For C_1, $l(1,0) = 0\ 0\ 0\ 1\ 1$ and $l(1,1) = 0\ 0\ 1\ 1\ 0$, which indicates that C_1 is inserted between vector numbers 3 and 4 in $L(2)$. For C_2, $l(1,0) = 0\ 0\ 1\ 0\ 1$ and $l(1,1) = 0\ 1\ 0\ 1\ 0$, which indicates that C_2 is inserted between vector numbers 9 and 10 in $L(2)$. This produces Table II for $L(3)$.

Finally, Table III gives values of $f(\nu, t)$ for small values of ν and t. The functions are written with a subset notation in this table. For example, function

$$f = 1 + x_3 + x_2 . x_4 \quad \text{is written as}$$

$$f = 1.\{\cdot\} + 1.\{3\} + 1.\{2,4\}.$$

In these expressions, "+" denotes exclusive-or and "." denotes the and operation.

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Comments on "Maximum-Rank Array Codes and Their Application to Crisscross Error Correction"

Ernst M. Gabidulin

The first four parts of R. M. Roth's paper,[1] are devoted to the theory of the maximum-rank array codes over finite fields. I have to point out that all his results are not new and were published earlier in [1]–[3]. The paper [1] contains the following results: the definition of the rank metric; necessary and sufficient conditions for some code to have a prespecified code rank distance; Singletone-like upper bound; the definition of the maximum-rank-distance codes (MRD codes); the rank weight distribution of MRD codes; the description of a class of MRD codes (just the same as in R. M. Roth's paper); a fast decoding algorithm using the Euclid's Division algorithm for a noncommutative ring of linearized polynomials (another fast matrix algorithm was described in Roth's paper); the description of a class of MRD codes which are analogs of cyclical codes; and use of MRD codes for correction of Hamming errors over the $(d-1)/2$ bound. (Note that the English translation [1] of the original Russian paper contains a few mistakes, in particular Lemma 1 was translated wrongly.)

It should be noted that an error model of crisscross errors (we used the terms "lattice-pattern errors" and "array errors") and the different metric matched to these errors—term rank of a matrix (term rank is equivalent to the weight of E defined in Introduction of R. M. Roth's paper)—were investigated too in a few papers since 1972, see for example [2].

Optimal codes for array ("crisscross") error correction were described in [3].

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[2] E. M. Gabidulin, V. I. Korzhik, "Codes correcting lattice-pattern errors," (in Russian) *Izvestiya VUZ. Radioelektronika*, 1972, vol. 15, no. 4, pp. 492–498. This magazine is translated into English, *Radioelectronics and Communications Systems*, Allerton Press.

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The author is with Moscow Institute of Physics and Technology, Institutskii per., 9, 141700 Dolgoprudnyi, Moscow Region, Russia E-mail: gab@ippi.msk.su.

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[1] R. M. Roth, "Maximum-rank array codes and their application to crisscross error correction," *IEEE Trans. Inform. Theory*, vol. IT-37, pp. 328–336, Mar. 1991.

Author's Reply[2]

Ron M. Roth, *Member, IEEE*

Most of the results in the first half of my paper are essentially contained in the three referenced papers of E. M. Gabidulin. Unfortunately, I was not aware of Gabidulin's prior work until very recently. There is nevertheless some material in the first sections of my paper which, with the different presentation, appears to add to the known results. For example, the decoding algorithm for maximum-rank array codes presented in Section IV mimics the Berlekamp–Massey (or, rather, the Peterson–Gornstein–Zierler) algorithm, whereas the one given by Gabidulin is of the Euclid type; both algorithms have the same computational complexity.

Although Sections II–IV in my paper concentrate on $n \times n$ maximum-rank array codes, the paper is not devoted merely to the theory of such codes (to this end Gabidulin's treatment is much more extensive), but rather to the relationship between the rank metric and crisscross error correction over various fields and with various dimensions of hyper-arrays. In Sections V–VII, it is shown that the "miracle"—that maximum-rank $n \times n$ array codes are optimal for crisscross error correction as well—no longer holds in general if we look at hyper-arrays (say, $n \times n \times n$ arrays), or at infinite fields.

[2] Manuscript received December 1991.

The author is with the Computer Science Department, Technion—Israel Institute of Technology, Haifa 32000, Israel.

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Correction to "New Lower Bounds for Asymmetric and Unidirectional Codes"

Tuvi Etzion, *Member, IEEE*

In the above correspondence,[1] there are many typo errors in Appendix A. The following is the correct Appendix.

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The author is with the Department of Computer Science, Technion—Israel Institute of Technology, Haifa 32000, Israel.

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[1] T. Etzion, *IEEE Trans. Inform. Theory*, vol. 37, pp. 1696–1704, Nov. 1991.

APPENDIX A

This appendix gives the partitions into disjoint asymmetric codes. The notation is as in Brouwer *et al.* [2]. We ordered the n-tuples in lexicographic order and the disjoint $a(n, 2)$ codes are numbered by 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, and so on. If

$\Pi^{**}(4)$:

1253412434155231

then there are 16 4-tuples in the partitions. The first 4-tuple (namely 0000) is in code 1, the second (namely 0001) in code 2, the third (namely 0010) in code 5, and so on.

$\Pi^{**}(5)$

52436412315423611521364342361425

$\Pi^{**}(6)$

1641375223751431756321464214632552364314614226571421573235731461

$\Pi^{**}(7)$

```
8314673126574263457326541421357252651427731256413146723567531467712653723548
8145662154167763423154734651321653724532186424276513
```

$\Pi^{**}(8)$

```
183645919352247554736724164158327247165381643291351249652356174361257342491
```

```
183645919352247554736724164158327247165381643291351249652356174361257342491
361848268165134972365943625175825463271433428621451792714826965381327915735
137264415653257142461354656431267485283917367154362742951943257247163184281
2846351317627835762153643956241
```

$\Pi^{**}(9)$

```
A6419874379541867532179384A7254154A32562621438579851467473621A368368A6275B4
3721429158462618638731A7691388521436232435719945721452A5662351821A372437451
A69253648781477413563521486528329A471459B27592184134685635A6372354754112682
35465261279841781654871362AAB9419875BA1236294386146325758147362762513464173
6214A4398123268148574234C5931657812973211645423B5781571224513845763A6574326
813A694233413862475461753279245393175261A92615872842735A1134664859238734B61
75326823124A765483712768943952463A13945163B2852712563945711768
```

$\Pi^{**}(10)$

```
1BA887929671A5B379B56BD18D2412564153BA7638496784249738A5CA619B326214593743A
87A653C8A469B69728341876123521527481952431A68349657A43A2C184A549639714639A7
8278A364B572145138B35225978526431A1749A86E95726413213A56281354297645871A93D
6859741A964325671A2853723154128543623C91A8748A2A7489A63369B751465B2841A2135
53781359214648276A93782512465413318AB13754292A58967C3A7865938241A4259614325
1753218B6B153752462491736241538A7516A298DE897A265741381426A315428925B351A47
618A5298754317A29361453421526923BA187457986A31496CA39B8643745225473468B93AC
69B13A689744785A123962512415416395A7132578923A8467CA153B529824114A624183257
563A798DE894A6357A832142657194261257351B6B81235715244269524A1538391687A4476
984A29135731BA853214564215287C9A671846352953187254312A1482B563217596346A984
7ABA8478A19C3263458214513573582A476521469A1279386EC9A37824679215318465A3125
1462759D68A9371A52462587942153B831541274156BA87387A936427936945A481C2A33A72
691386A1542593847251453216785B4837962964A8C316A78A3473954126143926ACBA83794
25876948367AB3514652142D81DB65B973B5A176929788AB1
```

Correction to "Source Coding for Average Rate and Average Distortion: New Variable-Length Coding Theorems"

T. Hashimoto, *Member, IEEE*

In the above paper,[1] Prof. Kieffer kindly informed the author that the crucial inequality before $(9)_{\text{orig}}$[2] does not hold in general. Specifically, the two-dimensional Lebesgue–Stieltjes integral

$$\int\int \chi\left[F^N(\rho, \delta \mid x^N) < \epsilon_1\right] dF^N(\rho, \delta \mid x^N)$$

may not be arbitrarily small as $\epsilon_1 \to 0$. An example is a distribution $F^N(\rho, \delta \mid x^N)$ which possesses all of its mass on a line $\rho + \delta = d_0$. Fortunately, we can circumvent this difficulty by a slight modification.

First, the encoding rule $(2)_{\text{orig}}$ is modified as follows.

A) Variable-Length Coding Scheme: For a given x^N, generate w^N randomly according to the conditional probability of W^N given that $X^N = x^N$, and encode x^N into a codeword y_m^N, $m = 1, 2, \cdots$, if

$$\frac{1}{N}\log m \leq \frac{1}{N} i_{XW}^N(x^N, w^N) + \epsilon$$

and

$$\frac{1}{N} d^N(x^N, y_m^N) \leq \frac{1}{N} d^N(x^N, w^N) + \Delta,$$

and into y_0^N if otherwise, where ϵ and Δ are positive numbers determined later.

With this modification, the bound $(5)_{\text{orig}}$ is now replaced with

$$r_X^N(c^N) \leq \frac{1}{N}\left(1 + \frac{1}{N}\right) I_{XW}^N + 2\epsilon + \frac{5}{N} + \frac{1}{N}\log N \qquad (1)$$

$$d_X^N(c^N) \leq \frac{1}{N} d_{XW}^N + \Delta + d_0 p_0^N(c^N), \qquad (2)$$

for $N \geq 2$, and it is straightforward to come up to the following bound.

$$Ep_0^N(\mathscr{C}^N) \leq E_X \int\int \chi\left[F^N(\rho, \delta + \Delta \mid X^N) < \epsilon_1\right] \cdot dF^N(\rho, \delta \mid X^N) + \exp\left[-\epsilon_1 2^{N\epsilon - 1}\right]. \qquad (3)$$

The bound is almost the same as the corresponding one in the original paper except that "δ" in the integrand is now replaced with "$\delta + \Delta$". The remaining task is to show the Lebesgue–Stieltjes integral in the bound becomes arbitrarily small as $\epsilon_1 \to 0$.

We are concerned with the distribution $F^N(\rho, \delta \mid x^N)$ which is nontrivial only in the doubly-infinite band $S \triangleq \{(\rho, \delta); 0 \leq \delta \leq d_0, -\infty < \rho < \infty\}$. Thus, hereafter, we suppose that we are totally confined to this parameter space S with the natural topology induced by the Euclidean distance. Moreover, we suppose, without the loss of generality, that $0 < \epsilon_1 < 1$. For an arbitrarily given, but fixed for a while, vector x^N, we let $S_1 \triangleq \{(\rho, \delta); F^N(\rho, \delta + \Delta \mid x^N) < \epsilon_1\}$ and let T be the boundary that separates S_1 from its complement (in S) S_1^c. Since $F^N(\rho, \delta \mid x^N)$ is upper semicontinuous (cf. [1]) and nondecreasing both in ρ and in δ, T is a nonincreasing continuous curve, in S_1^c, starting from a point p_0 and ending at a point p_1 as shown in Fig. 1. We have the following lemma, which is proved later.

Lemma 1: For $0 < \epsilon < 1$, there exist at most $\left\lceil \frac{d_0}{\Delta} \right\rceil$ points (ρ_i, δ_i), $i = 1, 2, \cdots$, on T such that the subsets (in S) $U_i \triangleq \{(\rho, \delta); \rho < \rho_i, \delta < \delta_i + \Delta\}$ satisfy $\int\int_{U_i} dF^N(\rho, \delta \mid x^N) \leq \epsilon_1$ and $S_1 \subset \cup_i U_i$.

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The author is with the Department of Electrical Engineering, University of Electro-Communications, Chofugaoka 1-5-1, Chofu, Tokyo 182, Japan.

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[1] T. Hashimoto, *IEEE Trans. Inform. Theory,* vol. IT-29, pp. 785–792, Nov. 1983.

[2] The equations in the original paper are distinguished by the subscript "orig" throughout this correction.