The on-line Heilbronn's triangle problem for 3 dimensions

From "The on-line Heilbronn's triangle problem" by Gill Barequet
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1 Introduction

At Gill's lecture we saw the on-line Heilbronn's triangle problem for 2 dimensions. This lesson we'll show a similar algorithm, but for the 3D problem. In the 3D case of Heilbronn's triangle problem, we need to maximize the least-volume tetrahedron, defined by 4 points in R^3.

The known lower bound for the 3D problem is $\Omega\left(\frac{\log n}{n^3}\right)$. In this lesson we'll show an on-line algorithm, which gives the lower bound of $\Omega\left(\frac{1}{n^{10/3}}\right)$. Of course, the on-line problem is harder than the off-line one, because n is not known in advance.

As in 2D algorithm, we assume that the k points, allready spread in the unit cube, satisfy few conditions, and we show that there exists a place where we can place the $k+1$th point, so the conditions will still hold. We show such a place exists, by summing the volumes of all the forbidden places for placing a new point, and showing that this volume is smaller than 1.
The conditions that the set of points, S, spread in the unit cube, is satisfying, are:

1. $|p_ip_j| \geq \frac{a}{n^{1/3}}$, $\forall p_i, p_j \in S, p_i \neq p_j$, and for some constant $a > 0$.
   $|p_ip_j|$ denotes the distance between p_i and p_j
2. $|p_ip_jp_k| \geq \frac{b}{n}$, $\forall p_i, p_j, p_k \in S, p_i \neq p_j \neq p_k$, and for some constant $b > 0$.
   $|p_ip_jp_k|$ denotes the area of the triangle $p_ip_jp_k$.
3. $|p_ip_jp_kp_l| \geq \frac{c}{n^{10/3}}$, $\forall p_i, p_j, p_k \in S, p_i \neq p_j \neq p_k$, and for some constant $c > 0$.
   $|p_ip_jp_kp_l|$ denotes the volume of the tetrahedron $p_ip_jp_kp_l$

2 Summing forbidden volumes

In the current iteration of the construction of points in the unit cube, we try to find a place for the new added point, p. We'll show such a place exists, by summing all the forbidden volumes for p to be located at, and showing this sum is less than 1. The forbidden volumes, are the ones that if p will be located at them, one of the above inequalitis is violated.

2.1 Forbidden balls

Condition no. 1 above defines, for every point in S, $|S| = v$, a forbidden ball for locating the point p. The ball's radius is $\frac{a}{n^{1/3}}$, and its volume is $\frac{4\pi}{3}\left(\frac{a}{n^{1/3}}\right)^3 = \frac{4\pi a^3}{3n}$.
The total forbidden balls volume is $v\frac{4\pi a^3}{3n} = O\left(\frac{a^3 v}{n}\right) = O(a^3)$

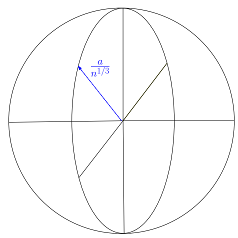

Figure 1: Forbidden ball, of radius $\frac{a}{n^{1/3}}$

2.2 Forbidden Cylinders

Condition no. 2 above defines, for every two points $p_i, p_j \in S$, a forbidden cylinder for locating the point p.
The cylinder's height is at most $\sqrt{3}$, and its radius is r, when the following exists:
$\frac{|p_i p_j| r}{2} = \frac{b}{n}$
$r = \frac{2b}{n|p_i p_j|}$
Thus, one cylinder's volume is at most $\sqrt{3}\pi r^2 = \sqrt{3}\pi \frac{4b^2}{n^2|p_i p_j|^2} = \frac{4\sqrt{3}\pi b^2}{n^2|p_i p_j|^2}$

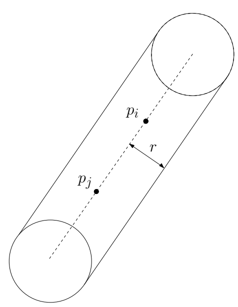

Figure 2: Forbidden cylinder

The total forbidden cylinders volume is $\sum_{1 \le i < j \le v} \frac{4\sqrt{3}\pi b^2}{n^2|p_i p_j|^2}$.
In order to bound this, we will use the spherical packing argument. We fix p_i and sum over p_j.
We define M_t to be the number of points of S that lie between two balls centered at p_i: inside the ball of radius $\frac{a(t+1)}{n^{1/3}}$, and outside the ball of radius $\frac{at}{n^{1/3}}$.
The distance between two such balls is $\frac{a}{n^{1/3}}$, and so we have $O\left(\frac{n^{1/3}}{a}\right)$ such balls.
Therefore:

$$\sum_{i\neq j}\frac{1}{|p_ip_j|^2}\leq\sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)}\frac{M_t}{\left(\frac{at}{n^{1/3}}\right)^2}=\sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)}\frac{M_tn^{2/3}}{t^2a^2}$$

The gap between the tth and the $t+1$th balls, is of volume $\frac{4\pi}{3}\left(\frac{a^3(t+1)^3}{n}-\frac{a^3t^3}{n}\right)=\Theta\left(\frac{a^3t^2}{n}\right)$.
In this volume, we pack balls, each of volume $\frac{4\pi a^3}{3n}$.
Therefore, the number of balls in the tth layer, M_t, is bounded by:
$M_t=\frac{a^3t^2}{n}/\frac{4\pi a^3}{3n}=O\left(t^2\right)$.
And so:

$$\sum_{i\neq j}\frac{1}{|p_ip_j|^2}\leq\sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)}\frac{M_tn^{2/3}}{t^2a^2}\leq\sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)}\frac{t^2n^{2/3}}{t^2a^2}=\sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)}\frac{n^{2/3}}{a^2}=\frac{n^{1/3}}{a}\cdot\frac{n^{2/3}}{a^2}=\frac{n}{a^3}$$

And if we get back to the original inequation, The total forbidden cylinders volume is:

$$\sum_{1\leq i<j\leq v}\frac{4\sqrt{3}\pi b^2}{n^2|p_ip_j|^2}\leq\sum_{1\leq i\leq v}\frac{4\sqrt{3}\pi b^2}{n^2}\cdot\frac{n}{a^3}=\frac{4\sqrt{3}\pi b^2}{a^3}\sum_{1\leq i\leq v}\frac{1}{n}=\frac{4\sqrt{3}\pi b^2}{a^3}\cdot\frac{v}{n}=O\left(\frac{b^2}{a^3}\right)$$

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2.3 Forbidden Slabs

Condition no. 3 above defines, for every three points $p_i, p_j, p_k \in S$, two forbidden slabs for locating the point p.
A slab's area is at most 3, and its height is h, when the following exists:
$\frac{|p_ip_jp_k|h}{3}=\frac{c}{n^{10/3}}$
$h=\frac{3c}{n^{10/3}|p_ip_jp_k|}$
Thus, two adjacent slabs' volume is at most:
$2\cdot 3\cdot\frac{3c}{n^{10/3}|p_ip_jp_k|}=\frac{18c}{n^{10/3}|p_ip_jp_k|}$

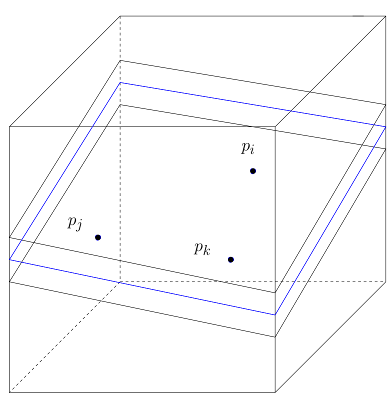

Figure 3: Forbidden slabs

The total forbidden slabs volume is at most

$$\sum_{1\le i<j<k\le v} \frac{18c}{n^{10/3}\,|p_ip_jp_k|}$$

In order to bound this, we will use the cylindrical packing argument. We fix p_i, p_j and sum over p_k.
We define N_t to be the number of points of S that lie between two cylinders centered at the line which passes through p_i and p_j: inside the cylinder of radius $\frac{a(t+1)}{n^{1/3}}$, and outside the cylinder of radius $\frac{at}{n^{1/3}}$.
Like in the previous case, the distance between two such cylinders is $\frac{a}{n^{1/3}}$, and so we have $O\left(\frac{n^{1/3}}{a}\right)$ such cylinders.
Unlike the previous case, here, in N_0 there is an option for points to be located. Each point located there defines a triangle of area at least b/n (becuase of condition no. 2). Therefore:

$$\sum_{k\ne i,j} \frac{1}{|p_ip_jp_k|} \le \frac{N_0}{b/n} + \sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)} \frac{N_t}{\frac{|p_ip_j|\left(\frac{at}{n^{1/3}}\right)}{2}} = \frac{N_0 n}{b} + \sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)} \frac{2N_t n^{1/3}}{at\,|p_ip_j|}$$

The volume of the most inner cylinder is at most $\sqrt{3}\pi\left(\frac{a}{n^{1/3}}\right)^2 = \Theta\left(\frac{a^2}{n^{2/3}}\right)$.
Therefore, the number of balls (points) it contains, is bounded by:
$N_0 = \frac{a^2}{n^{2/3}} / \frac{4\pi a^3}{3n} = O(\frac{n^{1/3}}{a})$.
The gap between the tth and the $t+1$th cylinders, is of volume:
$\sqrt{3}\pi\left(\left(\frac{a(t+1)}{n^{1/3}}\right)^2 - \left(\frac{at}{n^{1/3}}\right)^2\right) = \Theta\left(\frac{a^2t}{n^{2/3}}\right)$.
In this volume, we pack balls, each of volume $\frac{4\pi a^3}{3n} = \Theta\left(\frac{a^3}{n}\right)$.
Therefore, the number of balls in the tth layer, N_t, is bounded by:
$N_t = \frac{a^2t}{n^{2/3}} / \frac{a^3}{n} = O\left(\frac{tn^{1/3}}{a}\right)$.
And so:

$$\sum_{k\ne i,j} \frac{1}{|p_ip_jp_k|} \le \frac{N_0 n}{b} + \sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)} \frac{2N_t n^{1/3}}{at\,|p_ip_j|} \le \frac{n^{1/3}\cdot n}{ab} + \sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)} \frac{tn^{1/3}}{a}\cdot\frac{2n^{1/3}}{at\,|p_ip_j|}$$

$$= \frac{n^{4/3}}{ab} + \sum_{t=1}^{O\left(\frac{n^{1/3}}{a}\right)} \frac{2n^{2/3}}{a^2\,|p_ip_j|} \le \frac{n^{4/3}}{ab} + \frac{n^{1/3}}{a}\cdot\frac{2n^{2/3}}{a^2\,|p_ip_j|} = \frac{n^{4/3}}{ab} + \frac{2n}{a^3\,|p_ip_j|}$$

And if we get back to the original inequation, The total forbidden cylinders volume is:

$$\sum_{1\le i<j<k\le v} \frac{18c}{n^{10/3}\,|p_ip_jp_k|} \le 18c \sum_{1\le i<j\le v} \frac{1}{n^{10/3}\,|p_ip_jp_k|} \le 18c \sum_{1\le i<j\le v} \left(\frac{1}{n^2ab} + \frac{2}{n^{7/3}a^3\,|p_ip_j|}\right) =$$

$$O\left(c\sum_{1\le i<j\le v}\left(\frac{1}{n^2ab}+\frac{1}{n^{7/3}a^3\,|p_ip_j|}\right)\right)$$

First summand is $O\left(c\frac{v^2}{n^2ab}\right)=O\left(\frac{c}{ab}\right)$.
And the second is: (we fix p_i and sum over all p_j)

$$c\cdot\sum_{j\neq i}\frac{1}{a^3n^{7/3}\,|p_ip_j|}\le\frac{c}{a^3n^{7/3}}\sum_{t=1}^{O(n^{1/3}/a)}\frac{M_tn^{1/3}}{at}=\frac{c}{a^4n^2}\sum_{t=1}^{O(n^{1/3}/a)}\frac{O(t^2)}{t}=\frac{c}{a^4n^2}\sum_{t=1}^{O(n^{1/3}/a)}t$$

$$=O(\frac{c}{a^4n^2}\cdot\frac{n^{2/3}}{a^2})=O(\frac{c}{a^6n^{4/3}})$$

Summing this over all p_i, we get the bound $O(\frac{cv}{a^6n^{4/3}})=O(1)$
Thus, the total forbidden volumes for this stage is $O\left(\frac{c}{ab}\right)+O(1)=O\left(\frac{c}{ab}\right)$

The TOTAL forbidden area, is thus, $C_1a^3+C_2\frac{b^2}{a^3}+C_3\frac{c}{ab}$ And we can find such a,b,c that it will be less than 1.