Exercise 2 – Due 15.12.2005

Purpose

See limitations of the soundness gap amplification technique, using a counter example due to Andrej Bogdanov.

Definitions

For $G = (V, E)$ a $d$-regular graph, and $S \subset V$, let $e(S, \bar{S})$ be the number of edges crossing from $S$ to $\bar{S} = V \setminus S$. Formally,

$$e(S, \bar{S}) = \left| (S \times \bar{S}) \cap E \right|.$$  

$G$ is called a $(d, \lambda)$-edge expander if it is $d$-regular and for all $S \subset V$ with density $\theta = |S|/n$ we have

$$\left| e(S, \bar{S}) - \theta(1 - \theta)dn \right| \leq \lambda \sqrt{\theta(1 - \theta)} \cdot n.$$  

A cycle of length $r$ in $G$ is a sequence of vertices $v_0, \ldots, v_{r-1}$ such that $(v_i, v_{i+1 \mod r}) \in E$. Let the girth of $G$ be the minimal length of a cycle in $G$.

We use the following well Theorem about expander graphs (due to Lubotsky, Phillips and Sarnak).

**Theorem**  For infinitely many $d > 2$ and infinitely many $n$, there exists a $(d, 2\sqrt{d})$-edge expander on $n$ vertices with girth $\geq \frac{2}{3} \log_d n$.

Questions

Let $G$ be a $(d, 2\sqrt{d})$-edge expander with girth $\geq \frac{2}{3} \log_d n$, as in the above mentioned Theorem. Let $\mathcal{G}$ be the constraint graph over graph $G$, with alphabet $\Sigma = \{0, 1\}$ and each edge constraint is an inequality constraint, i.e. for every edge $e = (u, v) \in E$ and any assignment $A : V \to \Sigma$, we have $C_e(A(u), A(v)) = \text{accept}$ iff $A(u) \neq A(v)$.

Prove the following:

1. The initial soundness is large: $s(\mathcal{G}) \geq 1/2 - O(1/\sqrt{d})$.

2. Powering does not increase it significantly: If $n > d^{32a}$ then $s(\mathcal{G}^a) \leq 1/2$.

(Hint: (i) Start with a random assignment to vertices. (ii) Extend it to a locally consistent assignment to balls of radius $a$, using the large girth of $G$. (iii) Measure the expected number of violated constraints in $\mathcal{G}^a$. (iv) Use linearity of expectation.)