Iterative Hard Thresholding with Near Optimal Projection for Signal Recovery

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Abstract—Recovering signals that have sparse representations under a given dictionary from a set of linear measurements got much attention in the recent decade. However, most of the work has focused on recovering the signal’s representation, forcing the dictionary to be incoherent and with no linear dependencies between small sets of its columns. A series of recent papers show that such dependencies can be allowed by aiming at recovering the signal itself. However, most of these contributions focus on the analysis framework. One exception to these is the work reported in [1], proposing a variant of the CoSaMP for the synthesis model, and showing that signal recovery is possible even in high-coherence cases. In the theoretical study of this technique the existence of an efficient near optimal projection scheme is assumed. In this paper we extend the above work, showing that under very similar assumptions, a variant of IHT can recover the signal in cases where regular IHT fails.

I. INTRODUCTION

Recovering a sparse signal from a given set of linear measurements has been a major subject of research in recent years. In the basic setup, an unknown signal \( x_0 \in \mathbb{R}^d \) passes through a given linear transformation \( M \in \mathbb{R}^{n \times d} \) with an additive noise \( e \in \mathbb{R}^n \) providing a set of linear measurements \( y = Mx_0 + e \). The signal \( x_0 \) is assumed to have a \( k \)-sparse representation \( \alpha_0 \in \mathbb{R}^n \) under a given dictionary \( D \in \mathbb{R}^{d \times n} \), i.e., \( x_0 = Dx_0 \). \( \| \alpha_0 \|_0 \leq k \) and \( k \ll d \), where \( \| \cdot \|_0 \) is the “\( \ell_0 \)-norm” that counts the number of non-zero entries in a vector. The sparsity prior results with the following minimization problem

\[
\min_{\alpha} \| y - MD\alpha \|_2 \quad \text{s.t.} \quad \| \alpha \|_0 \leq k, \tag{1}
\]

in which we pursue the representation \( \alpha \) in order to recover the original signal \( x_0 \) from \( y \). Given a reconstructed representation \( \hat{\alpha} \), the estimation for the signal is simply given by \( \hat{x} = D\hat{\alpha} \).

Solving (1) is a NP-hard problem and many approximation techniques has been proposed for it [2]. One of these is the iterative hard thresholding (IHT) algorithm [3]. This approach, summarized in Algorithm 1, recovers the representation in an iterative way using two repeating steps: (i) Gradient step: moving in the optimal gradient direction for minimizing \( \| y - MD\alpha \|_2^2 \); (ii) Projection step: ensuring that the representation estimate is \( k \)-sparse. The operator \( \text{supp}(\cdot, k) \) returns the support of the largest \( k \) elements in a given vector and the subscript \( T \) for a vector/matrix means taking the entries/columns corresponding to the indices in \( T \).

In order to evaluate the performance of IHT, the restricted isometry property (RIP) [4] of the matrix \( M \) is used. A matrix \( A \in \mathbb{R}^{d \times n} \) satisfies the RIP with a constant \( \delta_k \) if for any \( k \) sparse vector \( \alpha \in \mathbb{R}^n \)

\[
(1 - \delta_k) \| \alpha \|_2^2 \leq \| MD\alpha \|_2^2 \leq (1 + \delta_k) \| \alpha \|_2^2. \tag{2}
\]

With this definition in hand it has been shown that if \( \delta_{4k} \leq 1/4 \) or \( \delta_{4k} \leq 1/\sqrt{3} \) then IHT recovers the representation stably, i.e.,

\[
\| \hat{\alpha}_{\text{IHT}} - \alpha_0 \|_2 \leq c_{\text{IHT}} \| e \|_2, \tag{3}
\]

where \( c_{\text{IHT}} > 2 \) is a function of \( \delta_{4k} \) and \( \delta_{4k} \) [3], [5], [6]. Note that with no prior on the noise distribution only a stable recovery is guaranteed with no noise reduction effect. The latter can be achieved by adding an assumption on the noise distribution [7]. This work deals only with the former case where \( e \) is an adversarial bounded noise.

Note that in the case where \( D \) contains \( k \) correlated columns we have \( \delta_k \geq 1 \). Then the above recovery conditions fail and (3) does not hold. The reason for this is that in the presence of linear dependencies between a small group of columns from \( D \), the representation is no longer unique [8] and the solution of (1) is no longer stable [4]. Though the recovery of the representation is not achievable in the presence of correlations within \( D \), we should keep in mind that our task is to estimate the signal and not the representation. Recovering the wrong support of \( \alpha \), but one that is closely related to the original signal may suffice for our needs.

This key point is contained in the union of subspaces literature [9], [10], [11]. However, it has been pointed out more clearly in a series of contributions for the analysis framework [12], [13], [14], [15], [16] assuming a different sparse model. As such, correlations in the analysis dictionary were found to pose no problem and it has been demonstrated that such are even an advantage [14], [15], [16].

The analysis results serve as a clue that the same may happen in the synthesis model when the signal is the objective. In particular, the condition in [12] are presented in terms of the D-RIP, which is a property of the measurement matrix \( M \) for the synthesis model. However, as indicated in [15], the results in [12] essentially hold true for signals emerging from the analysis model.

The work reported in [1] is very different from all the above, in addressing the synthesis model, providing signal recovery guarantees using the D-RIP. This work presents a modified version of CoSaMP, signal space CoSaMP (SSCoSaMP), that aims at recovering the signal, showing empirically that unlike the regular CoSaMP, the modified version gets a good recovery even in the presence of linear dependencies in \( D \). The authors of [1] use a similar proof technique to the one in [15] that was derived for the analysis CoSaMP (ACoSaMP). Just like [15], the work in [1] relies on the availability of near-optimal projection (this property will be defined clearly in the

Algorithm 1 Iterative hard thresholding (IHT)

Require: \( k, M, D, y \) where \( y = MD\alpha_0 + e \), \( k \) is the cardinality of \( \alpha_0 \) and \( e \) is an additive noise.

Ensure: \( \alpha_{\text{IHT}} \)-sparse approximation of \( \alpha_0 \).

Initialize representation \( \hat{\alpha}_0 = 0 \) and set \( t = 0 \).

while halting criterion is not satisfied do

1. Perform a gradient step: \( \alpha_t = \alpha_{t-1} + \mu MD^\top (y - MD\alpha_{t-1}) \)
2. Find a new support: \( T^d = \text{supp}(\alpha_t, k) \).
3. Calculate a new representation: \( \hat{\alpha}^t = (\alpha_t)_{T^d} \).

end while

Form the final solution \( \alpha_{\text{IHT}} = \hat{\alpha}^t \).
Next, another recent paper that exploits the D-RIP in the context of the synthesis $\ell_0$-minimization problem.

In this work, we continue with the same assumption as in [1] – the existence of a near optimal projection scheme\(^1\) and use the D-RIP too. In Section II we present notations, and the definitions of the D-RIP and the near-optimality of a projection. In Section III we introduce the signal space IHT (SSIHT) method for signal recovery and in Section IV we propose theoretical guarantees for it, relying on ideas taken from [15]. The SSIHT emerges from IHT as SSCoSaMP. Section V presents some numerical results showing ideas taken from [15]. The SSIHT emerges from IHT as SSCoSaMP. Section V presents some numerical results showing the advantage of SSIHT over IHT for the task of signal recovery.

II. Preliminaries

We start with the definition of the D-RIP. As indicated in [12], many types of random matrices satisfy this property with a small $\delta_d^2$.

**Definition 2.1:** A matrix $M$ obeys the D-RIP with a constant $\delta^2_d$, if $\delta_d^2$ is the smallest constant that satisfies

\[
(1 - \delta_d^2) \|z\|_2^2 \leq \|Mz\|_2^2 \leq (1 + \delta_d^2) \|z\|_2^2
\]

for any $z \in \mathbb{R}^d$ such that $z = D\alpha$ and $\|\alpha\|_0 \leq k$.

Another definition we need is the one of a near optimal projection.

In SSIHT we face the following problem: Given a general vector $\bar{z}$, where the closest vector with $k$-sparse representation is $P_\mathbb{D}\bar{z}$, our task is to recover $x_0$ from $\bar{z}$. The recovery result is denoted by $\hat{x}$.

We start with the definition of the near-optimality constant $C_k$. The first constant measures the energy kept in the projection.

**Definition 2.2:** A procedure $\hat{S}_k$ implies a near-optimal projection $P_{\hat{S}_k(\cdot)}$ with a constant $C_k$ if for any $z \in \mathbb{R}^d$

\[
\|z - P_{\hat{S}_k(\cdot)}z\|_2^2 \leq C_k \|z - P_{ST}(\cdot)z\|_2^2.
\]

In [1], a slightly different definition was used:

**Definition 2.3:** A procedure $\hat{S}_k$ implies a near-optimal projection $P_{\hat{S}_k(\cdot)}$ with constants $C_{k,1}$ and $C_{k,2}$ if for any $z \in \mathbb{R}^d$

\[
\|z - P_{\hat{S}_k(\cdot)}z\|_2^2 \leq C_{k,1} \|z - P_{S^*}(\cdot)z\|_2^2.
\]

Having these definitions we recall the problem we aim at solving:

\[\min_{y \in \mathbb{R}^m} \|y - Mx\|_2^2 + \mu \|x\|_0\]

Again, we assume $\mu$ is given by

\[\mu = \frac{K}{\sqrt{\delta_d^2}}\]

This constant's relation to the other two depends on the initial norm and the regular IHT is the projection scheme. As IHT works in the representation domain, its projection is performed also there and as the optimal support in the signal case seems to be a difficult task is to recover $x_0$ from $\bar{z}$.

The following guarantee has been proposed in [1] for SSCoSaMP.

\[\|\bar{z} - x_0\|_2 \leq C_1 \|\bar{z} - x_0\|_2 + C_2 \|\varepsilon\|_2,
\]

where $C_1 = ((2 + C_{k,1})\delta_d + C_{k,1})(2 + C_{k,2})\sqrt{\frac{1 + \mu}{\mu}}$ and $C_2 = 2(2 + C_{k,2})(2 + C_{k,1})(2 + \delta_d)\sqrt{\frac{1 + \mu}{\mu}}$.

III. Signal Space Iterative Hard Thresholding

SSIHT is presented in Algorithm 2. Its main difference from the regular IHT is the projection scheme. As IHT works in the representation domain, its projection is performed also there and as mentioned in the previous section, the projection is optimal in this case. For SSIHT that works in the signal domain no general projection procedure with an optimality guarantee is known.

The stopping criterion and the step size can be selected in the same way as in the regular IHT [19]. For the step size we consider three options: (i) Constant step-size selection $\mu = \mu$ in all iterations; (ii) Optimal changing step-size selection $\mu$ in each iteration by minimizing $\|y - Mx\|_2^2$; and (iii) Adaptive changing step-size.
selection that has a closed-form solution and uses

$$\mu^* := \arg\min_{\mu} \|y - \mathbf{M}(\mathbf{x}^k - 1 + \mu \mathbf{P}_{\mathcal{T}} \mathbf{M}^* (y - \mathbf{M} \mathbf{x}^k - 1))\|^2, \quad (10)$$

where $\hat{T} = \mathbf{P}_{\mathcal{T}}^{-1} \cup \mathbf{S}_0 (\mathbf{x}^k - 1))$. More details appear in [15], [19]. In our theoretical study we analyze the first two options. In the experimental part we use the third one as it works better than the first, and approximates the second that has no closed-form solution.

**Algorithm 2 Signal space iterative hard thresholding (SSIHT)**

**Require:** $k, M, D, y$ where $y = MD \alpha_0 + \epsilon$, $k$ is the cardinality of $\alpha_0$ and $\epsilon$ is an additive noise.

**Ensure:** $x_{smooth} := k$-sparse approximation of $x_0$.

Initialize estimate $\mathbf{x}^0 = \mathbf{0}$ and set $t = 0$.

while halting criterion is not satisfied $t = t + 1$.

Perform a gradient step: $\mathbf{x}_t = \mathbf{x}^{t-1} + \mu^* \mathbf{M}^* (y - \mathbf{M} \mathbf{x}^{t-1})$

Find a new support: $\mathcal{T}_t = \hat{T}_t (\mathbf{x}_t)$. Project to get a new estimate: $\mathbf{x}^t = \mathbf{D}_{\mathcal{T}_t} \mathbf{D}_{\mathcal{T}_t}^\dagger \mathbf{x}_t$.

end while

Form the final solution $x_{smooth} = \mathbf{x}^t$.

**IV. ALGORITHMS GUARANTEES**

A uniform guarantee for the idealized version of SISHT that has an access to the optimal projection and uses a constant step size $\mu = \mu$, is presented in [11]. The work in [11] deals with a general union of subspaces, $\mathcal{A}$, where in our case $\mathcal{A} = \{x| \mathbf{D} \alpha, \|\alpha\|_0 \leq k \}$. Using our notation Theorem 2 from [11] reads:

**Theorem 4.1 (Theorem 2 in [11]):** Consider the problem $P$ with $S_0 = S_0^*$ and apply SSHT with a constant step size $\mu$. If $1 + \delta_b \leq \frac{1}{\mu} < 1.5(1 - \delta_b)$ then after a finite number of iterations $t^*$

$$\|\mathbf{x}^t - \mathbf{x}_0\|_2 \leq c_3 \|\epsilon\|_2,$$

(11)

where the constant $c_3$ is a function of $\delta_b$ and $\mu$.

In our work we extend the above in several ways: First, we refer to the case where an optimal projection is not known, and show that the same flavor guarantees apply for a near-optimal projection.

The price we seemingly have to pay is that $\sigma_M$ enters the game. Second, we also consider the optimal step size and show that the same performance guarantees hold true in this case.

**Theorem 4.2:** Consider the problem $P$ and apply SSHT with a constant step size $\mu$ or an optimal step size. For any positive constant $\eta > 0$, let $b_1 := \frac{1}{\sqrt{1 + \eta}}$ and $b_2 := \frac{(C_0 - 1) \sigma_M^2}{\mu (C_0 - 1) + \Sigma}$, Suppose $\delta_{\mathcal{T}} < 1$, $\frac{1}{\mu} \leq \delta_{\mathcal{T}}$ and $1 + \delta_b \leq \frac{1}{\mu} < 1 + \frac{1}{\sqrt{1 + \frac{b_2}{b_1}}} b_1 (1 - \delta_b)$. Then for $t \geq t^* \triangleq \frac{(I + \frac{1}{\eta})^2 (\mu (C_0 - 1) + \Sigma) \log \left( \frac{\|\mathbf{x}_0\|_2}{\|\mathbf{e}\|_2} \right)}{\mu (C_0 - 1) + \Sigma}$

$$\|\mathbf{x}^t - \mathbf{x}_0\|_2 \leq \frac{(1 + \eta)^2}{1 - \delta_b} \|\mathbf{e}\|_2,$$

(12)

3Theorem 2 in [11] is more general and deals also with the case where $S_0$ is near-optimal up to an additive constant factor (in our definitions the factor is multiplicative). The error bound in the theorem has an additional constant factor that depends on the projection’s near-optimality additive constant.

4Our work in fact improves the condition of the idealized case in [11] to be $\delta_{\mathcal{T}} < \frac{1}{b_1}$ instead of $\delta_{\mathcal{T}} \leq \frac{1}{b_1}$.

This theorem is a variant of Theorem 6.5 in [15] for AIIHT and Theorem 2.1 in [20] for HT. If, for example, $\sigma_M^2 = 5$ and $C_k = 1.05$ then the conditions of Theorem 4.2 turn to be $\delta_{\mathcal{T}} \leq 0.289$ as mentioned before. For a better understanding of the nature of the theorem we refer the reader to the remarks after Theorems 6.2 and 6.5 in [15]. Briefly we comment on the selection of $\mu$ and $\eta$. For the stepsize selection, note that an optimal changing step-size has the same theoretical guarantees as the optimal constant step-size $\mu = \frac{1}{1 + \delta_{\mathcal{T}}}$.

The advantage of the changing step-size method is that it does not need to compute (or estimate) the value of $\delta_{\mathcal{T}}$. However, this comes at the cost of an additional complexity. Regarding the constant $\eta$, it gives a trade-off between satisfying the theorem conditions and the amplification of the noise. In particular, one may consider that the above theorem proves the convergence result for the noiseless case by taking $\eta$ to infinity. This result is included in Lemma 4.4, which we present later, that guarantees in the case $\epsilon = 0$ that $\sigma_M$ converges geometrically to $\sigma_M$. Due to the uniqueness property that appears in [17], this implies that $\mathbf{x}$ converges to $\mathbf{x}_0$.

We prove the theorem by presenting two key lemmas. The proofs rely on the ones in [15] that adopted ideas from [20] and [11]. Recall that the iterative algorithm tries to reduce the objective $\|y - \mathbf{M} \mathbf{x}\|^2_2$ over iterations $t$. Thus, the progress of the algorithm can be indirectly measured by how much the objective $\|y - \mathbf{M} \mathbf{x}^t\|^2_2$ is reduced at each iteration $t$. The two lemmas that we present capture this idea. The first lemma relates $\|y - \mathbf{M} \mathbf{x}^t\|^2_2$ to $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2$ and similar quantities at iteration $t-1$. We remark that the constant $\frac{1}{\mu} \leq \sigma_M^2$ in Theorem 4.2 may not be necessary and it is added only for having a simpler derivation of the results in this theorem. Furthermore, this is a very mild condition compared to $\frac{1}{\mu} < 1 + \sqrt{1 + \frac{b_2}{b_1}} b_1 (1 - \delta_b)$ and can only limit the range of values that can be used with the constant step size version of the algorithm.

**Lemma 4.3:** Consider the problem $P$ and apply SSHT with a constant step size $\mu$ satisfying $\frac{1}{\mu} > 1 + \delta_b$ or an optimal step size.

Then, at the $t$-th iteration, the following holds:

$$\|y - \mathbf{M} \mathbf{x}^{t+1}\|^2_2 - \|y - \mathbf{M} \mathbf{x}^t\|^2_2 \leq C_k \|y - \mathbf{M} \mathbf{x}_0\|^2_2 \quad (13)$$

where $C_k = \frac{1}{\mu (1 - \delta_b)} - 1$ and $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2 \leq \eta^2 \|\mathbf{e}\|_2^2$.

The proof of the above lemma is exactly the same as the proof of Lemma 6.6 in [15] with the change that here we use the $D$-RIP instead of the $\Omega$-RIP and the near-optimal projection scheme for synthesis instead of the one for analysis. The second lemma shows that once the objective $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2$ at iteration $t-1$ is small enough, then we are guaranteed to have small $\|y - \mathbf{M} \mathbf{x}^{t}\|^2_2$. As given the presence of the noise, this is quite natural; one cannot expect it to approach 0 but may expect it not to become worse. Moreover, the lemma also shows that if $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2$ is not small, then the objective in iteration $t$ is necessarily reduced by a constant factor.

**Lemma 4.4:** Suppose that the same conditions of Theorem 4.2 holds true. If $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2 \leq \eta^2 \|\mathbf{e}\|_2^2$, then $\|y - \mathbf{M} \mathbf{x}^{t}\|^2_2 \leq \eta^2 \|\mathbf{e}\|_2^2$.

Furthermore, if $\|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2 > \eta^2 \|\mathbf{e}\|_2^2$, then

$$\|y - \mathbf{M} \mathbf{x}^{t}\|^2_2 \leq c_4 \|y - \mathbf{M} \mathbf{x}^{t-1}\|^2_2,$$

(14)

where $c_4 < 1$ and

$$c_4 := \left(1 + \frac{1}{\eta}\right)^2 \frac{1}{\mu (1 - \delta_b)} - 1 \frac{C_k + (C_k - 1) \mu \sigma_M^2 - 1 + C_k}{\eta^2}$$

For an optimal step size the bound is achieved with the value $\mu = \frac{1}{1 + \delta_{\mathcal{T}}}$.\]
The Lemma’s proof is similar to the one of Lemma 6.7 in [15]. The needed adaptations are similar to those done for Lemma 4.3. Having the two lemmas above, the proof of the theorem is straightforward.

**Proof of Theorem 4.2:** Since \( \hat{x} = 0 \), \( \| y \|_2 = \| y - Mx \|_2 \). Assuming that \( \| y \|_2 > \eta \| e \|_2 \) and applying Lemma 4.4 repeatedly, we obtain \( \| y - Mx \|_2^2 \leq \max (c^2, \| y \|_2^2, \| e \|_2^2) \). Since \( c^2 \| y \|_2^2 \leq \eta^2 \| e \|_2^2 \) for \( t \geq t' \), we have

\[
\| y - Mx^t \|_2^2 \leq \eta^2 \| e \|_2^2
\]

for \( t \geq t' \). If \( \| y - Mx^t \|_2 = \| y \|_2 \leq \eta \| e \|_2 \) then according to Lemma 4.4, (15) holds for every \( t > 0 \). Finally, we observe

\[
\| \hat{x}^t - x_0 \|_2 \leq \frac{1}{1 - \delta_k} \| M(\hat{x}^t - x_0) \|_2^2
\]

and by the triangle inequality,

\[
\| M(\hat{x}^t - x_0) \|_2 \leq \| y - Mx^t \|_2 + \| e \|_2.
\]

By plugging (15) into (17) and then the resulting inequality into (16), the claim of the Theorem follows.

\( \square \)

V. NUMERICAL PERFORMANCE

We turn to check numerically whether SSHT can recover signals in scenarios where IHT cannot. We perform a synthetic test similar to the one in [17] for signals that are sparse under a dictionary which is highly coherent and with linear dependencies between its columns. We generate a dictionary \( D = [D_1, D_2] \) where \( D_1, D_2 \in \mathbb{R}^{d \times d} \), \( d = 200 \). \( D_1 \) contains sparse columns with 2 non-zero entries which are 1 or \(-1\) with probability 0.5 and \( D_2 \) contains columns which are linear combinations of random 3 columns from \( D_1 \) with random zero-mean white Gaussian weights. Each entry of the measurement matrix \( M \in \mathbb{R}^{m \times d} \) is distributed according to a normal Gaussian distribution, where \( m = \gamma d \) and \( \gamma \) is the sampling rate – a value in the range \([0, 1]\). We set \( k \) to be \( \lfloor m \rho \rfloor \) (\( \rho < 0.1 \)) and measure the recovery rate of the representation \( \mathbf{a} \) and the signal \( \mathbf{x} \) for various values of \( \gamma \in [0.1, 0.2, 0.3, 0.4, 0.5] \) and \( \rho \in [0.01, 0.02, 0.03, 0.05] \). We compare SSHT also to SSCoSamp, where both uses projection by thresholding. The adaptive changing step-size selection rule is used for IHT and SSHT. Similar to what is done in [15], by uniqueness conditions it is better to apply the algorithms with sparsity \( k = \max (k, m/2) \).

Figure 1 presents the recovery performance over 100 realizations per each parameter setting. As expected, IHT fails almost always in recovering the signal since it focuses on the representation, while SSHT and SSCoSamp succeed in several cases and their performance are similar. At a first glance, some would think that the SSHT phase diagram implies that for a fixed \( k/m \) (e.g. 0.03) one may improve the recovery result if he uses less samples, i.e. smaller \( m/d \). However, this observation misses the fact that for a fixed \( k/m \), \( k \) is reduced together with \( m \). Note that the recovery results of SSHT and SSCoSamp can be improved by using other techniques for the projection, rather than thresholding, as done in [1] for SSCoSamp.

VI. CONCLUSION

In this paper we have proposed a variant of the IHT algorithm – the Signal-Space IHT (SSHT) – for recovering signals with sparse representations under highly coherent dictionaries. We have shown that IHT fails in recovering such signals, as it operates in the representation domain. SSHT, on the other hand, targets the signal. A uniform recovery guarantee has been derived for the SSHT, assuming the availability of a near optimal projection. Numerical simulations show that SSHT succeeds in recovering signals for which IHT fails, even when the projection is not near-optimal.

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