

Davenport-Schinzel Sequences and Their Geometric Applications

Advanced Topics in Computational Geometry

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Outline

- Definition of DS -sequences
- The complexity of DS -sequences
- Geometric applications of DS -sequences:
 - Lower envelopes
 - Nearest neighbors in a set of dynamic points
 - A cell in an arrangement of line segments
 - Geometric graphs

References

- [BY98] Jean-Daniel Boissonnat and Mariette Yvinec.
Algorithmic geometry. Cambridge University Press,
1998.
- [KL98] Meir Katchalski and Hagit Last. On geometric graphs
with no two edges in convex position. *Discrete and
Computational Geometry*, 19:399–404, 1998.
- [SA95] Micha Sharir and Pankaj K. Agarwal.
*Davenport-Schinzel sequences and their geometric
applications*. Cambridge University Press, 1995.

Definition

Let n and s be positive integers. A sequence of integers $U = \langle u_1, u_2, \dots, u_m \rangle$ is an (n, s) -*Davenport-Schinzel sequence* if it satisfies the following conditions:

1. $1 \leq u_i \leq n$ for $1 \leq i \leq m$;
2. $u_i \neq u_{i+1}$ for every $1 \leq i < m$; and
3. There do not exist $s + 2$ indices $1 \leq i_1 < i_2 < i_3 < \dots < i_{s+2} \leq i_m$ such that $u_{i_1} = u_{i_3} = u_{i_5} = \dots = a$, $u_{i_2} = u_{i_4} = u_{i_6} = \dots = b$, and $a \neq b$.

Example

A $DS(4, 3)$ -sequence:

$\langle 1, 2, 1, 3, 1, 3, 2, 4, 2, 4, 3, 4 \rangle$

This is **not** a $DS(4, 3)$ -sequence:

$\langle 1, \mathbf{2}, 1, \mathbf{3}, 1, 3, \mathbf{2}, 4, 2, 4, \mathbf{3}, 4, \mathbf{2} \rangle,$

since the third condition is violated.

Complexity

$$\lambda_s(n) = \max\{|U| : U \text{ is a } DS(n, s)\text{-sequence}\}$$

Simple bounds:

- $\lambda_1(n) = n$ $\quad \quad \quad \langle 1, 2, 3, \dots, n \rangle$
- $\lambda_2(n) = 2n - 1$
 - $\langle 1, 2, \dots, n - 1, n, n - 1, \dots, 2, 1 \rangle \Rightarrow \lambda_2(n) \geq 2n - 1$
 - $\lambda_2(n) \leq 2n - 1$ $\quad \quad \quad$ Proof by induction on n .

A Simple Bound on $\lambda_3(n)$

Lemma 1.11 [SA95]. $\lambda_3(n) \leq 2n(\ln(n) + O(1))$

Proof. Let U be a $DS(n, 3)$ -sequence, and let a be the least frequent symbol in U . Then a appears in U at most $|U|/n \leq \lambda_3(n)/n$ times.

We remove all the appearances of a in U , and one of every pair of identical symbols that become adjacent by the removal of the a 's. For example ($a='1'$):

$\langle 1, 2, 3, 1, 3, 2, 4, 2, 4, 3, 4 \rangle$

\downarrow

$\langle /, 2, 3, /, /, 2, 4, 2, 4, 3, 4 \rangle$

A Simple Bound on $\lambda_3(n)$ (cont.)

The latter removal might happen only around the first and last appearances of a , since otherwise we get:

$$< \dots, a, \dots, x, a, x, \dots, a, \dots >$$

The result is a $DS(n-1, 3)$ -sequence. Thus we obtain the recurrence relation:

$$\lambda_3(n) \leq \lambda_3(n-1) + \lambda_3(n)/n + 2$$

that yields $\lambda_3(n) \leq 2n(\ln(n) + O(1))$ \square

A Tight Bound on $\lambda_3(n)$

It can be shown* that

$$\lambda_3(n) = \Theta(n\alpha(n)).$$

- $\alpha(n)$ is the extremely slow-growing functional inverse of Ackermann's function.
- $\alpha(n) \rightarrow \infty$ when $n \rightarrow \infty$, but for any practical value of n $\alpha(n) \leq 4$.

*The full (long and technical) proof can be found in Sections 2.2 and 2.3 of [SA95].

Known Bounds for $s > 3$

- $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$

-

$$\lambda_s(n) \leq \begin{cases} n \cdot 2^{\alpha(n)^{(s-2)/2} + C_s(n)} & \text{if } s \text{ is even,} \\ n \cdot 2^{\alpha(n)^{(s-3)/2} \log \alpha(n) + C_s(n)} & \text{if } s \text{ is odd,} \end{cases}$$

where $C_s(n)$ is a function of $\alpha(n)$ and s .

- for even s ,

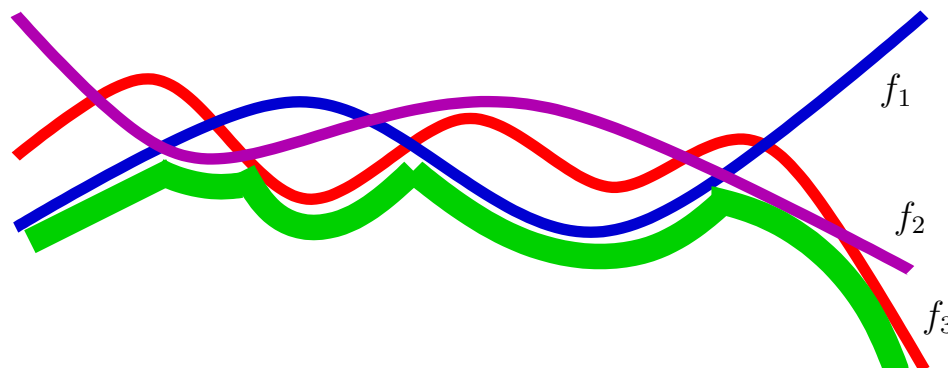
$$\lambda_s(n) = \Omega(n \cdot 2^{K_s \alpha(n)^{(s-2)/2} + Q_s(n)}),$$

where $K_s = \left(\frac{s-2}{2}\right)!$ and $Q_s(n)$ is a polynomial in $\alpha(n)$ of degree at most $(s-4)/2$.

Lower Envelopes

Let $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ be a set of n real-valued continuous functions defined over a common interval I . The *lower envelope* of \mathcal{F} is defined as:

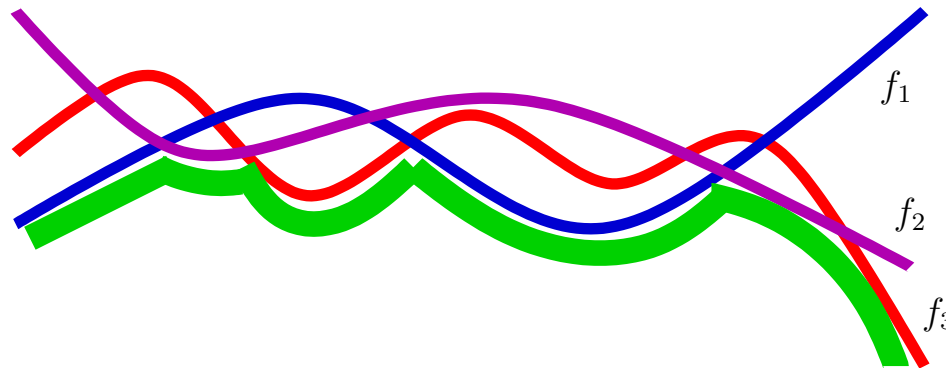
$$E_{\mathcal{F}}(x) = \min_{1 \leq i \leq n} f_i(x), \quad x \in I$$



Note: We will assume that for every $i \neq j$, f_i and f_j intersect in at most s points.

Lower Envelope Sequence

A *lower envelope sequence* is simply the sequence of the indices of the functions along the lower envelope.



$$U(f_1, f_2, f_3) = \langle 1, 2, 3, 1, 2, 3 \rangle$$

Lower Envelope Sequence and DS -Sequences

Lemma 1.2 [SA95]. *Given n real-valued continuous functions (f_1, f_2, \dots, f_n) defined on a common interval, such that f_i and f_j intersect at most s times ($i \neq j$), then $U(f_1, f_2, \dots, f_n)$ is a $DS(n, s)$ -sequence.*

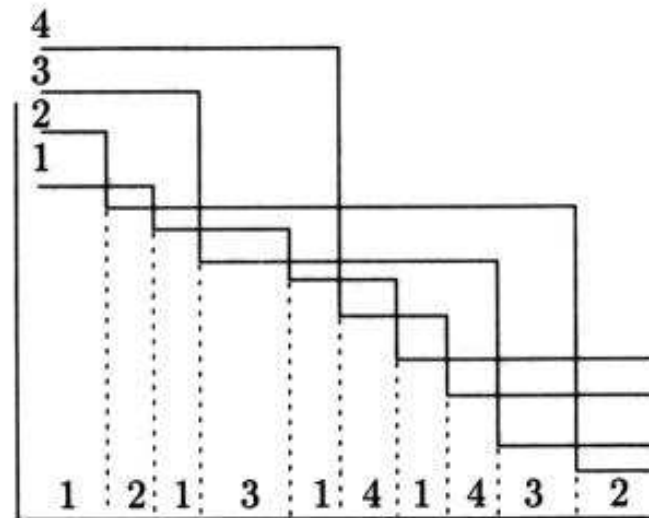
Conversely, for any given $DS(n, s)$ -sequence U , one can construct a collection f_1, f_2, \dots, f_n of functions for which $U(f_1, f_2, \dots, f_n) = U$.

Proof - $U(f_1, f_2, \dots, f_n)$ is a $DS(n, s)$ -Sequence

- By definition $U(f_1, f_2, \dots, f_n)$ satisfies the first two conditions.
- Suppose the third condition is violated, i.e., there is a forbidden subsequence $\langle \dots, a, \dots, b, \dots, a, \dots, b, \dots \rangle$ of length $s + 2$, then f_a and f_b intersect at least $s + 1$ times.

Proof of the Converse Direction

Given a $DS(n, s)$ -sequence $U = (u_1, u_2, \dots, u_m)$ we define $m - 1$ “transition points”, and $n + m - 1$ “horizontal levels”.



Let a be the $(i + 1)$ st element in U , $1 \leq i \leq m - 1$.

At the i th transition point f_a drops to the next free level.

The Lower Envelope of Partially-Defined Functions

Lemma 1.4 [SA95]. *If f_1, f_2, \dots, f_n is a collection of **partially defined** continuous functions, such that f_i and f_j intersect at most s times ($i \neq j$), then $U(f_1, f_2, \dots, f_n)$ is a $DS(n, s + 2)$ -sequence. Conversely, ...*

Proof.

\Rightarrow By reduction to totally-defined functions.

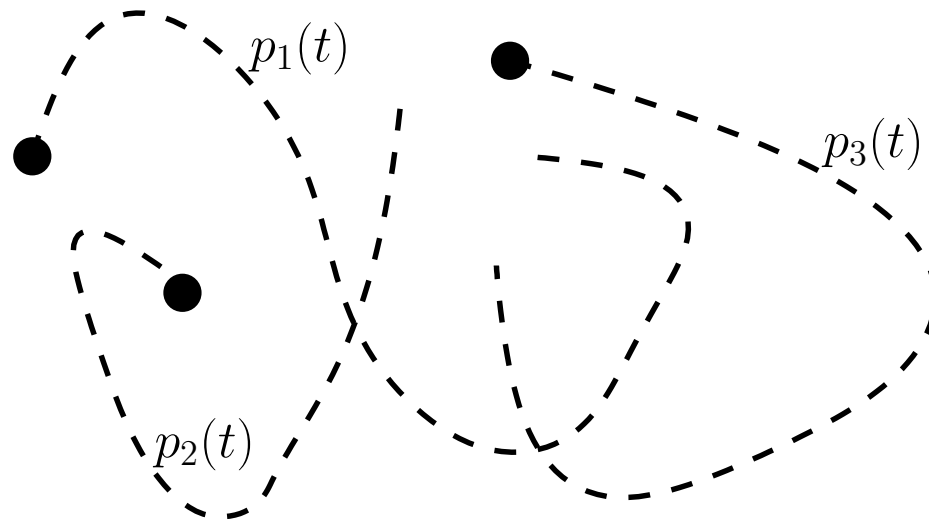
\Leftarrow By a similar construction as in totally-defined functions.

Corollary 1.5 [SA95]. *The lower envelope sequence of n (nonvertical) line segments in the plane is a $DS(n, 3)$ -sequence.*

Dynamic Points in the Plane

Let $P = \{p_1, p_2, \dots, p_n\}$ be a set of n points in the plane such that the coordinates of every p_i are functions of time:

$$p_i(t) = (x_i(t), y_i(t)).$$



Assume $x_i(t)$, $y_i(t)$ are polynomials of maximum degree s .

Nearest Neighbors in a Set of Dynamic Points

Let $P(t)$ be the configuration of the points at time t .

For a point $p_i(t) \in P(t)$, let $q_i(t) \in P(t)$ denote its *nearest neighbor*, that is

$$\text{dist}(p_i(t), q_i(t)) = \min_{j \neq i} \text{dist}(p_i(t), p_j(t))$$

Question: How many times does the nearest neighbor of a point $p_i \in P$ change?

Changes in the Nearest Neighbor of p_i

For every $i \neq j$, let

$$g_{ij}(t) = \text{dist}^2(p_i(t), p_j(t)) = (x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2$$

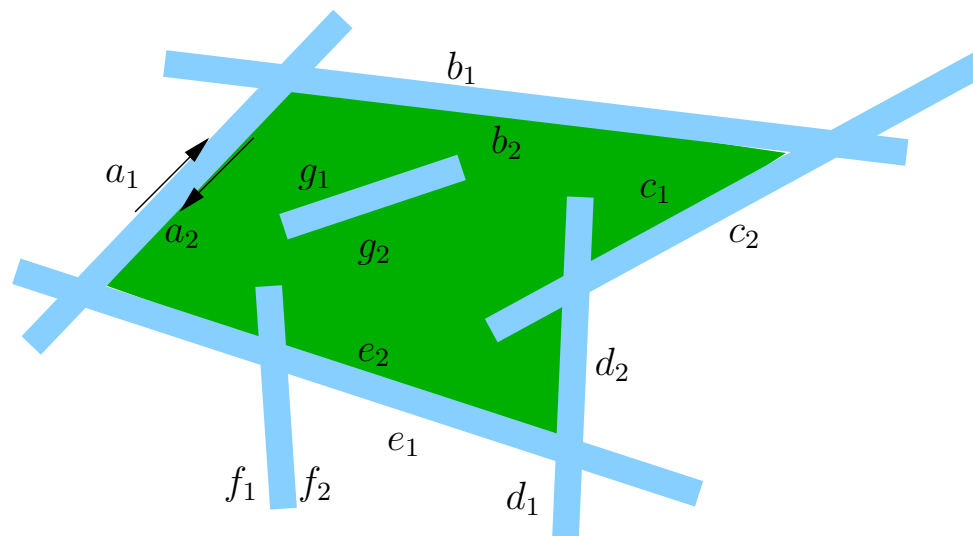
- The degree of every g_{ij} is at most $2s$.
- The complexity of the lower envelope of g_{ij} equals the number of times the nearest neighbor of p_i changes.

\Rightarrow The nearest neighbor of p_i changes at most $\lambda_{2s}(n)$ times (Theorem 8.40 [SA95]).

An Arrangement of Line Segments in the Plane

- \mathcal{S} is a set of n line segments in the plane.
- Every segment has two sides.
- The *arrangement* of \mathcal{S} , $\mathcal{A}(\mathcal{S})$, is formed by the
 - **vertices** – intersection and endpoints of the segments;
 - **edges** – maximal portions of the segments between two vertices; and
 - **cells** – connected components of $\mathbb{E}^2 \setminus \mathcal{S}$,of the planar subdivision induced by \mathcal{S} , and their incidence relationships.

A Cell in an Arrangement of Line Segments



The boundary of this cell is

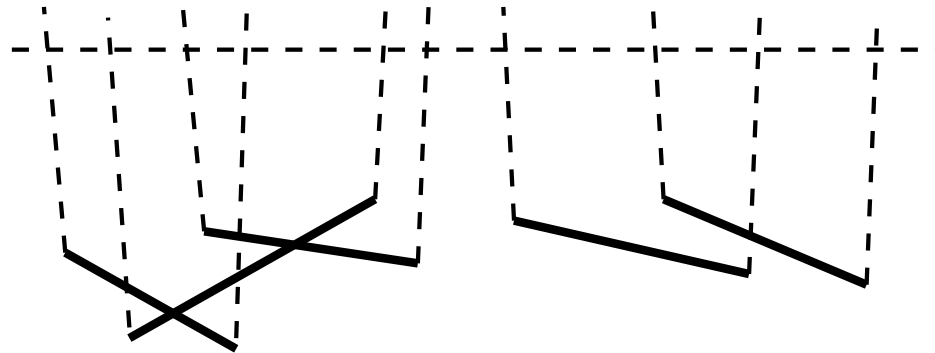
$\langle a_2, e_2, f_1, f_2, e_2, d_1, c_2, c_1, d_1, d_2, c_1, b_2 \rangle$ and $\langle g_1, g_2 \rangle$.

Question: What is the complexity of a single cell?

The Complexity of a Single Cell is $\Omega(n\alpha(n))$

- According to Corollary 1.5, the complexity of the lower envelope of n segments in the plane is $O(n\alpha(n))$.
- In fact, this bound can be realized [SA95, §4].

\Rightarrow The complexity of a single cell is $\Omega(n\alpha(n))$:



Assumptions

- We assume the segments are in **general position**:
 - No three segments intersect in a common point; and
 - Two segments intersect in at most one point.

These assumptions are taken since the complexity is maximal in that case.

Assumptions (cont.)

- The boundary of the cell is a **single connected component**. This is because every segment is contained in at most one connected component of the boundary of the cell. Thus, suppose the boundary is composed of k connected components, and n_i segments contribute edges to the i th component, then the complexity of the cell is $\sum_{i=1}^k O(n_i \alpha(n)) = O(n \alpha(n))$.

The Complexity of a Single Cell is $O(n\alpha(n))$

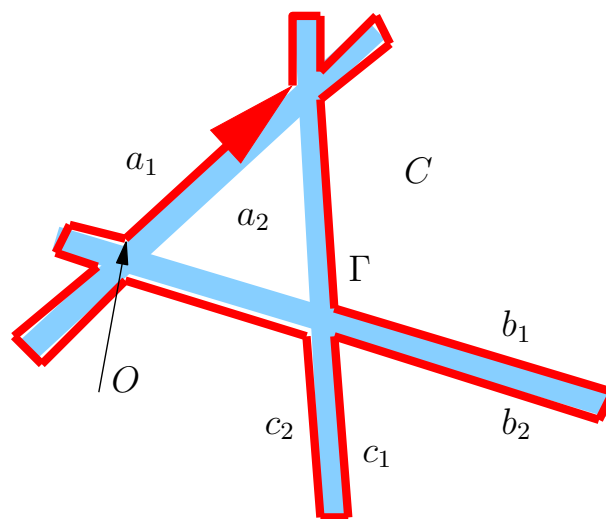
Let Γ be a connected component of the boundary of some cell C .

Lemma 15.4.1 [BY98]. *Let s be a segment that contains at least one edge of Γ . The edges of Γ contained in s are traversed on the boundary of Γ in the same order as they are traversed on the boundary of s .*

Proof. By drawing...

The Complexity of a Single Cell is $O(n\alpha(n))$ (cont.)

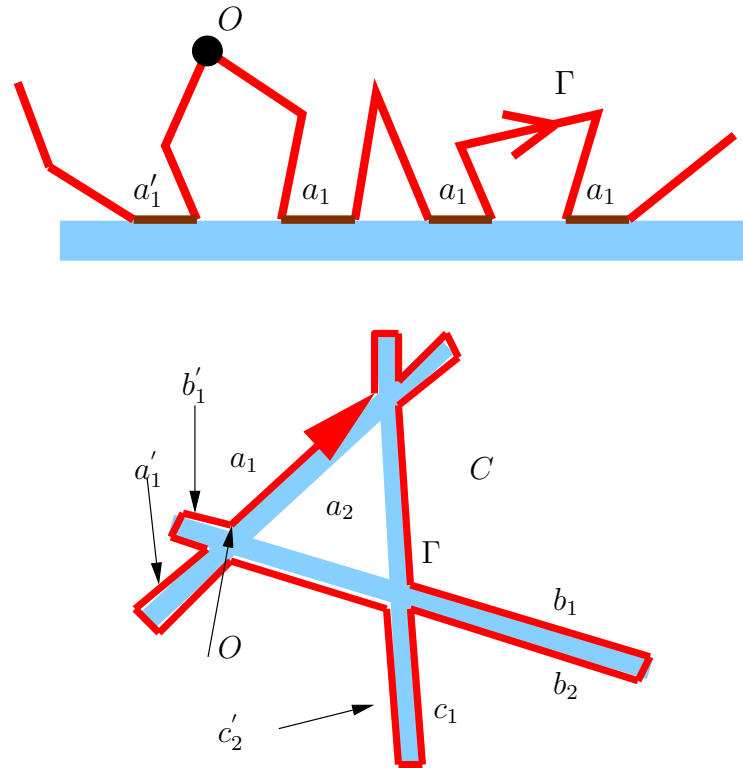
The labeling of Γ defines a circular sequence that can be turned into a linear sequence by choosing an arbitrary starting point O . However, it is not always possible to choose O such that the resulting sequence is a $DS(2n, 3)$ -sequence:



$$\Sigma_{\Gamma} = a_1 c_2 c_1 a_1 a_2 c_1 b_1 b_2 c_1 c_2 b_2 a_2 a_1 b_2 b_1$$

The Complexity of a Single Cell is $O(n\alpha(n))$ (cont.)

To solve this problem we add more labels:

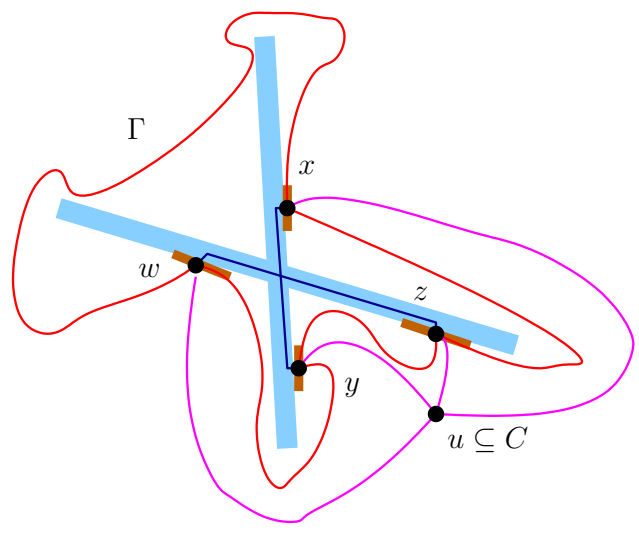


$$\Sigma_{\Gamma}^* = a_1 c_2 c_1 a_1 a_2 c_1 b_1 b_2 c_1 c'_2 b_2 a_2 a'_1 b_2 b'_1$$

The Complexity of a Single Cell is $O(n\alpha(n))$ (cont.)

Lemma. *If $abab$ is a subsequence of Σ_Γ^* , a and b intersect.*

Proof. Let x, z, y, w be points on Γ such that $x, y \in a$, $z, w \in b$ and Γ passes through these points in the order x, z, y, w . If a and b do not intersect then we can draw K_5 in the plane:



which is a contradiction. \square

The Complexity of a Single Cell is $O(n\alpha(n))$ (cont.)

Lemma 15.4.2 [BY98]. Σ_{Γ}^* is a $DS(4n, 3)$ -sequence.[†]

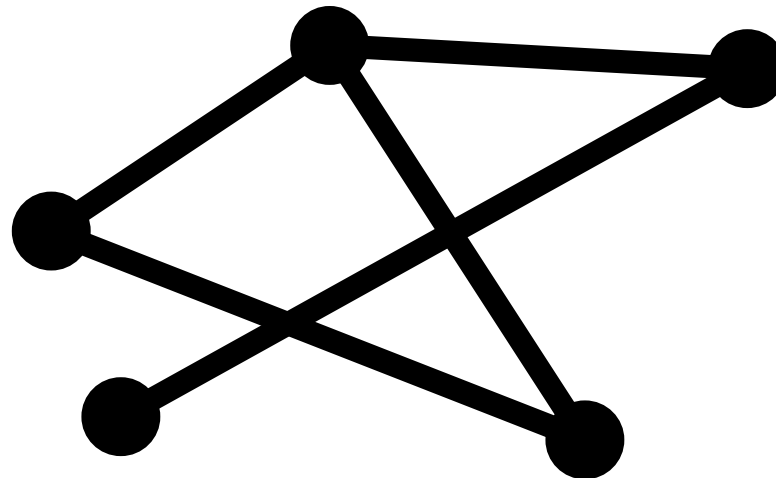
Proof. It is clear that Σ_{Γ}^* satisfies the first two conditions. Assume for contradiction that Σ_{Γ}^* contains a subsequence $ababa$. Then the previous lemma guarantees that there are two intersection points between a and b , and by lemma 15.4.1 and the labeling these intersection points are distinct. \square

Theorem 15.4.3 [BY98]. *The complexity of a cell in the arrangement of n line segments in the plane is $O(n\alpha(n))$.*

[†]Actually a $DS(3n, 3)$ -sequence

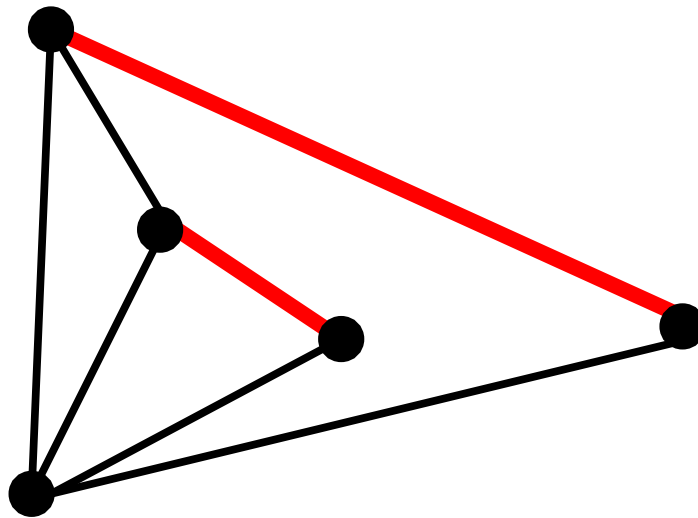
Geometric Graphs

A *geometric graph* is a graph $G = G(V, E)$ drawn in the plane, whose vertex set V consists of points in general position (no three are collinear) and whose edge set E consists of straight line-segments between points of V .



Convex Edges in a Geometric Graph

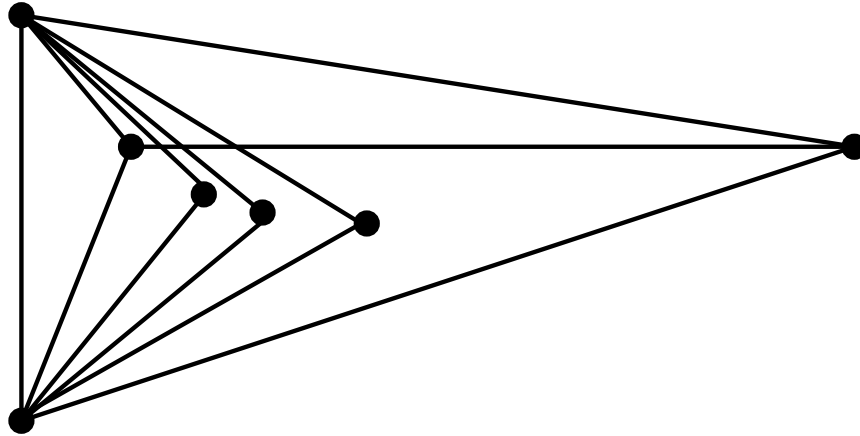
Two segments are in *convex position* (parallel) if they are disjoint edges of a convex quadrilateral.



Question: What is the maximal number of edges e in a geometric graph of n vertices, in which no two edges are in convex position?

A Lower Bound on e

According to the following construction $e \geq 2n - 2$ (for $n \geq 4$):



- In fact, this bound is tight.
- We will only prove $e \leq 2n - 1$.

Proof of the Upper Bound of e

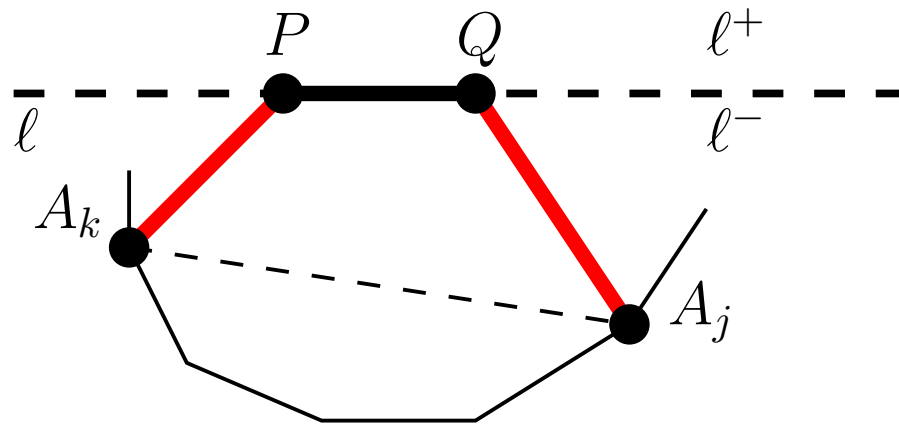
Lemma 2 [KL98]. *Let A_i, A_j, A_k, A_l be four points appearing in this order on a closed convex curve γ . Let P, Q be two points in the interior of the region bounded by γ . Consider the four (closed) segments*

$$PA_i, QA_j, PA_k, QA_l,$$

and assume that among them there is no segment s that contains only one of the points P, Q and such that the line supporting s contains both of them. Then two of the four segments are in convex position.

Proof of Lemma 2

Assume that P is the origin of coordinates and Q is to the right of P on the x -axis. Let $\ell = \ell(P, Q)$ be the line through P and Q . Let ℓ^+ and ℓ^- be the two halfspaces defined by ℓ . Then, if ℓ^+ (or ℓ^-) contains two disjoint segments (from PA_i, QA_j, PA_k, QA_l), they are in convex position.



Proof of Lemma 2 (cont.)

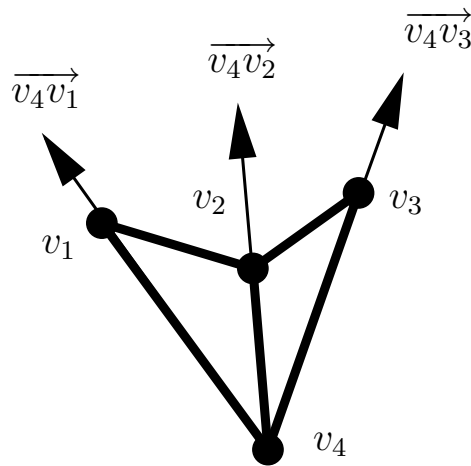
Consider all the possible locations of the four points with respect to ℓ :

- One of the points is on ℓ ;
- ℓ^+ (ℓ^-) contains more than two of the points; or
- ℓ^+ and ℓ^- contain two of the points.

Simple case analysis shows the claim.

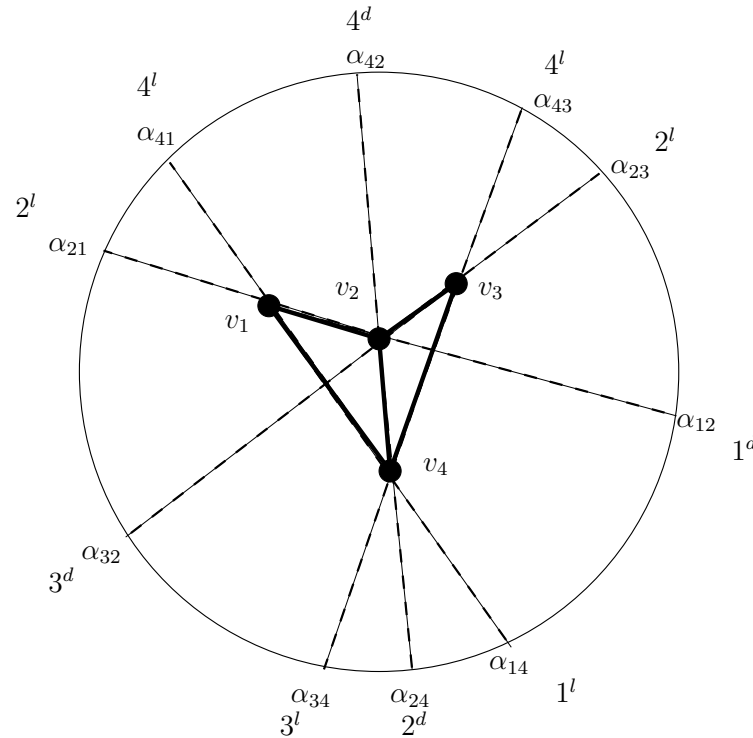
Proof of the Upper Bound of e (cont.)

An edge xy is *right* (resp., *left*) of an edge xz if \overrightarrow{xy} is obtained from \overrightarrow{xz} by a clockwise (resp., counter-clockwise) rotation around x by a positive angle smaller than π .



If there is no edge to the right (resp., left) of xy , then xy is the *rightmost* (resp., *leftmost*) edge of x .

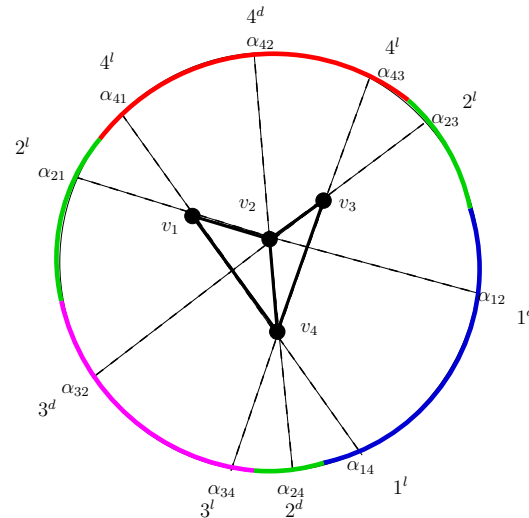
Proof of the Upper Bound of e (cont.)



$$\text{color}(\alpha_{ij}) = \begin{cases} \text{light } i & v_j v_i \text{ is the rightmost or leftmost edge of } v_j \\ \text{dark } i & \text{otherwise} \end{cases}$$

Proof of the Upper Bound of e (cont.)

$D(G)$ = the circular sequence of points. An *arc* of $D(G)$ is a maximal sequence of points of the same color (irrespective of dark or light). $PS(G)$ is the sequence of the colors of the arcs along $D(G)$.



arcs: $(\alpha_{41}, \alpha_{42}, \alpha_{43}), (\alpha_{23}), (\alpha_{12}, \alpha_{14}), (\alpha_{24}), (\alpha_{34}, \alpha_{32}), (\alpha_{21})$.
 $PS(G) = (4, 2, 1, 2, 3, 2)$

Proof of the Upper Bound of e (cont.)

Corollary 1.10 [SA95]. *The maximum length of a $DS(n, 2)$ -**cycle** is $2n - 2$.*

Proof. Similar to the case of a $DS(n, 2)$ -sequence.

Lemma 3 [KL98]. *$PS(G)$ is a $DS(n, 2)$ -cycle.*

Lemma 4 [KL98]. *An arc of $D(G)$ contains at most one dark point.*

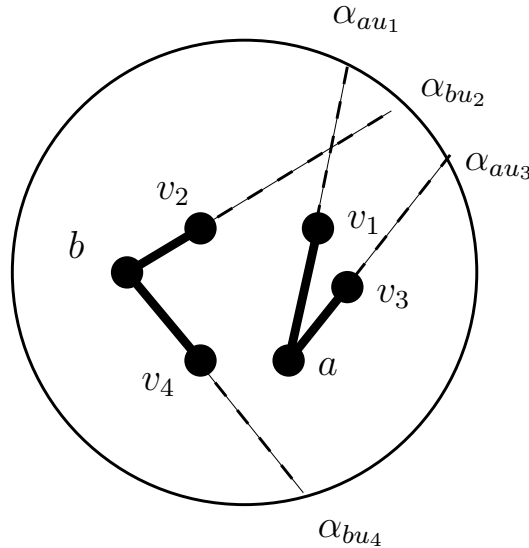
(The proofs of lemmata 3 and 4 are found below.)

Proof of the Upper Bound of e (cont.)

- $|D(G)| = 2e = \# \text{ light color points} + \# \text{ dark color points}$
- Every vertex has at most one rightmost edge and one leftmost edge, therefore, $\# \text{ light color points} \leq 2n$.
- By lemma 4: $\# \text{ dark color points} \leq |PS(G)|$.
- By corollary 1.10 and lemma 3: $|PS(G)| \leq 2n - 2$.
- Therefore, $e \leq 2n - 1$ \square

Proof of Lemma 3

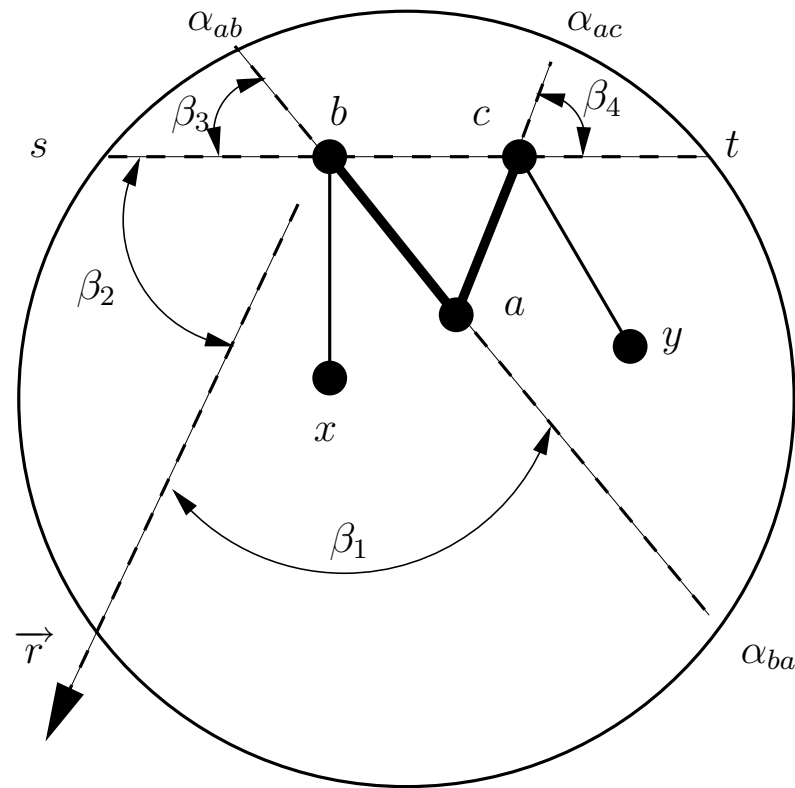
Assume for contradiction that there are four points $\alpha_{au_1}, \alpha_{bu_2}, \alpha_{au_3}, \alpha_{bu_4}$ along the cycle:



According to lemma 2 there must be two edges in convex position, a contradiction. \square

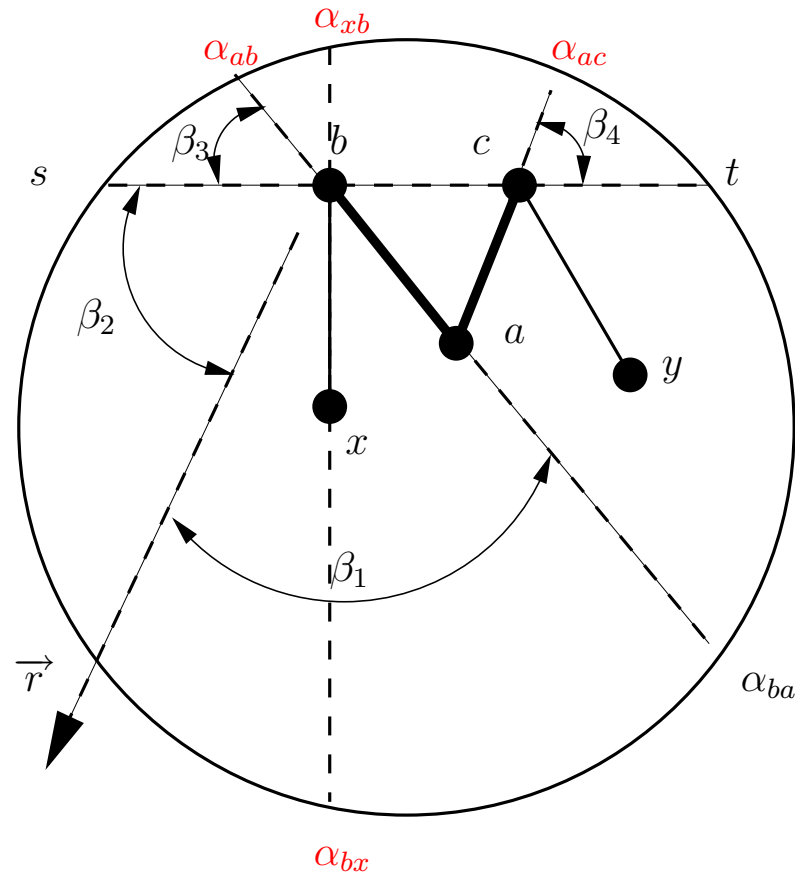
Proof of Lemma 4

Let α_{ab} and α_{ac} be two dark points on the same arc. Let bx (resp., cy) be an edge to the right (resp., left) of ba (resp., ca). Also let $\vec{r} \parallel \vec{ca}$.



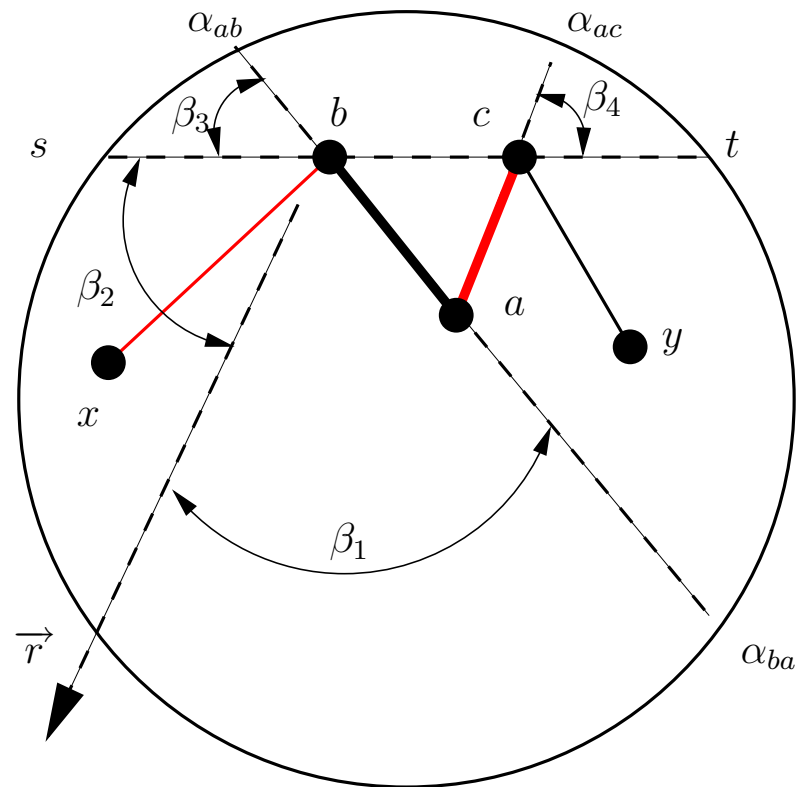
Proof of Lemma 4 (cont.)

If $x \in \beta_1$, then α_{ab} and α_{ac} are not on the same arc:



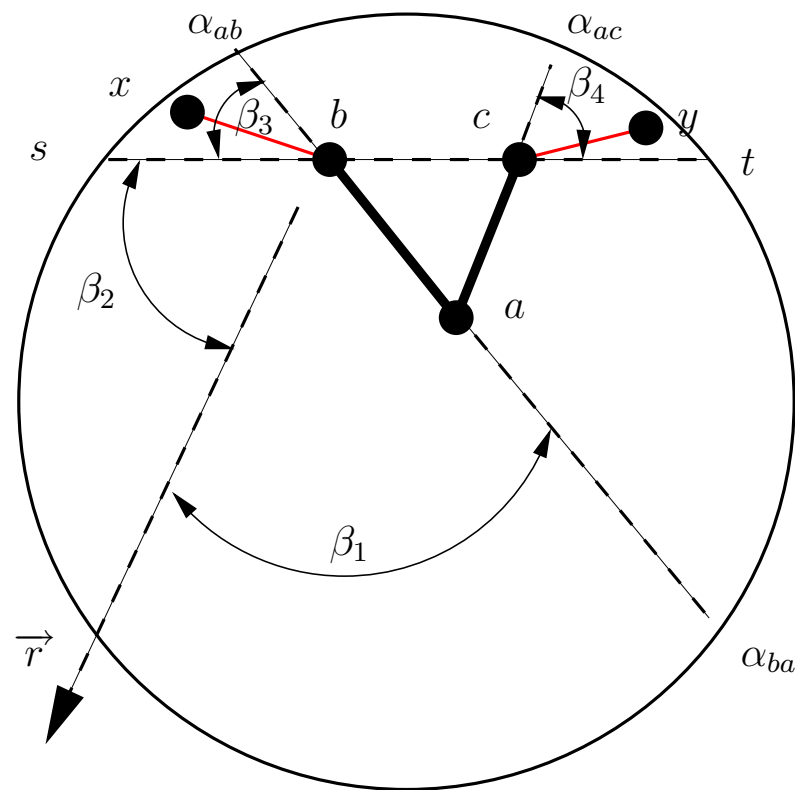
Proof of Lemma 4 (cont.)

If $x \in \beta_2$, then bx and ca are in convex position:



Proof of Lemma 4 (cont.)

Therefore, $x \in \beta_3$. In a similar manner we obtain $y \in \beta_4$, thus bx and cy are in convex position:



Summary

- Definition of Davenport-Schinzel sequences.
- Complexity of DS -sequences:

$$\lambda_1(n) = n, \quad \lambda_2(n) = 2n - 1, \quad \lambda_3(n) = \Theta(n\alpha(n)).$$

- DS -sequences and lower envelopes.
- Nearest neighbors in a set of dynamic points.
- The complexity of a single cell in an arrangement of n line segments in the plane is $\Theta(n\alpha(n))$.
- The maximal number of edges in a geometric graph of n vertices and no convex edges is at most $2n - 1$. In fact, it is exactly $2n - 2$.