
Figure 8: An example of a union shape with a concave “corner”.

but not to each other, see Figure 8. The boundary point is then a concave “corner”. The situation described above is, however, not possible under the assumptions, since a concave corner in the boundary corresponds to an axis of non trivial length. (The end points of the axis segment corresponding to a concave corner are located in a direction perpendicular to the boundary’s tangents from both sides). In such cases, we therefore have $l_a = r_a = 0$. From (6) it is evident that this implies that (11) is true with equality, in contradiction with the assumptions. □

References


two arc segments connecting them at both ends (the arc segments of the disks at the ends of the proposed infinitesimal axis). Next we show that the boundary described above is the boundary of the union shape. The boundary is closed. Furthermore, since each boundary point is on the boundary of at least one axis disk, none of the points of the boundary described above is outside the union shape. What remains to be shown is that none of the boundary points is inside the union shape, or equivalently, none of the points is inside any of the axis disks.

From Lemma 4, all proposed axis disks are tangent to the boundary, and the radius of curvature at every boundary point is larger than the radius of the axis disk touching it. Therefore, the constructed boundary points are outside the axis disks corresponding to neighboring boundary points. Since the axis segment is infinitesimal, we conclude that every point of the boundary is outside all other axis disks of the axis segment. Hence, the boundary constructed above is the boundary of the union shape.

Since all the disks of the proposed axis touch the boundary at two different points, they are all maximal disks in the union shape. The proposed axis function describes therefore a collection of centers of maximal disks. All that is left to be shown is that the collection is complete, and that there exists no other maximal disk in the union shape.

Assume the contrary is correct, and there is another maximal disk in the union shape. Being a maximal disk it has to either touch the boundary at two different points, or at one point in which case, its radius has to be equal to the radius of curvature at that point. In any case, the maximal disk has to be tangent to the the boundary at the touching points.

If an internal disk touches a point of the two arc segments connecting $\mathcal{L}$ with $\mathcal{R}$, its radius is bound to be less or equal to the radius of the axis disk at the corresponding end of the axis segment. Hence, such a disk may not be a maximal disk. If an internal disk touches a point of either $\mathcal{L}$ or $\mathcal{R}$, it has to be tangent to the boundary at the point. Also the axis disk corresponding to the same point is tangent to the boundary. Hence, either the new maximal disk contains the axis disk or vice versa. In both cases we get contradiction to the assumptions.

The only alternative allowing the two disks to be simultaneously maximal in the union shape is if at the relevant boundary point the boundary has a negative discontinuity of the tangent. In this case both disks may be "tangent" to the boundary
**proof:** First note that a boundary reconstruction is possible only for parametric functions obeying (10), otherwise the expressions in (3) do not exist.

From the reconstruction equations it is evident that every axis disk touches the reconstructed boundary at two points. The disk centered at $A(a)$ touches the left and right boundary segments at $L(a)$ and $R(a)$ respectively. The tangent vectors to the boundary segments in (5) are perpendicular to the directions of their respective axis disk radii. Those are given by substituting (3) into $\frac{1}{R}(L - A)$ and $\frac{1}{R}(R - A)$. Hence every axis ball is tangent to the boundary at two points. (The points are different or else $R_a = 1$).

To prove the second part of the lemma we have to show that the axis disk curvature ($\frac{1}{R}$) is larger then both boundary curvatures $K_L$ and $K_R$ in (8). If either of the boundary curvatures is negative, then the respective problem is solved. Therefore, we show that if $K_L > 0$ then $\rho_L > R$ and if $K_R > 0$ then $\rho_R > R$.

If

$$K_L = \frac{K_A \sqrt{1 - R_a^2 - R_{aa}}}{1 - R_a^2 - R_{aa} + K_A R_a \sqrt{1 - R_a^2}} > 0$$

then from restriction (11) the denominator is positive, and

$$K_A \sqrt{1 - R_a^2 - R_{aa}} > 0$$

(17)

We have to show that in this case $\rho_L > R$ or by substitution from (9)

$$\frac{1 - R_a^2}{K_A \sqrt{1 - R_a^2 - R_{aa}}} > 0$$

But this is a direct conclusion from (17) and the restriction (10).

A similar argument asserts that if $K_R > 0$ then $\rho_R > R$. □

**Proof of the theorem:** The first part of the theorem is: All axis functions obey (10) and (11). This part is directly derived from Lemmas 2 and 3.

The second part is: An infinitesimal axis like function $M$ obeying (10) and (11) with a strict inequality is a legal axis function. In order to show this, we have to show that $M$ is the axis of the union shape (1).

Let us first describe a shape by its boundary. The boundary is composed of the two boundary segments $L$ and $R$ defined by the reconstruction equations (3), and of
to the axis function. Consider the line segment connecting the axis point $A(a_0)$ to $L(a_0)$, the left boundary point corresponding to it according to (3). Now extend the line segment to a line (the dotted line in Figure 7), obtaining two half planes. By (5), the tangent $L_1$ to the $L$ at $a_0$, is pointing towards the same half plane as $A_a$ (the tangent to $A$). However, the derivative $L_a$ in (4) is pointing to the same side only if $l_a$ in (6) is positive.

![Figure 7: Boundary parameter is monotone in axis parameterization.](image)

Suppose, however, that $l_a < 0$. Then, for a sufficiently small $\Delta a$, the radius connecting the axis point at parameter $a_0 + \Delta a$ on one side of the line, to the boundary point $L(a_0 + \Delta a)$ on the other side of the line, intersects it, see Figure 7. According to Lemma 5 this means that either $L(a_0)$ is in the axis disk of parameter $a_0 + \Delta a$ or $L(a_0 + \Delta a)$ is in the axis disk of parameter $a_0$. Hence, either $L(a_0)$ or $L(a_0 + \Delta a)$ is not a boundary point, contradicting the assumption that the function is an axis function.

A similar argument can be made for $r_a < 0$. Note that if $(1 - R_a^2) - RR_{aa} < |K_a|R \sqrt{1 - R_a^2}$, as in the hypothesis of the lemma, then by (6) either $l_a < 0$ or $r_a < 0$. 

**Lemma 4** If a parametric function obeys restrictions (10) and (11) with a strict inequality, then any axis disk is tangent to a boundary reconstruction as in (3) on two points. The disk’s curvature in this case is larger than the reconstructed boundary’s curvature at those points.
Theorem  Every medial axis obeys (10) and (11). Consider an infinitesimally short 3D parametric $C^1$ function $(X(a), Y(a), R(a))$, with a the arc-length of $(X(a), Y(a))$. If that segment obeys (10) and (11) with a strict inequality, it is a legal medial axis segment.

We separate the theorem to a few lemmas.

Lemma 2  If in a parameter $a_0$ of the parametric function, $|R_a| > 1$, then the parametric function is not an axis function.

Proof: Suppose $R_a(a_0) > 1$, then for a sufficiently small $\Delta a$, we have $\frac{\Delta R}{\Delta a} > 1$, with $\Delta R = R(a_0 + \Delta a) - R(a_0)$. Since $\Delta R > \Delta a$, the axis disk of parameter $a_0$, is totally contained in the axis disk of parameter $a_0 + \Delta a$. See Figure 6.

A similar argument can be made for the assumption $R_a < -1$. 

Figure 6: If the radius function changes too quickly, the proposed function can not be an axis function.

Lemma 3  If in a parameter $a_0$ of the parametric function, $(1 - R_a^2) - RR_{aa} < |K_A|R\sqrt{1 - R_a^2}$, then the parametric function is not an axis function.

Proof: Suppose the contrary is correct, and the function is an axis function. By Lemma 1 the curves $\mathcal{L}$ and $\mathcal{R}$ in (3) are on the boundaries of the shape corresponding
Let us examine the distance $R + \Delta d$ from $A(a + \Delta a)$ to $L(a)$. By the cosine law,

$$(R + \Delta d)^2 = R^2 + \Delta a^2 - 2 R \Delta a \cos(180 - \phi)$$

removing second order infinitesimal terms, we get

$$\frac{\Delta d}{\Delta a} = \cos \phi$$

(16)

from (15) and (16)

$$\Delta d < \Delta R$$

Therefore, points lying on azimuth $\psi > \theta$ from $A_a$, are inside the axis disk of a neighboring points, contradicting the assumption that it is a boundary point of the shape.

A similar argument is valid for the assumption of an azimuth $\psi < \theta$. ■

**Lemma 5** Let $\overline{AB}$ and $\overline{CD}$ be two segments in the plane, intersecting at a point $E$. Then, either $B \in C_B^R$ or $D \in C_B^R$. Where $C_A^R$ denotes the circle centered at $A$ and passing through $B$, and $C_B^R$ denotes the circle centered at $C$ and passing through $D$.

**Proof:** We have to show (see Figure 5), that either $BC \leq CD$ or $DA \leq AB$. Suppose the contrary is true, i.e. $BC > CD$ and $DA > AB$. Adding the two inequalities we get $BC + DA > CD + AB = CE + ED + AE + EB = (CE + EB) + (DE + EA)$ in contradiction to the triangle inequality $BC \leq CE + EB$ and $DA \leq DE + EA$. ■

![Figure 5: If the radii intersect, at least one of their end points is inside the other circle.](image)
Appendix

Lemma 1 If a segment of a $C^1$ three dimensional parametric function is an axis segment, then the curves $L$ and $R$ of (3) are on the boundary of the shape that the axis describes.

Proof: Every axis disk touches the boundary of the shape at two points. Those points are $R(a)$ distant from the axis point $A(a)$. The only exceptions to the two point correspondence may be found at the axis end point. (Explanations about boundary axis point correspondences, may be found in [1], and more formally in [6] and [7]). What we shall show is that the azimuth of the boundary points on the disk, is the angle $\theta$ on both sides of the tangent $A_a$ to the axis curve, with $\theta$ as in (2).

Let us assume the contrary, and suppose that the azimuth of a boundary point is at angle $\psi$ from $A_a$, such that $\psi > \theta$ with $\cos \theta = R_a$. Since the tangent $A_a$ to the axis is continuous, for a sufficiently small $\Delta a$ also the angle $\phi$ of the azimuth from the infinitesimal line segment connecting axis points $a$ and $a + \Delta a$ will be, such that $\phi > \theta$. See Figure 4.

![Diagram](image_url)

Figure 4: The azimuth of the boundary from the axis point corresponding to it.

We have $\cos \phi < \cos \theta = R_a$. Since also $R_a$ is continuous, we can find a sufficiently small $\Delta a$, so that

$$\frac{\Delta R}{\Delta a} > \cos \phi$$

where $\Delta R = R(a + \Delta a) - R(a)$. 

11
5 Summary

In this paper, we dealt with a representation of planar shape boundaries, based on the curve representation of the medial axes of the shape. This representation has been shown efficient in calculating boundary features. The proposed scheme has also led to two local restrictions on axis functions. We have stated a theorem, asserting conditions that are necessary and locally sufficient for an axis function. Finally we have proposed a new skeletonization approach in which the medial axis is a solution of a first order system of ordinary differential equations.
This system should be enough to solve for the three unknown functions \( X(a) \), \( Y(a) \), and \( R(a) \).

Taking the difference of the equations in (3) we get
\[
\mathcal{L} - \mathcal{R} = 2R \sqrt{1 - R_a^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_a
\]
Since \( A_a \) is a unit vector, we are interested only in the directional information implied by the above equation
\[
A_a \perp \mathcal{L} - \mathcal{R}
\]  
(12)

Summing \( \mathcal{L} \) and \( \mathcal{R} \) from (3) we get
\[
\frac{2A - (\mathcal{L} + \mathcal{R})}{2R} = R_a A_a
\]
Considering the norm of the above vector equation we get
\[
R_a = \frac{\|2A - (\mathcal{L} + \mathcal{R})\|}{2R} 
\]  
(13)

By now we have obtained equations describing the axis from corresponding boundary information. All that is left to do is to make sure that the correspondence between the boundary segments, and between them and the axis segment, is maintained. The information about the changes in the correspondence may be extracted from (6). We substitute the axis terminology of (6), to boundary terminology using (8), thereby obtaining
\[
l_a = \sqrt{1 - R_a} \frac{1}{1 - K_LR} \quad r_a = \sqrt{1 - R_a} \frac{1}{1 - K_R R}
\]  
(14)

The initial condition for the axis is naturally located normal to the boundary's curvature maximum, at a distance of \( \frac{1}{K_M} \), where \( K_M \) is the maximal curvature, see Figure 2. The initial condition for the radius parameter is the local radius of curvature \( R(0) = \frac{1}{K_M} \). The same point of maximal curvature on the boundary, is the point where we “cut” the boundary, defining the two curves: \( \mathcal{L} \) to the left of the axis and \( \mathcal{R} \) to its right.

The equation system in (12), (13) and (14) may serve as the engine of the proposed skeletonization algorithm. The engine must be applied to the boundary via a control process. This process should find axis segments that cross. Handling a situation like this involves: Stopping the axis segments at the crossing point, checking what pair of boundary segments meet, and initiating a new axis segment driven by the remaining two boundary segments, see Figure 3.
• The arc-length sign in (6) should always be positive (or more accurately, the tangent of the boundary can not flip its direction). Hence the second restriction.

\[(1 - R_a^2) - RR_{aa} \geq |K_A| R \sqrt{1 - R_a^2}\]  (11)

Indeed the following lemmas are proven in the appendix.

**Lemma 2** If in a parameter \( a_0 \) of the parametric function, \( |R_a| > 1 \), then the parametric function is not an axis function.

**Lemma 3** If in a parameter \( a_0 \) of the parametric function, \( (1 - R_a^2) - RR_{aa} < |K_A| R \sqrt{1 - R_a^2} \), then the parametric function is not an axis function.

The next theorem formalizes the above two intuitive restrictions. The theorem is formally proven in the appendix.

**Theorem** Every medial axis obeys the above two restrictions (10) and (11). Consider an infinitesimally short 3D parametric \( C^1 \) function \((X(a), Y(a), R(a))\), with a the arc-length of \((X(a), Y(a))\). If that segment obeys (10) and (11) with a strict inequality, it is a legal medial axis segment.

The proof of the theorem is based on Lemmas 2, 3, and 4:

**Lemma 4** If a parametric function obeys restrictions (10) and (11) with a strict inequality, then any axis disk is tangent to a boundary reconstruction as in (3) at two points. The disk's curvature in this case is larger than the reconstructed boundary's curvature at those points.

4 **Skeletonization**

Skeletonization is a procedure that finds the medial axis of a given shape. It is also sometimes referred to as “the medial axis transformation”. We propose that the axis function \( M \) of a shape is a solution of a system of ordinary differential equations. Note that the reconstruction formulae (3) constitute a system of four first order equations having the left and right boundary segment coordinates as inputs.
If we differentiate (5) with respect to \( a \), and divide the result by (6), we get the second derivatives of the boundary:

\[
\mathcal{L}_{ll} = \frac{K_A \sqrt{1-R_a^2} - R_{aa}}{1-R_a^2 - RR_{aa} + K_A R \sqrt{1-R_a^2}} \left( -\frac{R_a}{\sqrt{1-R_a^2}} \frac{\sqrt{1-R_a^2}}{R_a} \right) A_a
\]

\[
\mathcal{R}_{rr} = \frac{-K_A \sqrt{1-R_a^2} - R_{aa}}{1-R_a^2 - RR_{aa} - K_A R \sqrt{1-R_a^2}} \left( \frac{R_a}{\sqrt{1-R_a^2}} - \frac{\sqrt{1-R_a^2}}{R_a} \right) A_a
\]

From (7) we can get the curvature of the left and right boundary segments, \( K_L \) and \( K_R \) respectively. The curvature is the length of the second derivative of the curve with respect to its arc-length. Hence

\[
K_L = \frac{K_A \sqrt{1-R_a^2} - R_{aa}}{1-R_a^2 - RR_{aa} + K_A R \sqrt{1-R_a^2}} \quad K_R = -\frac{K_A \sqrt{1-R_a^2} + R_{aa}}{1-R_a^2 - RR_{aa} - K_A R \sqrt{1-R_a^2}}
\]

The radius of curvature of a boundary, is the inverse of its curvature \( \rho = \frac{1}{K} \); therefore

\[
\rho_L = R + \frac{1 - R_a^2}{K_A \sqrt{1-R_a^2} - R_{aa}} \quad \rho_R = R - \frac{1 - R_a^2}{K_A \sqrt{1-R_a^2} + R_{aa}}
\]

The above results are not new. They correspond to results obtained by Blum and Nagel [2] who derived them geometrically. Nevertheless, the technique presented here also enables the derivation of other (more complex) boundary features that do not have a simple geometric meaning, such as derivatives of the curvature or higher order derivatives of the boundary.

3 Local Restrictions on the Medial Axis

As mentioned in the introduction, the way to verify that an axis-like function is indeed an axis of a shape, is a global problem involving reconstruction of the shape via either (1) or (3). The question raised in this section is whether local restrictions on the axis function can help us reject some of the candidates without relying on global verification. Two such local restrictions are intuitively apparent:

- The cosine in (2) is bounded to \([-1, 1]\). Hence the first restriction.

\[
|R_a| \leq 1
\]
always continuous, the axis functions \( R(a) \) and \( A(a) \) always are. Approaching a discontinuity at \( a_0 \) from one side of the axis, the boundary reaches a certain point in a direction corresponding to the limit value \( (a \to a_0) \) of \( R_a(a) \) and \( A_a(a) \) from that side. While approaching from the other side, the boundary similarly reaches a different point. Both points are, however, on the same circle of radius \( R(a_0) \) centered on \( A(a_0) \). The gap between the two points is completed by a circular arc segment of this circle.

### 2.2 Boundary Features from Medial Axis Description

Having the boundary description all boundary features are easily derived. The derivatives of (3) with respect to \( a \), are

\[
\mathcal{L}_a = \left( \frac{1 - R_a^2 - RR_{aa}}{\sqrt{1 - R_a^2}} + K_A R \right) \left( \frac{\sqrt{1 - R_a^2}}{R_a} - \frac{R_a}{\sqrt{1 - R_a^2}} \right) A_a
\]

\[
\mathcal{R}_a = \left( \frac{1 - R_a^2 - RR_{aa}}{\sqrt{1 - R_a^2}} - K_A R \right) \left( \frac{\sqrt{1 - R_a^2}}{R_a} + \frac{R_a}{\sqrt{1 - R_a^2}} \right) A_a
\]

Where \( K_A \) is the curvature of the axis curve \( A \). The unit tangent vectors to \( \mathcal{L} \) and \( \mathcal{R} \) are the derivatives of each curve segment with respect to its own arc-length, \( (l \text{ and } r \text{ respectively}) \). From (4) it is easy to derive the left and right arc-length information.

Note that by the chain rule: \( \mathcal{L}_a = \mathcal{L}_l \cdot l_a \) and \( \mathcal{R}_a = \mathcal{R}_r \cdot r_a \). \( \mathcal{L}_l \) and \( \mathcal{R}_r \) are both unit vectors, and \( l_a \) and \( r_a \) are the scalar length coefficients of \( \mathcal{L}_a \) and \( \mathcal{R}_a \) respectively.

In (4), the vector part and the magnitude part are easy to separate. Note that \( A_a \) multiplied by a rotation matrix is of unit length, and therefore:

\[
\mathcal{L}_l = \left( \frac{\sqrt{1 - R_a^2}}{R_a} - \frac{R_a}{\sqrt{1 - R_a^2}} \right) A_a \quad \mathcal{R}_r = \left( \frac{\sqrt{1 - R_a^2}}{R_a} + \frac{R_a}{\sqrt{1 - R_a^2}} \right) A_a
\]

The derivatives of the mappings of axis arc-length into boundary arc-lengths, are the scalar magnitude parts of (4).

\[
l_a = \left( \frac{1 - R_a^2 - RR_{aa}}{\sqrt{1 - R_a^2}} + K_A R \right) \quad r_a = \left( \frac{1 - R_a^2 - RR_{aa}}{\sqrt{1 - R_a^2}} - K_A R \right)
\]
After removing second order infinitesimal terms:

$$\Delta R = \Delta a \cos \theta$$

which in the limit, results in (2).

2.1 Boundary from Medial Axis Description

Each axis point usually corresponds to two boundary points. Hence, each axis segment corresponds to two boundary segments, \( L(a) \) and \( R(a) \) located on the left side and on the right side of the medial axis respectively. The boundary point corresponding to point \( a \) on the medial axis, is located in a distance \( R(a) \) from the axis curve point \( A(a) \). The azimuth of the boundary point is \( \theta \) degrees from the direction of the tangent \(-A_a(a)\) to the curve part at \( a \). Here \( \theta \) is determined by (2). Hence, the following reconstruction formula

\[
L = A - R \left( \frac{R_a}{\sqrt{1 - R_a^2}} \sqrt{1 - R_a^2} \right) A_a \tag{3}
\]

\[
R = A - R \left( \frac{R_a}{\sqrt{1 - R_a^2}} \right) A_a \tag{3}
\]

Note that \( A_a \) is a unit vector indicating a direction tangent to \( A \). The matrix multiplying it from the left is a unit size rotation matrix, rotating \( A_a \) by \( \theta = \arccos R_a \) degrees in the clockwise direction for \( L \) (or counter clockwise for \( R \)). The directions obtained are the directions of \( L(a) \) and \( R(a) \) from the axis point \( A \). The distance to the boundary is the radius value \( R(a) \).

The reconstruction formula (3) is formalized in the following lemma.

**Lemma 1** If a segment of a \( C^1 \) three dimensional parametric function is an axis segment, then the curves \( L \) and \( R \) of (3) are on the boundary of the shape that the axis describes.

The lemma is proved in the appendix. Relying on the above explanation of (3) the proof concentrates on proving (2).

The only exception to this reconstruction rule is the case when either of the derivatives \( R_a(a) \) or \( A_a(a) \) is not continuous. While the axis derivatives are not
2 Reconstruction

Suppose we have a medial axis of a certain shape and we want to reconstruct the shape. We can obtain the shape using (1). This would give us an area description of the shape, whose boundary we seek. We can, however, reconstruct a boundary description of the shape directly from the axis description. This is made possible using a result by Blum.

In [1], [2], Blum asserted that each boundary point has at least one medial axis disk tangent to it, and that generally, each medial axis disk is tangent to the shape boundary in two points. Of the two points, one is located on the left of the medial axis and the other on its right. Blum also indicated that each of the two points is located at an angle \((180^\circ - \theta)\) from the tangent to the axis so that

\[
R_a(a) = \cos \theta
\]  

(2)

Let us examine this statement in an intuitive way. A proof can be found in the appendix. Consider an infinitesimal axis segment as depicted in Figure 1. The lower line segment in Figure 1 is the infinitesimal axis segment, and the upper line segment is a corresponding segment of the boundary. The lines connecting the ends of the segments are the radii corresponding to the axis segments end points. Since the upper right triangle is nearly right angle, we can approximate

![Diagram of an axis segment and boundary](image)

Figure 1: The azimuth from the axis to the boundary.

\[
D \approx \sqrt{\Delta l^2 + (R + \Delta R)^2} \approx R + \Delta R
\]

and by Pythagoras:

\[
D^2 = (R + \Delta R)^2 = R^2 \sin^2 \theta + (R \cos \theta + \Delta a)^2
\]
in some cases it is possible to detect an illegal axis function by local inspection of the axis. We present two local conditions for an axis function to be legal, and prove that they are the only necessary local conditions. All further investigations would have to be of a global nature, involving some kind of shape or boundary reconstruction.

The next three sections address the three questions raised above, though in different order. In Section 2 we address the problem of boundary reconstruction, and show how we can use the new boundary representation to extract boundary features from the axis function. In Section 3 we present the local restrictions on the axis function. A theorem stating the main result of the section is cited, its proof being referred to the appendix. In Section 4 we address the problem of axis generation or skeletonization. We conclude with a summary in Section 5. The rest of the introduction is devoted to the notation and terminology that we use.

A medial axis of a simple planar shape is a collection of axis segments, each being a continuous three dimensional parametric function \( \mathcal{M}(a) = (X(a), Y(a), R(a))^T \). The first two coordinates of the medial axis function \( X(a) \) and \( Y(a) \), are the parametric description of a planar curve, the third coordinate \( R(a) \), being the radius or so-called quenching function. In the following we sometimes refer to the curve part of the medial axis as \( \mathcal{A}(a) = (X(a), Y(a))^T \). The parameter \( a \) that we normally use, is the standard arc-length parameter of the curve \( \mathcal{A} \). If \( p \) is an arbitrary parameterization of \( \mathcal{A} \), then

\[
a(p) = \int_0^p \sqrt{X'^2(\tau) + Y'^2(\tau)} d\tau
\]

From now on we use subscript to indicate derivatives, thus \( R_a = \frac{\partial}{\partial a} R \) and \( A_a = \frac{\partial}{\partial a} \mathcal{A} \).

The trivial way to define the shape \( S \) of a given axis function \( \mathcal{M} \) is through the union of axis disks.

\[
S = \bigcup_{a \in \text{Domain}(\mathcal{M})} B_{\mathcal{M}}(a)
\]  \hspace{1cm} (1)

An axis disk \( B_{\mathcal{M}}(a) \) is a disk of radius \( R(a) \) centered on \( (X(a), Y(a))^T \). Of course, not every three dimensional parametric function is a legal axis function. In order to be one, it would have to be an axis of some shape. Since a union shape (1) is defined for every axis-like function, we may say that in order to be a legal axis function, an axis-like function would have to be the axis of its union shape.

3
1 Introduction

The medial axis of a planar shape consists of the locus of centers of maximal discs in the shape, and of their corresponding radii. A maximal disc in the shape, is a disc contained in the shape such that there is no other disc in the shape that contains it [1]. The medial axis is considered an attractive representation of the shape. In addition to it being a lossless representation, it often provides an intuitively appealing thin version of the shape.

This paper addresses three questions. The first is: Can we go directly from the axis representation of the shape to its boundary representation? Due to the ease of the transformation from the axis to a direct representation of the spatial contents of the shape, this question has not gained a lot of interest, ever since the initial work of Blum and Nagel [2]. Blum and Nagel where interested in extracting boundary features from the axis, rather then a full boundary representation. While the feature extraction in [2] is based on geometric considerations, the algebraic boundary representation proposed herein enables the extraction of the basic features as well as more complex features having less obvious geometric meaning. In [5] Bruce and Giblin use similar algebraic representations to derive the generic forms of deforming symmetry sets.

The second question we address is: Can we go directly from a boundary representation to an axis representation of the shape? This is the problem of skeletonization. Skeletonization is a complex problem, that suffers from many implementation problems. Due to this fact it has been attacked from many different directions. Skeletonization from boundary descriptions has been addressed by Montanari [9] and by Lee [8] who suggested exact solutions for polygonal shapes. Some good algorithms approximating the above exact solutions where suggested by Bookstein [3] and by Brandt et al. [4] respectively. We show that the medial axis is the solution to a system of first order ordinary differential equations driven by the boundaries. Solving the system would result in a skeletonization technique for a general shape boundary. We then sketch the outline of a skeletonization algorithm based on this approach.

The third question addressed here is: Is there an easy way to tell whether a given axis representation is legal? By a legal axis function we mean a function that is indeed the axis of some real shape, and by an easy way we mean avoiding an attempt to reconstruct the corresponding shape. This problem has been previously addressed by Rosenfeld [10]. Rosenfeld argued that ultimately some reconstruction process has to be carried out, in order to approve a proposed axis function. He also indicated that
On Symmetry Axes and Boundary Curves

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Abstract

The paper deals with a representation of shape boundary, based on the medial axis of the shape. This representation is shown to be efficient in calculating local boundary features. The problem of deciding whether an axis-like function is an axis of a shape is addressed. The proposed representation scheme provides a basis for two local restrictions on axis functions. A theorem is stated, asserting these restrictions are necessary and locally sufficient for an axis function. Finally a new skeletonization approach is proposed, in which the medial axis is the solution of a first order ordinary differential equation system driven by the boundaries.

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