a rational surface [6]. Although the bisector of two rational surfaces, in general, is non-rational, there are some special cases in which the bisector is a rational surface. Dutta and Hoffmann [2] considered the bisector of simple CSG primitives (planes, spheres, cylinders, cones, and tori). Note that these CSG primitives are surfaces of revolution. When two CSG primitives have the same axis of rotation, their bisector is a quadratic surface of revolution, which is rational. Elber and Kim [6] showed that the bisector of a sphere and a rational surface with a rational offset is a rational surface; moreover, the bisector of two circular cones sharing the same apex is also a rational conic surface with the same apex. In a recent work, Peternell [16] investigated algebraic and geometric properties of curve-curve, curve-surface, and surface-surface bisector surfaces. Based on these properties, Peternell [16] proposed elementary bisector constructions for various special pairs of rational curves and surfaces, using dual geometry and representing bisectors as envelopes of symmetry lines or planes.

This paper outlines the computational procedures that construct the rational bisector curves and surfaces discussed above (except some material discussed by Peternell [16]). The basic construction steps are important since a similar technique will be employed in extending the bisector to a more general concept, the so-called *α-sector*. Instead of taking an equal distance from two input varieties, the α-sector allows different relative distances from the two varieties. Even in the simple case of a point and a line, the α-sector may assume the form of any type of conic, depending on the value of α $(0 < \alpha < 1)$. Exact α-sectors are non-rational even in the special cases where the bisectors are rational. We also present the pseudo α-sectors which approximate exact α-sectors with rational varieties. Both the exact and pseudo α-sectors reduce to bisectors when $\alpha = \frac{1}{2}$.

The rest of this paper is organized as follows. In Section 2, we consider special cases where the bisectors of two varieties are rational curves and surfaces (in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively). In Section 3, we consider bisectors in higher dimensions. In Section 4, we extend the bisector ('$\frac{1}{2}$-sector') to the more general concept of α-sector. We conclude this paper with some final remarks in Section 5.

2 Rational Bisectors

There are some special cases in $\mathbb{R}^2$ and $\mathbb{R}^3$ where the bisector has a simple closed form or a rational representation. In this section we survey some important results already known.

2.1 Point-Curve Bisectors in $\mathbb{R}^2$

Farouki and Johnstone [10] showed that the bisector of a point and a rational curve in the plane is a rational curve. Consider a fixed point $Q \in \mathbb{R}^2$ and a regular C^1 rational curve $C(t) \in \mathbb{R}^2$. Let $\mathcal{B}(t)$ denote the bisector point of Q and $C(t)$. Then we have

$$\left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt} \right\rangle = 0, \qquad (1)$$

$$\|\mathcal{B}(t) - Q\| = \|\mathcal{B}(t) - C(t)\|, \qquad (2)$$

where $\|\cdot\|$ denotes the length of a vector (in the L_2 norm).

Equation (1) means that the bisector point $\mathcal{B}(t)$ belongs to the normal line of the curve $C(t)$, while Equation (2) implies that $\mathcal{B}(t)$ is at an equal distance from Q and $C(t)$. We can square both sides of Equation (2) and cancel out $\|\mathcal{B}(t)\|^2$, to obtain the equation

$$\langle \mathcal{B}(t), C(t) - Q \rangle = \frac{\|C(t)\|^2 - \|Q\|^2}{2}. \tag{3}$$

Equations (1) and (3) are *linear* in $\mathcal{B}(t)$. Using Cramer's rule, we can solve these equations for $\mathcal{B}(t) = (b_x(t), b_y(t))$ and compute a rational representation of $\mathcal{B}(t)$. Note that the resulting bisector curve $\mathcal{B}(t)$ has its supporting foot points at Q and $C(t)$. In other words, the bisector curve $\mathcal{B}(t)$ has the same parameterization as the original curve $C(t)$.

2.2 Point-Curve, Curve-Curve, and Point-Surface Bisectors in $\mathbb{R}^3$

Elber and Kim [4] showed that the bisector of two rational space curves is a rational surface; moreover, the bisector of a point and a rational space curve in $\mathbb{R}^3$ is a rational ruled surface. Consider a fixed point $Q \in \mathbb{R}^3$ and a regular C^1 rational space curve $C(t) \in \mathbb{R}^3$. Let $\mathcal{B}(t)$ be the bisector point of Q and $C(t)$. Then we have

$$\begin{aligned} \left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt} \right\rangle &= 0, & (4)\\ \|\mathcal{B}(t) - Q\| &= \|\mathcal{B}(t) - C(t)\|. & (5) \end{aligned}$$

Since $\mathcal{B}(t)$ is a three-dimensional point, there is one degree of freedom in these equations.

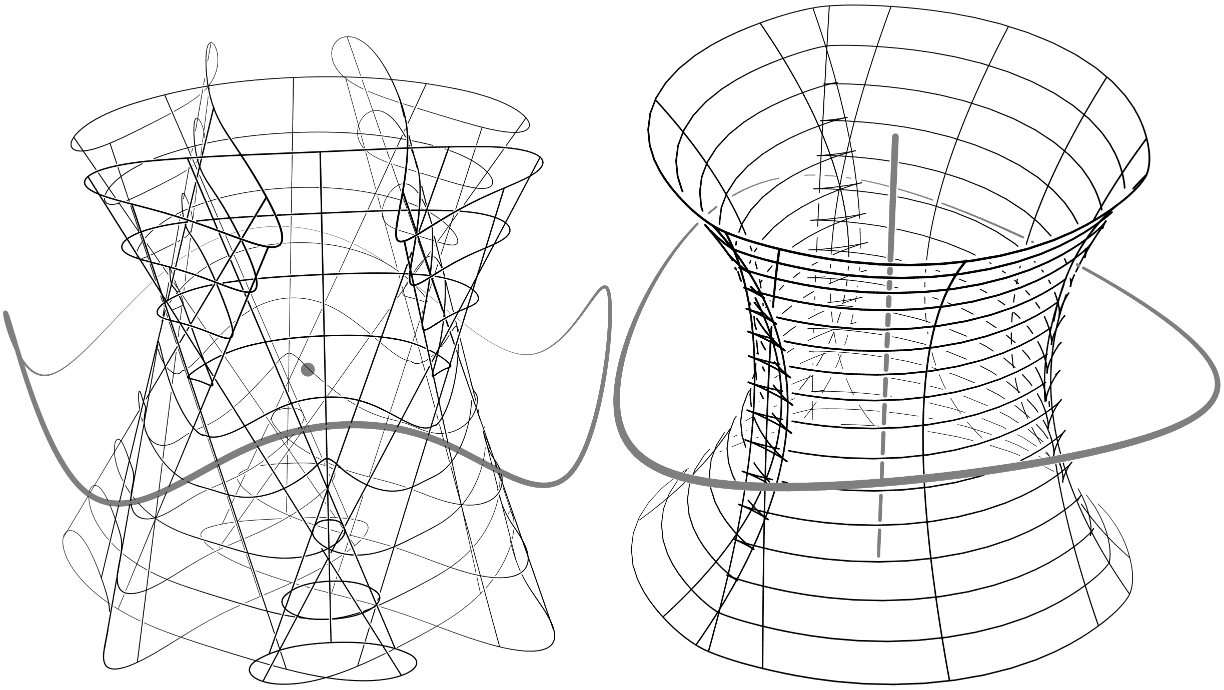
Consider a fixed location $C(t_0)$ on the space curve $C(t)$. Clearly $\mathcal{B}(t_0) \in \mathcal{P}_n(t_0)$, where $\mathcal{P}_n(t_0)$ is the normal plane of the curve at the fixed point $C(t_0)$. Furthermore, $\mathcal{B}(t_0)$ is at an equal distance from Q and $C(t_0)$. Hence, $\mathcal{B}(t_0)$ must belong to the plane $\mathcal{P}_d(t_0)$ which bisects Q and the point $C(t_0)$. Any point on the line $\mathcal{L}_{nd}(t_0) = \mathcal{P}_n(t_0) \bigcap \mathcal{P}_d(t_0)$ satisfies both Equations (4) and (5). Thus, the bisector surface $S(u,t)$ of the point Q and the curve $C(t)$ must be a ruled surface, where each ruling line $\mathcal{L}_{nd}(t)$ is parameterized by a linear parameter u. Figure 1(a) shows an example of such a rational ruled bisector surface generated in this case from a point and a periodic rational space curve in $\mathbb{R}^3$. Based on the concept of dual geometry, Peternell [16] showed that the ruled surface $S(u,t)$ is in fact a developable surface.

The bisector surface (in $\mathbb{R}^3$) of two regular C^1 rational space curves $C_1(u)$ and $C_2(v)$ is also rational. Let $\mathcal{B}(u,v)$ be the bisector point of $C_1(u)$ and $C_2(v)$. Then, the bisector must satisfy the following three equations:

$$\begin{aligned} \left\langle \mathcal{B}(u,v) - C_1(u), \frac{dC_1(u)}{du} \right\rangle &= 0, & (6)\\ \left\langle \mathcal{B}(u,v) - C_2(v), \frac{dC_2(v)}{dv} \right\rangle &= 0, & (7)\\ \|\mathcal{B}(u,v) - C_1(u)\| &= \|\mathcal{B}(u,v) - C_2(v)\|. & (8) \end{aligned}$$

Equations (6) and (7) mean that the bisector point $\mathcal{B}(u,v)$ is simultaneously contained in the two normal planes of $C_1(u)$ and $C_2(v)$, while Equation (8) implies that $\mathcal{B}(u,v)$ is at an equal distance from $C_1(u)$ and $C_2(v)$.

The constraints in Equations (6)–(8) are all *linear* in $\mathcal{B}(u,v)$. (Note that the quadratic terms in Equation (8) cancel out.) Using Cramer's rule, we can solve these equations for $\mathcal{B}(u,v) = (b_x(u,v), b_y(u,v), b_z(u,v))$ and compute a rational surface representation of $\mathcal{B}(u,v)$. The resulting



(a) (b)

Figure 1: (a) The bisector surface of a point and a space curve in $\mathbb{R}^3$. (b) The bisector surface of a line and a round triangular periodic cubic curve in $\mathbb{R}^3$. The original curves are shown in gray.

bisector surface follows the parameterization of the two original curves. In other words, for each point on the first curve, $C_1(u_0)$, and each point on the second curve, $C_2(v_0)$, $\mathcal{B}(u_0, v_0)$ is the bisector point. Figure 1(b) shows a rational bisector surface of a line and a rounded triangular periodic cubic curve in $\mathbb{R}^3$.

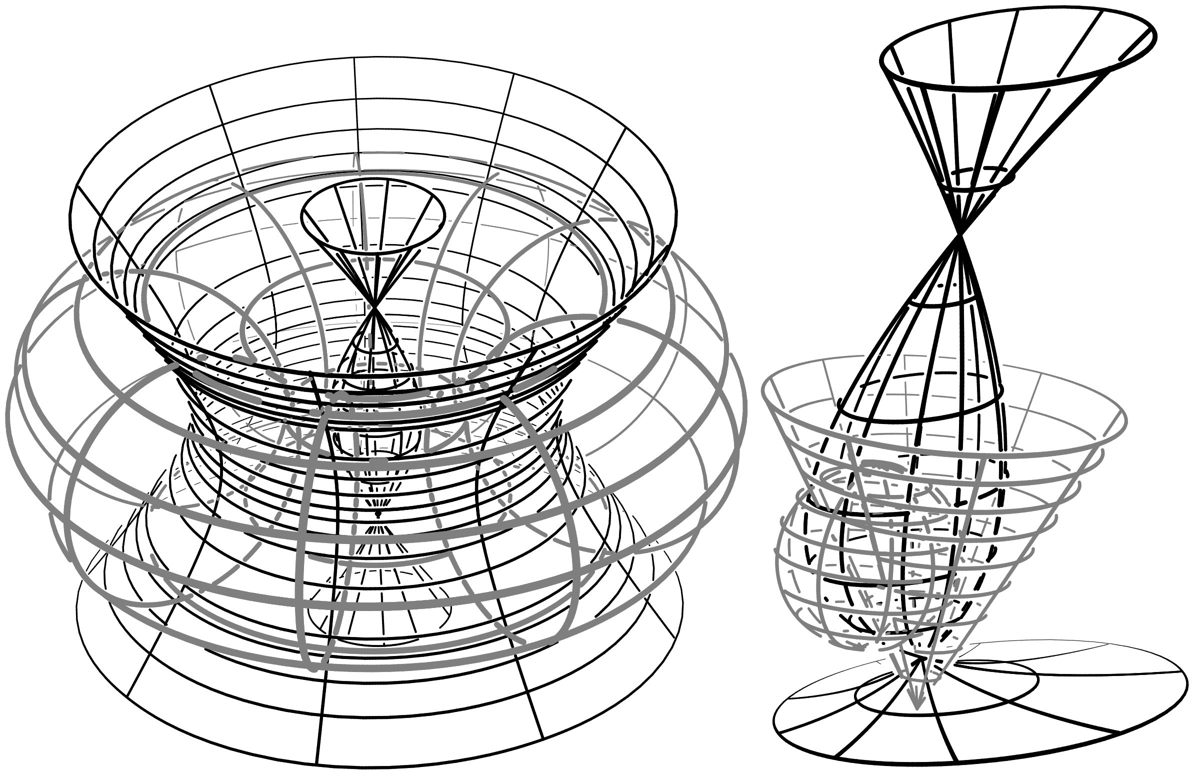
The bisector of a point and a rational surface in $\mathbb{R}^3$ is also rational [6]. Consider a fixed point $Q \in \mathbb{R}^3$ and a regular C^1 rational surface $S(u,v) \in \mathbb{R}^3$. Let $\mathcal{B}(u,v)$ be the bisector point of Q and $S(u,v)$. Then we have,

$$\left\langle \mathcal{B}(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle = 0, \quad (9)$$

$$\left\langle \mathcal{B}(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle = 0, \quad (10)$$

$$\|\mathcal{B}(u,v) - Q\| = \|\mathcal{B}(u,v) - S(u,v)\|. \quad (11)$$

The constraints in Equations (9)–(11) are also all *linear* in $\mathcal{B}(u,v)$. Using Cramer's rule again, we can solve these equations for $\mathcal{B}(u,v) = (b_x(u,v), b_y(u,v), b_z(u,v))$ and compute a rational surface representation of $\mathcal{B}(u,v)$. The resulting bisector surface follows the parameterization of the original surface. Figure 2(a) shows the rational bisector surface of a torus and a point located at the center of the torus.



(a) (b)

Figure 2: (a) The bisector of a torus and a point at the center of the torus, in $\mathbb{R}^3$. (b) The bisector of a cone and a sphere in $\mathbb{R}^3$. Original surfaces are shown in gray. Both bisector surfaces are infinite.

2.3 Special Cases of Surface-Surface Bisectors in $\mathbb{R}^3$

In general, the bisector of two rational surfaces is non-rational in $\mathbb{R}^3$, as we have already noted. However, there are some special cases where the bisector surface is rational. For example, when one of the initial surfaces is a sphere, the problem reduces to finding the bisector of a point and an offset surface. Thus, the bisector is rational when the offset surface is rational. This special case is discussed in Section 2.3.1. Moreover, when the two surfaces are given as surfaces of revolution sharing a common axis of rotation, the problem reduces to finding the planar bisector of the generating curves of the two surfaces. The bisector surface is rational if and only if the bisector of two generating curves is rational. This special case is discussed in Section 2.3.2. The bisector of two conic surfaces sharing the same apex is closely related to the bisector of two spherical curves; Section 2.3.3 considers the bisectors of points and curves on the unit sphere. A plane is a special case of a cone with $\frac{\pi}{2}$ as its spanning angle. Moreover, the set of all planes is closed under the offset operation. Section 2.3.4 combines the results of Sections 2.3.2 and 2.3.3 to compute the line-plane and cone-plane bisectors.

2.3.1 Sphere-Surface Bisectors in $\mathbb{R}^3$

In Section 2.2 we showed that the bisector of a point and a rational surface in $\mathbb{R}^3$ is a rational surface; this immediately implies that the bisector of a sphere and a surface with a rational offset is also a rational surface. Simultaneously offsetting both varieties by the same distance does not change the bisector of the two varieties. Figure 2(b) shows the bisector surface of a sphere and a cone computed by offsetting the cone by the radius of the sphere.

Pottmann [17] classified the class of all rational curves and surfaces that admit rational offsets. An important subclass of all polynomial curves having rational offsets includes the Pythagorean Hodograph (PH) curves [9]. Simple surfaces (that is, planes, spheres, cylinders, cones, and tori), Dupin cyclides, rational canal surfaces, and non-developable rational ruled surfaces, all belong to this special class of rational surfaces with rational offsets [3, 15, 18]. Thus, our results can be used to construct a wide range of bisectors in $\mathbb{R}^2$, where one curve is a circle and the other is a rational curve having rational offsets, and in $\mathbb{R}^3$, where one surface is a sphere and the other is a rational surface having rational offsets.

Even the simple rational bisector of two spheres, or the bisector of a point and a sphere, has many important applications in practice. The bisector of two spheres of different radii can be used for finding an optimal path of a moving object (e.g., an airplane) which attempts to avoid radar detection. Different radar devices have different intensities, and thus their regions of influence may be modeled by spheres of different radii. The optimal path lies on the bisector surface of the spheres.

2.3.2 Special Cases of Simple Surfaces with Rational Bisectors in $\mathbb{R}^3$

Dutta and Hoffmann [2] considered the bisectors of simple surfaces (CSG primitives), such as natural quadrics and tori, in particular configurations. Note that these CSG primitives are surfaces of revolution, which can be generated by rotating lines or circles about an axis of rotation. When two primitives share the same axis of rotation, their bisector construction essentially reduces to that of the generating curves of two primitives. The bisectors of lines and circles are conics, which are rational. Thus, the bisector of two primitives sharing the same axis of rotation is a rational quadratic surface of revolution.

We can extend this result to a slightly more general case. Consider a rational surface of revolution generated by a planar curve with a rational offset. When the axis of rotation is identical with that of a torus (or a sphere), the bisector of the surface of revolution and the torus (or the sphere) is a rational surface of revolution. This is because the bisector of a circle and a planar rational curve with a rational offset is the same as the bisector of the center of the circle and the rational offset curve; therefore the latter curve is also rational. Peternell [16] showed that the bisector of a line and a rational curve with a rational offset is also a rational curve. Similar arguments also apply to the cylinder, cone, and plane, when the axis of rotation is shared with the surface of revolution.

Dutta and Hoffmann [2] also considered the bisector of two cylinders of the same radius, and the bisector of two parallel cylinders. The bisector of two cylinders of the same radius is the same as the bisector of their axes, which is a hyperbolic paraboloid and therefore rational. Moreover, the bisector of two parallel cylinders is a cylindrical surface which is obtained by linearly extruding the bisector of two circles. Thus, the bisector of two parallel cylinders is an elliptic or hyperbolic cylinder, which is also rational.

Again, we can slightly extend this result. Consider two rational canal surfaces obtained by sweeping a sphere (of a fixed radius) along two rational space curves. The bisector of these canal surfaces is the same as that of their skeleton space curves, which is a rational surface. Moreover, two parallel cylindrical rational surfaces have a rational bisector surface if their cross-sectional curves have a rational bisector curve. In particular, when one cross-section is a circle and the other cross-section is a planar rational curve with a rational offset, the bisector must be a rational cylindrical surface.

2.3.3 Bisectors on the Unit Sphere S^2

Consider two conic surfaces that share the same apex. Their bisector surface is another conic surface with the same apex, which we may assume to be located at the origin. Thus the conic surfaces are ruled surfaces with their directrix curves fixed at the origin. The intersection of these conic surfaces with the unit sphere S^2 generates spherical curves; the curve corresponding to the bisector surface is indeed the bisector of the two spherical curves obtained from the original conic surfaces. Thus, the bisector curve construction on S^2 is equivalent to the bisector surface construction for two conic surfaces sharing the same apex. In the present section we consider the construction of bisector curves on S^2.

Given two points P and Q on S^2, let their spherical (geodesic) distance $\rho(P,Q)$ on S^2 be the angle between P and Q: $\rho(P,Q) = \arccos \langle P, Q\rangle$, where P and Q are two unit vectors. Consequently, for three points $P, Q, R \in S^2$, we have $\rho(P,Q) = \rho(P,R)$, if and only if $\langle P, Q\rangle = \langle P, R\rangle$. Let $Q \in S^2$ be a point and $C(t) \in S^2$ be a regular C^1 rational spherical curve. Their spherical bisector curve $\mathcal{B}(t) \in S^2$ must satisfy the following three constraints:

$$\begin{aligned} \langle \mathcal{B}(t), Q\rangle &= \langle \mathcal{B}(t), C(t)\rangle, & (12)\\ \left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt}\right\rangle &= 0, & (13)\\ \langle \mathcal{B}(t), \mathcal{B}(t)\rangle &= 1. & (14) \end{aligned}$$

Equation (12) locates the bisector curve $\mathcal{B}(t)$ at an equal spherical geodesic distance from Q and $C(t)$. Since the normal plane $\mathcal{P}_n(t)$ of a spherical curve $C(t) \in S^2$ contains the origin, it intersects S^2 in a great circle that is orthogonal to $C(t)$. Equation (13) implies that the bisector point is contained in the normal plane $\mathcal{P}_n(t)$. Finally, Equation (14) constrains the bisector curve to the unit sphere S^2.

Unfortunately, Equation (14) is quadratic in $\mathcal{B}(t)$; thus the spherical curve is, in general, non-rational. Fortunately, the ruling directions of conic surfaces may be represented by nonunit vectors. Thus, for the construction of rational direction curves, we replace the unitary condition of Equation (14) by the following *linear* equation:

$$\langle \mathcal{B}(t), (0,0,1)\rangle = 1. \quad (15)$$

Equation (15) constrains the bisector curve to the plane $Z = 1$. Equations (12), (13), and (15) form a system of three linear equations in $\mathcal{B}(t)$, whose solution is a rational curve on the plane $Z = 1$, which we denote as $\overline{\mathcal{B}}(t)$. Normalizing $\overline{\mathcal{B}}(t)$, we obtain a spherical bisector curve: $\mathcal{B}(t) = \overline{\mathcal{B}}(t)/\|\overline{\mathcal{B}}(t)\| \in S^2$. Because of the square root in the denominator, the bisector curve $\mathcal{B}(t) \in S^2$ will be, in general, non-rational.

Given two regular C^1 rational curves $C_1(u)$ and $C_2(v)$ on S^2, their bisector curve $\mathcal{B}(u(v)) \in S^2$ must satisfy the following three conditions:

$$\begin{aligned}
\langle \mathcal{B}(u(v)), C_1(u) - C_2(v) \rangle &= 0, & (16)\\
\langle \mathcal{B}(u(v)) - C_1(u), C_1'(u) \rangle &= 0, & (17)\\
\langle \mathcal{B}(u(v)) - C_2(v), C_2'(v) \rangle &= 0. & (18)
\end{aligned}$$

Equation (16) is the constraint of equal distance. Equations (17) and (18) imply that the bisector is simultaneously on the normal planes of the two curves. All three planes pass through the origin and they intersect, in general, only at the origin. However, there is a singular case where the three planes intersect in a line and their normal vectors are coplanar:

$$\lambda(u,v) = \begin{vmatrix} C_1(u) - C_2(v) \\ C_1'(u) \\ C_2'(v) \end{vmatrix} = 0. \qquad (19)$$

In fact, it is a necessary and sufficient condition for a bisector point $\mathcal{B}(u(v)) \in S^2$ to have its foot points at $C_1(u)$ and $C_2(v)$ [7]. The bisector point $\mathcal{B}(u(v)) \in S^2$ is then computed as one of the intersection points between the line and the unit sphere. Because of this extra constraint $\lambda(u,v) = 0$, the spherical bisector curve is, in general, non-rational (see also Elber and Kim [5]). However, the spherical bisector curve of two circles on S^2 is an interesting special case which allows a rational bisector.

In a slightly more general case, let us assume that one curve $C_1(u)$ is a circle and the other curve $C_2(v)$ has a rational spherical offset (e.g. a circle on the sphere). Then the curve-curve bisector on the unit sphere is the same as the bisector of a point and an offset curve on S^2. To obtain this bisector, we first offset both curves on S^2 until the circular offset degenerates to a point, and then solve this simplified system of equations for the spherical point-curve bisector. Using this technique, we can reduce the spherical circle-circle bisectors to the spherical point-circle bisectors.

2.3.4 Line-Plane and Cone-Plane Bisectors

A plane is a special case of a circular cone with $\frac{\pi}{2}$ as its spanning angle. Moreover, the set of all planes is closed under offsetting. Based on these two properties, and by combining results discussed in Sections 2.3.2 and 2.3.3, we can construct the line-plane and cone-plane bisectors.

Consider the bisector of a line $\mathcal{L}$ and a plane $\mathcal{P}$. Without loss of generality, we may assume that $\mathcal{P}$ is the XY-plane and $\mathcal{L}$ intersects $\mathcal{P}$ at the origin. (We assume that $\mathcal{P}$ and $\mathcal{L}$ are not parallel, since the parallel case reduces to the point-line bisector.) Let $Q = \mathcal{L} \bigcap S^2$ and $C(t) = \mathcal{P} \bigcap S^2$ be a point and a great circle, respectively, both on S^2. Moreover, let $\overline{\mathcal{B}}(t)$ be the bisector of Q and $C(t)$ on the plane $Z = 1$. Then, the bisector surface of $\mathcal{L}$ and $\mathcal{P}$ is given by

$$\mathcal{B}(t,r) = r\overline{\mathcal{B}}(t), \qquad r \in \mathbb{R}.$$

Next we consider the bisector of a circular cone $\mathcal{C}$ and a plane $\mathcal{P}$. Without loss of generality, we may assume that $\mathcal{P}$ is the XY-plane and that the apex of the circular cone $\mathcal{C}$ is located at the origin. Let $C_1(u) = \mathcal{C} \bigcap S^2$ and $C_2(t) = \mathcal{P} \bigcap S^2$ be a circle and a great circle, respectively, both on S^2. Moreover, let $\overline{\mathcal{B}}(t)$ be the bisector of $C_1(u)$ and $C_2(t)$ on the plane $Z = 1$. (Note that the bisector curve is constructed by the spherical offset technique discussed at the end of Section 2.3.3.) Then, the bisector surface of $\mathcal{C}$ and $\mathcal{P}$ is again given by

$$\mathcal{B}(t,r) = r\overline{\mathcal{B}}(t), \qquad r \in I\!R.$$

If the apex of the cone $\mathcal{C}$ is not contained in $\mathcal{P}$, we can offset both the cone and the plane until the apex is contained in $\mathcal{P}$. A translation moves both varieties so that the new apex is now located at the origin. All cone-plane bisectors can thus be reduced to the standard form discussed above. Note that the same technique can be applied to non-circular cones $\mathcal{C}$ as well if their spherical curves $\mathcal{C} \cap S^2$ have rational spherical offsets.

3 Bisectors in Higher Dimensions

We now examine the existence of rational bisectors in higher dimensions. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties of dimensions d_1 and d_2, respectively, both in $I\!R^d$. The bisector $\mathcal{B}$ of $\mathcal{V}_1$ and $\mathcal{V}_2$ must be located in the normal subspaces of the two varieties. Hence, there are $d_1 + d_2$ orthogonality constraints to be considered. The bisector must, of course, also be at an equal distance from the two varieties, so there are in total $d_1 + d_2 + 1$ linear constraints. When the two varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are in general position, their bisector $\mathcal{B}$ has a rational representation if

$$d_1 + d_2 + 1 \leq d.$$

For example, consider two curves in $I\!R^3$. Each curve contributes one orthogonality constraint; that is, the bisector must be contained in the normal plane of each curve. Together with the requirement of equidistance from two input curves, the total number of constraints is three, which is equal to the dimension of the space. Thus, the bisector has a rational representation.

In contrast, a bivariate surface imposes two orthogonality constraints; namely that the bisector of two surfaces must be contained in the normal line of each. Including equidistance, the total number of constraints is therefore five. Hence the bisector of two bivariate surfaces has a rational representation in $I\!R^d$, for $d \geq 5$, but not in $I\!R^3$. Similarly, the bisector of a bivariate surface and a univariate curve has a rational representation in $I\!R^d$, for $d \geq 4$, but not in $I\!R^3$.

The bisector curve of two curves in $I\!R^2$, the bisector surface of a curve and a surface in $I\!R^3$, and the bisector of two surfaces in $I\!R^3$ are all, in general, non-rational; therefore we need to approximate them numerically. Methods for approximating the bisectors of two curves were presented by Farouki and Ramamurthy [11] and by Elber and Kim [5]. Additionally, methods for approximating the bisector of two surfaces or that of a curve and a surface in $I\!R^3$ were recently proposed by the latter authors [8].

4 α-Sectors

By definition, the shortest distances from a bisector point to the two varieties being bisected are always equal. Consider an intermediate surface with weighted distances from the two varieties,

$$\alpha\|\mathcal{B} - \mathcal{V}_1\| = (1-\alpha)\|\mathcal{B} - \mathcal{V}_2\|, \tag{20}$$

where $0 \leq \alpha \leq 1$. We denote the locus of points that are at relative distances α and $(1-\alpha)$ from the two varieties as the *α-sector*. Unfortunately, the square of Equation (20) is linear in $\mathcal{B}$ only for $\alpha = \frac{1}{2}$. Nevertheless, there is a nice property that the two special α-sectors are identical with the original varieties when $\alpha = 0$ or $\alpha = 1$. Note that the α-sector reduces to the bisector when $\alpha = \frac{1}{2}$.

The ability to change α continuously could be a useful tool in a range of applications, e.g., to produce metamorphosis between two freeform shapes. In the next sections we consider a few simple examples of the α-sectors of two varieties. While Equation (20) is quadratic, we later 'linearize' this constraint and introduce the *pseudo α-sector* which is simple to represent as a rational function.

4.1 The Point-Line α-Sector in $\mathbb{R}^2$

We may assume without loss of generality that the line is the Y-axis, that is, the parametric line $C(t) = (0, t)$, and that the point is $Q = (1, 0)$. We choose α so that $\alpha = 0$ corresponds to the line and $\alpha = 1$ corresponds to the point.

The α-sector $\mathcal{B} = (b_x, b_y)$ between the Y-axis and the point Q satisfies the line-orthogonality constraint

$$0 = \left\langle \mathcal{B} - C(t), \frac{dC(t)}{dt} \right\rangle = \langle (b_x, b_y) - (0, t), (0, 1) \rangle = b_y - t, \tag{21}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + b_y^2 \right) = (1-\alpha)^2 (b_x^2 + (b_y - t)^2). \tag{22}$$

Solving Equations (21) and (22) and replacing (b_x, b_y) with (x, y), we obtain the quadratic curve

$$\left(\frac{2\alpha - 1}{\alpha^2} \right) x^2 + y^2 - 2x + 1 = 0. \tag{23}$$

Figure 3 shows the α-sectors of the line $(0, t)$ and the point $(1, 0)$ for various different values of α. When $\alpha < \frac{1}{2}$, the coefficients of x^2 and y^2 have opposite signs, and so the α-sector is a hyperbola. When $\alpha = \frac{1}{2}$, the coefficient of x^2 vanishes, and so the bisector is a parabola. When $\alpha > \frac{1}{2}$, the coefficients of x^2 and y^2 have the same sign, and so the α-sector is an ellipse.

4.2 The Point-Plane α-Sector in $\mathbb{R}^3$

A similar α-sector exists for a point and a plane in three dimensions. We may assume without loss of generality that the plane is the YZ-plane, that is, the parametric plane $S(u, v) = (0, u, v)$, and that the point is $Q = (1, 0, 0)$. We choose α such that $\alpha = 0$ corresponds to the plane and $\alpha = 1$ corresponds to the point.

Let $\mathcal{B} = (b_x, b_y, b_z)$ be the α-sector of $S(u, v)$ and Q. As in the two-dimensional case we have the two plane-orthogonality constraints

$$0 = \left\langle \mathcal{B} - S(u, v), \frac{\partial S(u, v)}{\partial u} \right\rangle = \langle (b_x, b_y, b_z) - (0, u, v), (0, 1, 0) \rangle = b_y - u, \tag{24}$$

$$0 = \left\langle \mathcal{B} - S(u, v), \frac{\partial S(u, v)}{\partial v} \right\rangle = \langle (b_x, b_y, b_z) - (0, u, v), (0, 0, 1) \rangle = b_z - v, \tag{25}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + b_y^2 + b_z^2 \right) = (1-\alpha)^2 (b_x^2 + (b_y - u)^2 + (b_z - v)^2). \tag{26}$$

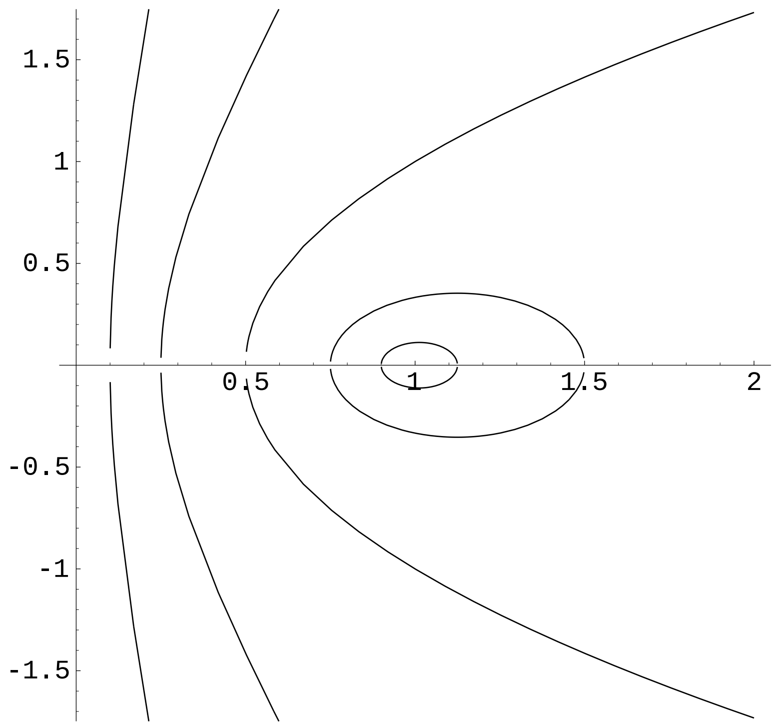


Figure 3: The α-sectors of the point (1, 0) and the line (0, t) for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$.

Solving Equations (24)–(26) and replacing (b_x, b_y, b_z) with (x, y, z), we obtain the quadratic surface

$$\left(\frac{2\alpha - 1}{\alpha^2}\right) x^2 + y^2 + z^2 - 2x + 1 = 0. \tag{27}$$

This is a hyperboloid of two sheets for $0 < \alpha < \frac{1}{2}$, an elliptic (circular) paraboloid for $\alpha = \frac{1}{2}$, and an ellipsoid for $\frac{1}{2} < \alpha < 1$.

4.3 The Line-Line α-Sector in $\mathbb{R}^3$

Yet another simple example is the α-sector of two straight lines $C_1(u) = (1, u, 0)$ and $C_2(v) = (0, 0, v)$. We choose α such that $\alpha = 0$ corresponds to $C_2(v)$ and $\alpha = 1$ corresponds to $C_1(u)$. Now let $\mathcal{B} = (b_x, b_y, b_z)$ be the α-sector of $C_1(u)$ and $C_2(v)$, and we have the two line-orthogonality constraints

$$0 = \left\langle \mathcal{B} - C_1(u), \frac{dC_1(u)}{du} \right\rangle = \langle (b_x, b_y, b_z) - (1, u, 0), (0, 1, 0) \rangle = b_y - u, \tag{28}$$

$$0 = \left\langle \mathcal{B} - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = \langle (b_x, b_y, b_z) - (0, 0, v), (0, 0, 1) \rangle = b_z - v, \tag{29}$$

and the distance constraint

$$\alpha^2 \left((b_x - 1)^2 + (b_y - u)^2 + b_z^2 \right) = (1 - \alpha)^2 (b_x^2 + b_y^2 + (b_z - v)^2). \tag{30}$$

The solution of Equations (28)–(30) is the quadratic surface

$$\left(\frac{2\alpha - 1}{\alpha^2}\right) x^2 - \left(\frac{1 - \alpha}{\alpha}\right)^2 y^2 + z^2 - 2x + 1 = 0. \tag{31}$$

Thus the $\frac{1}{2}$-sector (bisector) of $C_1(u)$ and $C_2(v)$ is the surface

$$y^2 - z^2 + 2x - 1 = 0,$$

whose parametric form is given as $(\frac{1-u^2+v^2}{2}, u, v)$. This confirms the result of [4, §2.2].

When $\alpha = \frac{1}{2}$, Equation (31) yields a hyperbolic paraboloid. Otherwise, when $0 < \alpha < 1$, but $\alpha \neq \frac{1}{2}$, it yields a hyperboloid of one sheet, which reduces to a line for $\alpha = 0$ or 1. However, the α-sector of two general rational curves in $\mathbb{R}^3$ is usually a non-rational surface.

4.4 The Pseudo α-Sector

In the case of a spherical bisector, we resorted to the linear constraint $Z = 1$. Similarly, we now seek a linear constraint that replaces the quadratic L_2-norm of Equation (20) while yielding similar properties to the α-sector in constraining the relative distances to the two given varieties. We choose the plane that is at relative distances of α and $(1-\alpha)$ from the closest point on each variety.

For example, for the pseudo α-sector of a curve $C(t)$ and a point Q in $\mathbb{R}^2$, we impose the two linear constraints

$$\left\langle \mathcal{B}(t) - C(t), \frac{dC(t)}{dt} \right\rangle = 0, \tag{32}$$

$$\langle \mathcal{B}(t) - (\alpha Q + (1-\alpha)C(t)), C(t) - Q \rangle = 0. \tag{33}$$

Equation (32) is the regular orthogonality constraint, and Equation (33) ensures that the bisector is on the plane containing the point $\alpha Q + (1-\alpha)C(t)$ and orthogonal to the vector $C(t) - Q$. If $C(t)$ has a rational representation, we can easily use Cramer's rule to obtain a rational representation for $\mathcal{B}(t) = (b_x(t), b_y(t))$.

Figure 4 shows three examples of planar pseudo α-sectors of: (i) a point and a line (Figure 4(a)), (ii) a point and a cubic curve (Figure 4(b)), and (iii) a point and a circle (Figure 4(c)). These examples were all created using the IRIT solid-modeling environment [12].

The extension to $\mathbb{R}^3$ follows the same guidelines. The pseudo α-sector of two curves $C_1(u)$ and $C_2(v)$ in $\mathbb{R}^3$ imposes the three linear constraints

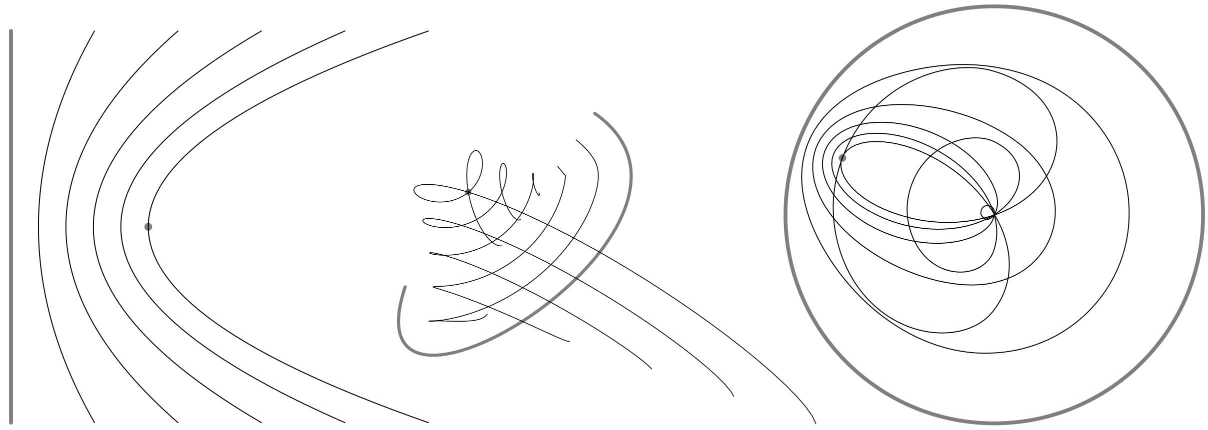
$$\left\langle \mathcal{B}(t) - C_1(u), \frac{dC_1(u)}{du} \right\rangle = 0, \tag{34}$$

$$\left\langle \mathcal{B}(t) - C_2(v), \frac{dC_2(v)}{dv} \right\rangle = 0, \tag{35}$$

$$\langle \mathcal{B}(t) - (\alpha C_1(u) + (1-\alpha)C_2(v)), C_1(u) - C_2(v) \rangle = 0. \tag{36}$$

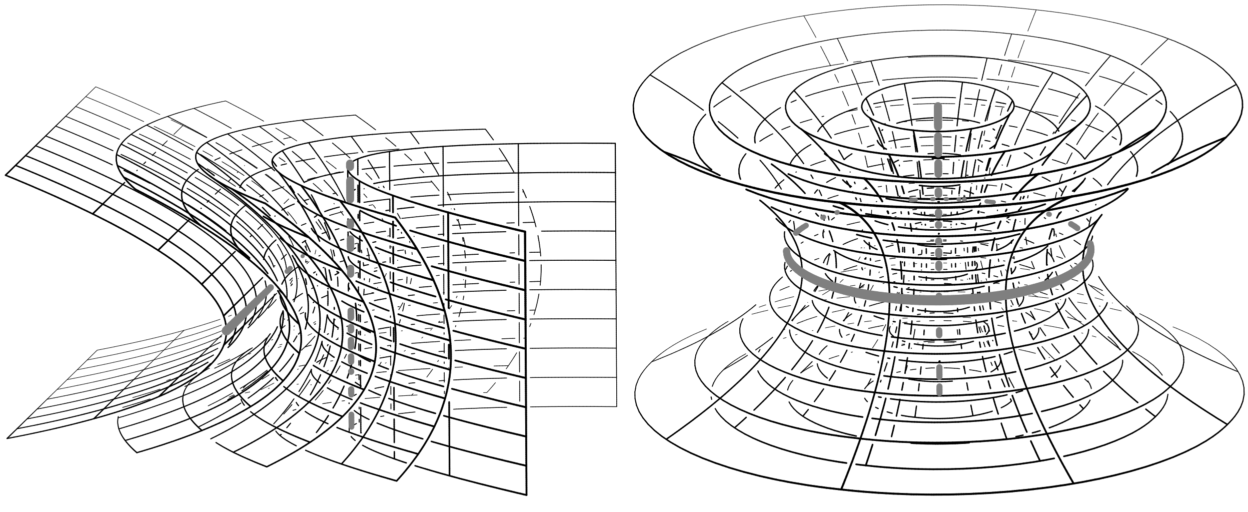
Again, if $C_1(u)$ and $C_2(v)$ have rational representations, we can use Cramer's rule to obtain a rational representation for $\mathcal{B}(t)$. Figure 5 shows two such pseudo α-sectors in $\mathbb{R}^3$, for (i) two lines (Figure 5(a)), and (ii) a line and a circle (Figure 5(b)).

The pseudo α-sector is identical with the α-sector only when $\alpha = \frac{1}{2}$; in that case, they are both equivalent to the bisector. Note also that the pseudo 0- and 1-sectors are only approximations to the original varieties. This is because of the approximate distance constraint: points on the pseudo α-sector do not satisfy the $\alpha : (1-\alpha)$ distance ratio; instead, this property constrains only their projections on the lines joining the respective points on the varieties.



(a) (b) (c)

Figure 4: (a) The pseudo α-sectors of a point and a line in $\mathbb{R}^2$ for $\alpha = 0.10,\ 0.25,\ 0.50,\ 0.75,\ 0.90$ (cf. Figure 3). (b) The α-sectors of a point and a cubic curve in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. (c) The α-sectors of a point and a circle in $\mathbb{R}^2$ for $\alpha = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$. The original curves and points are shown in gray.



(a) (b)

Figure 5: (a) The pseudo α-sectors of two lines in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. (b) The α-sectors of a line and a circle in $\mathbb{R}^3$ for $\alpha = 0.0,\ 0.25,\ 0.5,\ 0.75,\ 1.0$. The original curves are shown in gray.

5 Conclusion

In this paper we have examined various special cases for which rational bisectors exist. We showed constructively that the point-curve bisectors in $\mathbb{R}^2$, and all point-curve, point-surface, and curve-curve bisectors in $\mathbb{R}^3$, have rational representations. We have also considered some special cases where the surface-surface bisectors are rational.

Further, we describe the exact and pseudo α-sectors, extensions of the bisector that should be useful in various applications, such as metamorphosis between the pseudo α-sector.

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