Voronoi Diagrams for Non-Euclidean Metrics

Algorithmic Geometry

J.D. Boissonnat and M. Yvinec

Chapter 18

Presented by: Iddo Hanniel

Talk Overview

- Voronoi diagrams reminder – reviewing results on lifting to the paraboloid P in E^{d+1}.
- Power diagrams and higher order power diagrams.
- Affine diagrams and diagrams for a general quadratic distance.
- Weighted diagrams – additive weights.
- Weighted diagrams – multiplicative weights.
- L_1 and L_∞ metrics diagrams.
- Voronoi diagrams in hyperbolic space.

Voronoi Diagrams Reminder

<u>Objective</u>: To compute Voronoi diagrams (VDs) of n sites in E^d.

Σ: a sphere in E^d centered at point C with radius r.

- The **power** of point X with respect to Σ:

$$\Sigma(X)=\|X-C\|^2-r^2$$

- The power $\boldsymbol{\sigma}$ of the origin with respect to Σ:

$$\sigma=\Sigma(0)=\|C\|^2-r^2$$

Voronoi Diagrams Reminder (cont.)

We map spheres in E^d into points in E^{d+1} by the mapping:

$$\Phi(\Sigma)=(C,\sigma)$$

Φ maps points in E^d onto the paraboloid P:

$$X_{d+1}=\|X\|^2=\sum X_i^2$$

In a homogeneous system of coordinates P is represented as:

$X\Delta_P X^t=0$ where $\Delta_p=\begin{pmatrix} I_d & 0 & 0 \\ 0 & 0 & -0.5 \\ 0 & -0.5 & 0 \end{pmatrix}$

Voronoi Diagrams Reminder (cont.)

The **polar hyperplane** of $A \epsilon E^{d+1}$, with respect to P is:

$$A^* = \{X \epsilon E^{d+1} : X \Delta_P A^t = 0\}, \text{ i.e.,}$$

$$A^* = \{X \in E^{d+1} : \; x_{d+1} = 2\sum_{i=1}^{d} A_i x_i - A_{d+1}\}$$

Two hyperspheres Σ_1 and Σ_2 are **orthogonal** if $\Sigma_1(C_2) = r_2^2$

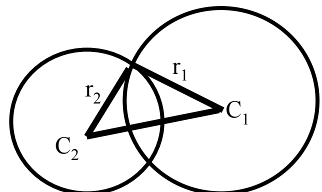
$$\|C_1 - C_2\|^2 = r_1^2 + r_2^2$$

$$2C_1 \cdot C_2 - \sigma_1 - \sigma_2 = 0$$

$$\Phi(\Sigma_1)\, \Delta_P\, \Phi(\Sigma_2) = 0$$

Geometric interpretation:

Note: In particular, points **on** the sphere are orthogonal spheres of zero radius.



Voronoi Diagrams Reminder (cont.)

Lemma 17.2.1: The set of all spheres that are orthogonal to Σ is mapped by Φ to the polar hyperplane $\Phi(\Sigma)^*$ of $\Phi(\Sigma)$.

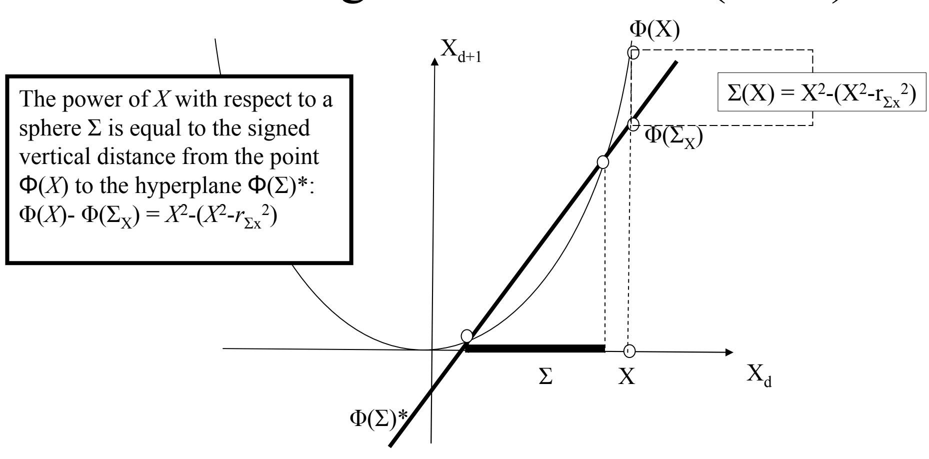
Lemma 17.2.2: The points of a sphere Σ in E^d lifted on the paraboloid P, belong to the polar hyperplane $\Phi(\Sigma)^*$ of $\Phi(\Sigma)$.

Voronoi Diagrams Reminder (cont.)

The power of a point X with respect to a sphere Σ is equal to the square of the radius of the sphere Σ_X orthogonal to Σ and centered at X.

Lemma 17.2.3: The power of X with respect to a sphere Σ is equal to the signed vertical distance from the point $\Phi(X)$ to the hyperplane $\Phi(\Sigma)^*$.

Voronoi Diagrams Reminder (cont.)



Voronoi Diagrams Reminder (cont.)

$X \epsilon \Sigma \Leftrightarrow \Phi(X) \epsilon\ \Phi(\Sigma)^* \Leftrightarrow \Phi(\Sigma) \epsilon\ \Phi(X)^*$

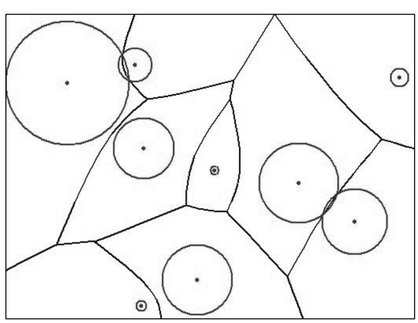
$X \epsilon \mathrm{ext}(\Sigma) \Leftrightarrow \Phi(X) \epsilon\ \Phi(\Sigma)^{*+} \Leftrightarrow \Phi(\Sigma) \epsilon\ \Phi(X)^{*+}$

$X \epsilon \mathrm{int}(\Sigma) \Leftrightarrow \Phi(X) \epsilon\ \Phi(\Sigma)^{*-} \Leftrightarrow \Phi(\Sigma) \epsilon\ \Phi(X)^{*-}$

Conclusion

The complexity of the Voronoi diagram of n points in E^d is $\Theta(n^{\lceil d/2 \rceil})$ in the worst case. We can compute such a diagram in $O(n\log n + n^{\lceil d/2 \rceil})$ time, which is optimal in the worst case.

Voronoi Diagrams with Non-Euclidean Metrics



Power Diagrams

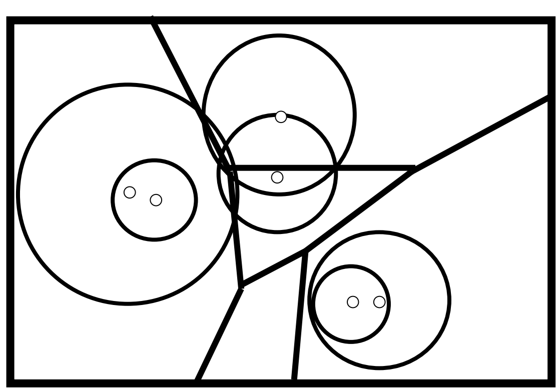
<u>Definition:</u>

Let $S=\{\Sigma_1,\ldots,\Sigma_n\}$ be a set of n spheres in E^d. To each Σ_i corresponds a region defined by:

$$P(\Sigma_i)=\{X\epsilon E^d : \Sigma_i(X)\leq \Sigma_j(X),\ i\neq j\}$$

The regions $P(\Sigma_i)$ and their faces are called the **power diagram** of S and denoted by Pow(S).

Power Diagrams (cont.)



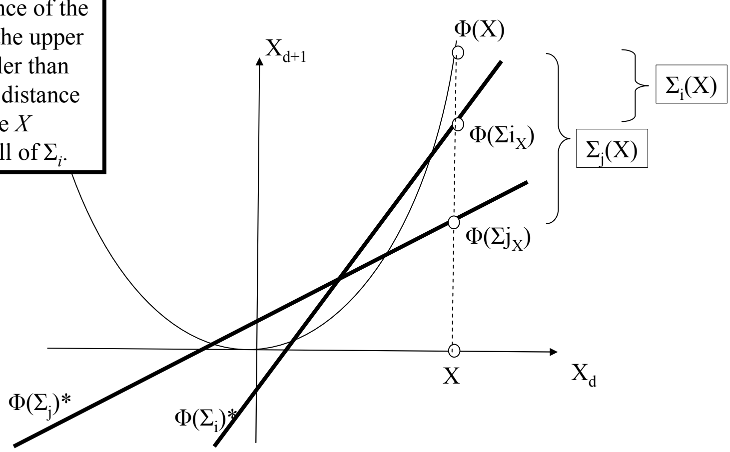
Power Diagrams (cont.)

Let $P(S)$ be the intersection of the halfspaces bounded below by the polar hyperplanes $\Phi(\Sigma_1)^*, \ldots, \Phi(\Sigma_n)^*$.

Theorem 18.1.1: Pow(S) is a cell complex in E^d. Its faces are obtained by projecting $P(S)$ from E^{d+1} to E^d.

Power Diagrams (cont.)

The signed distance of the point $\Phi(\Sigma_{iX})$ on the upper envelope is smaller than any other signed distance $\Phi(\Sigma_{jX})$. Therefore X belongs to the cell of Σ_i.



Power Diagrams - Conclusion

Theorem 18.1.2: The complexity of the power diagram of n spheres in E^d is $\Theta(n^{\lceil d/2 \rceil})$. The diagram can be computed in $O(n\log n + n^{\lceil d/2 \rceil})$ time.

Comments

- There can be redundant hyperplanes (spheres with an empty region).
- The spheres may be imaginary (i.e., $\|X-C\|^2 + r^2 = 0$).
- Any polytope in E^{d+1} corresponds to a power diagram.

Higher-Order Power Diagrams

Definition:

Let S_k be a subset of S of size k.

$$P(S_k)=\{X \epsilon E^d : \Sigma_i(X) \leq \Sigma_j(X),\ \Sigma_i \epsilon S_k,\ \Sigma_j \epsilon S \backslash S_k\}$$

The structure $P(S_k)$ is called the **power cell of S_k**. The union of non-empty power cells is called the **power diagram of order k of S** and is denoted by $\mathrm{Pow}_k(S)$.

Higher-Order Power Diagrams and Levels of Arrangements

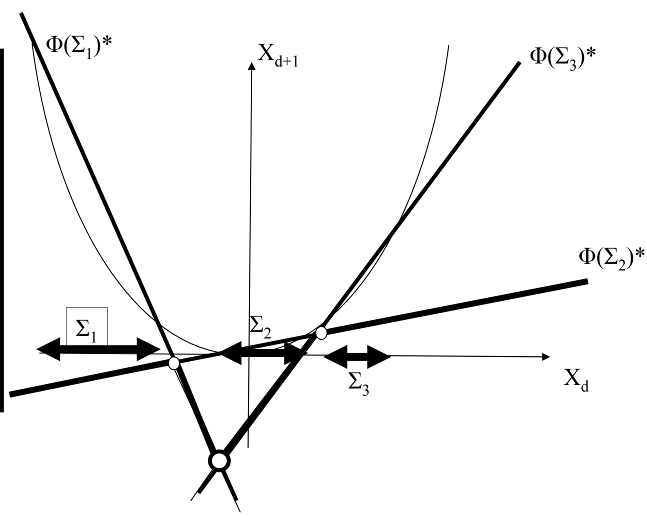
Let $A(S)$ be the arrangement of $\Phi(\Sigma_1)^*, \ldots, \Phi(\Sigma_n)^*$, the polar hyperplanes of spheres in S.

Theorem 18.1.3: The k-order diagram of S $\mathrm{Pow}_k(S)$ is obtained by projecting the faces of the cells at level k of the arrangement $A(S)$, onto E^d.

The l-faces of $\mathrm{Pow}_k(S)$ ($l<d$) are obtained by projecting the l-faces common to at least two cells of $A(S)$ at level k.

Higher-Order Power Diagrams and Levels of Arrangements (cont.)

The 2-order diagram corresponds to level 2 in the arrangement of polar hyperplanes. Note that not all the l-faces (vertices in the figure) are projected to the diagram.



Higher-Order Power Diagrams Conclusions

Theorem 18.1.4: The complexity of the first k power diagrams of a set of n spheres in E^d is $O(n^{\lceil d/2 \rceil} k^{\lceil d/2 \rceil})$.

These k-diagrams can be computed in $O(n^{\lceil d/2 \rceil} k^{\lceil d/2 \rceil})$ time if $d>2$ and in $O(nk^2 \log(n/k))$ time if $d=2$.

Affine Diagrams

An **affine diagram** is a diagram in which the locus of points equidistant from two sites (bisector) is a hyperplane.

To any affine diagram correspond a set of bisectors H_{ij} that satisfy the relation:

$$H_{ij} \cap H_{jk} = H_{ij} \cap H_{ik} = H_{ik} \cap H_{jk} = I_{ijk}$$

The diagram is said to be **simple** if the I_{ijk} are disjoint and not empty.

Affine Diagrams and Power Diagrams

Theorem 18.2.1: Any simple affine diagram in E^d is the power diagram of a set of spheres in E^d.

General Idea of Proof:

Constructing a set of n hyperplanes $P_1,..,P_n$ in E^{d+1} such that the vertical projection of $P_i \cap P_j$ is H_{ij}. Then for each P_i corresponds a sphere $\Sigma_i = \Phi^{-1}(P_i^*)$ whose polar hyperplane is exactly P_i and H_{ij} is the radical hyperplane between Σ_i and Σ_j.

Affine Diagrams and Power Diagrams

Theorem 18.2.2: The affine diagram whose hyperplanes H_{ij} have equations:

$$-2(C_i - C_j) \cdot X + \sigma_i - \sigma_j = 0$$

is the power diagram of the spheres Σ_i with center C_i and power σ_i.

Proof: H_{ij} can be written as $\Sigma_i(X) - \Sigma_j(X) = 0$.

Voronoi Diagrams for Quadratic Distance

For two points X, $A \in E^d$, the **general quadratic distance** from A to X is:

$$\delta_Q(X,A) = (X-A)\Delta\,(X-A)^t + \rho(A)$$

Where $\rho(A) \in \boldsymbol{R}$ and Δ is a real symmetric dxd matrix.

Voronoi Diagrams for Quadratic Distance (cont.)

All diagrams we have met so far have been special cases of Voronoi diagrams for quadratic distances.

- Standard Voronoi diagrams: $\Delta = I_d$ and $\rho(A)=0$.
- Power diagrams: $\Delta = I_d$ and $\rho(A) \neq 0$.
- Furthest-neighbor diagrams: $\Delta = -I_d$ and $\rho(A)=0$.

Voronoi Diagrams for Quadratic Distance - Conclusion

Theorem 18.2.3: The Voronoi diagram of n points for an arbitrary general quadratic distance in E^d has complexity $\Theta(n^{\lceil d/2 \rceil})$.

It can be computed in $O(n \log n + n^{\lceil d/2 \rceil})$ time in the worst case.

Proof: For any pair of points, the bisector is a hyperplane and thus (by 18.2.1) the diagram is an affine diagram.

Weighted Diagrams: Introduction

<u>An alternative representation for Voronoi diagrams</u>:

For each site M_i there is a cone

$$C(M_i): X_{d+1} = \delta(X, M_i) = \|X - M_i\|$$

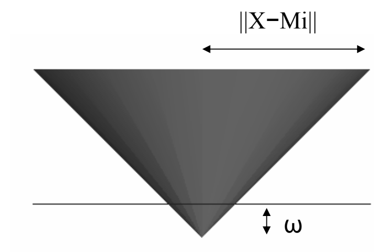
Which has apex $(M_i, 0)$ and an opening angle of $\pi/4$.

Weighted Diagrams: Introduction (cont.)

Observation 1: For an additive distance function

$$\delta(X, M_i) = \|X - M_i\| - \omega,$$

The cone has apex $(M_i, -\omega)$

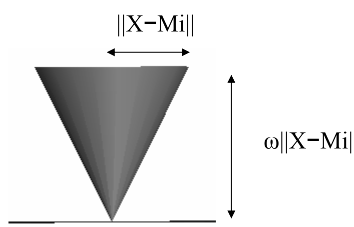


Weighted Diagrams: Introduction (cont.)

Observation 2: For a multiplicative distance function

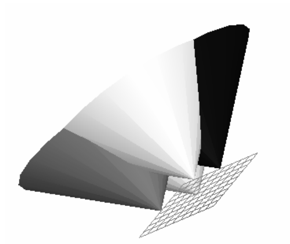
$$\delta(X, M_i) = \omega \|X - M_i\|,$$

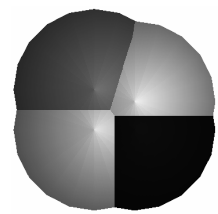
The cone has an opening angle of $arctan(1/\omega)$.



Weighted Diagrams: Introduction (cont)

The projection of the lower envelope of the cones onto E^d is the Voronoi diagram Vor(M).





Weighted Diagrams: Additive Weight

Let $M = \{M_1, ..M_n\}$ be a set of n points in E^d. To each point M_i corresponds a real r_i called the weight of M_i. The additive weight distance from X to M_i is:

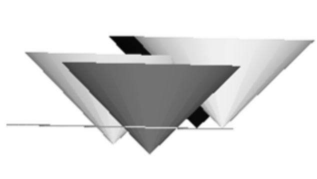
$$\delta(X, M_i) = \|X - M_i\| - r_i,$$

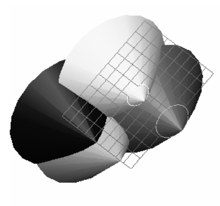
The Voronoi diagram of M with additive weights is denoted $\mathrm{Vor}_+(M)$.

Weighted Diagrams: Additive Weight

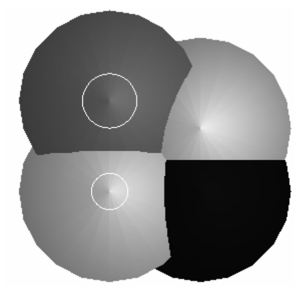
To get $\mathrm{Vor}_+(M)$ we will set each cone apex at (M_i, r_i). The projection of the lower envelope of the cones onto E^d is $\mathrm{Vor}_+(M)$.

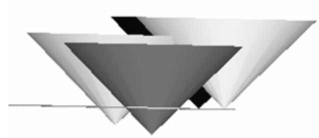
We will show that this can be done by computing a power diagram in E^{d+1}.

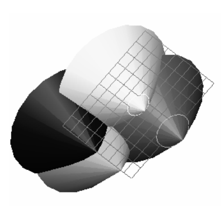




Weighted Diagrams: Additive Weight







Weighted Diagrams: Additive Weight

The set of equidistant points from two points of M is the projection of the intersection of two cones:

C_1: $(X_{d+1} + r_1)^2 = \|X - M_1\|^2$, $\quad X_{d+1} + r_1 > 0$

C_2: $(X_{d+1} + r_2)^2 = \|X - M_2\|^2$, $\quad X_{d+1} + r_2 > 0$

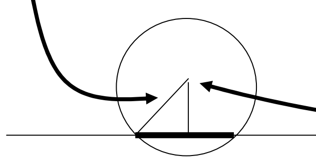
The intersection of the two cones is contained in the hyperplane H_{12}:

H_{12}: $-2(M_1 - M_2)\cdot X - 2(r_1 - r_2)\, X_{d+1} + M_1^2 - M_2^2 + r_2^2 - r_1^2$

Q: In R^2, what does that make the bisector curves?

Weighted Diagrams: Additive Weight

Let Σ_i' be the sphere (in E^{d+1}) centered at (M_i, r_i) and of radius $r_i\sqrt{2}$ (the intersection of Σ_i' with E^d is Σ_i).



What do we need Σ_i' for? We will see that the Voronoi cell $\mathrm{Vor}_+(M_i)$ corresponds to the power diagram cell of Σ_i'.

Weighted Diagrams: Additive Weight

Theorem: The cell of $\mathrm{Vor}_+(M)$ that corresponds to M_i is the projection of the intersection of the cone C_i and the cell of Σ_i' in the power diagram.

Proof: Let X_i be the point $(X, x_{d+1}) \epsilon E^{d+1}$.

$X \epsilon\ \mathrm{Vor}_+(M_i) \Leftrightarrow$

$$\left.\begin{array}{l}(x_{d+1}+r_i)^2 = \|X - M_i\|^2 \textbf{ and} \\ (x_{d+1}+r_j)^2 \leq \|X - M_j\|^2\ ,\ j \neq i\end{array}\right\} \begin{array}{l}X_i \text{ is on } C_\mathrm{i} \textbf{ and} \\ X_i \text{ is below all } C_\mathrm{j}\end{array}$$

$$\Leftrightarrow \Sigma_i'(X_i) \leq \Sigma_j'(X_i)\ \ ,\ j \neq i \quad \Big\}\ X_\mathrm{i} \text{ is in the cell of } \Sigma_i'.$$

Weighted Diagrams: Additive Weight

The additive diagram can thus be computed as follows:

1. Compute Σ_i' for i=1,…, n.
2. Compute the E^{d+1} power diagram of the Σ_i's.
3. For all i=1, …, n project onto E^d the intersection with the cone C_i of the cell of the power diagram that corresponds to Σ_i'.

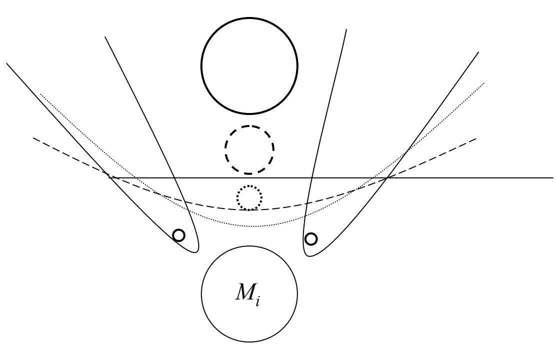
Weighted Diagrams: Additive Weight - Conclusions

Theorem 18.3.1: The Voronoi diagram of a set of n points in E^d with additive weights, has complexity $O(n^{\lfloor d/2 \rfloor + 1})$ and can be computed in $O(n^{\lfloor d/2 \rfloor + 1})$ time.

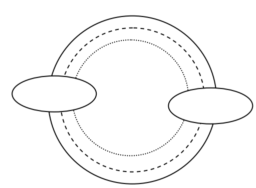
Note: For d=2, this is not optimal since each cell is connected (cones of $\pi/4$ angle), and thus (why?) the diagram has complexity $O(n)$.

Weighted Diagrams: Additive Weight - Conclusions

An example of an $O(n^2)$ cell in dimension 3:



Side view



View from below

Weighted Diagrams: Multiplicative Weight

Let $M = \{M_1, ..M_n\}$ be a set of n points in E^d. To each point M_i corresponds a positive real number $\rho(M_i)$ called the weight of M_i. The **multiplicative distance** from X to M_i is:

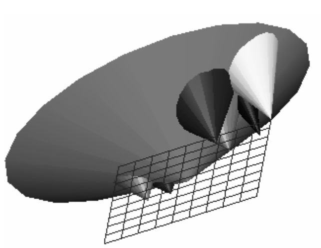
$$\delta(X, M_i) = \rho(M_i)\|X - M_i\|,$$

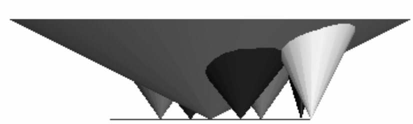
The Voronoi diagram of M with multiplicative weights is denoted $\text{Vor}_*(M)$.

Weighted Diagrams: Multiplicative Weight

For each M_i there is a cone given by:

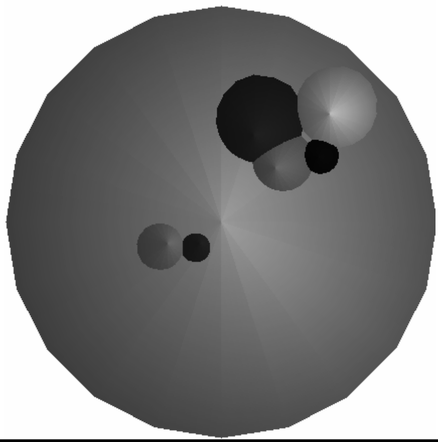
$$C(\Sigma_i)\text{: } X_{d+1} = \rho(M_i)\|X - M_i\|,$$





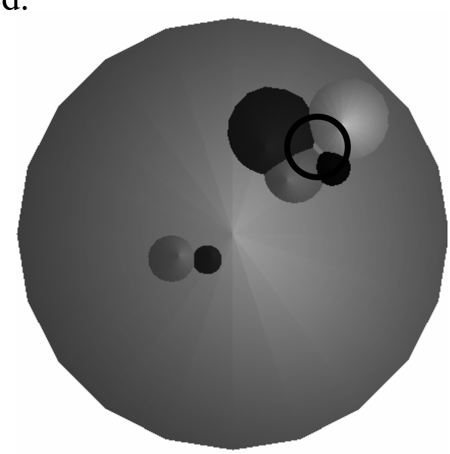
Weighted Diagrams: Multiplicative Weight

- The projection of the lower envelope of the cones C_i onto E^d is exactly $\mathrm{Vor}_*(M)$.



Weighted Diagrams: Multiplicative Weight

- Note that the cell of the diagram need not be connected.



Weighted Diagrams: Multiplicative Weight

The set of all points at equal distance from the sites M_i and M_j (the bisector) is the sphere:

$$X^2 - 2\frac{\rho_i M_i - \rho_j M_j}{\rho_i - \rho_j} \cdot X + \frac{\rho_i M_i^2 - \rho_j M_j^2}{\rho_i - \rho_j} = 0$$

$X^2 - 2\frac{\rho_i M_i - \rho_j M_j}{\rho_i - \rho_j} \cdot X + \frac{\rho_i M_i^2 - \rho_j M_j^2}{\rho_i - \rho_j} = 0$

Weighted Diagrams: Multiplicative Weight

The bisector sphere is represented in E^{d+1} as the point:

$$\Phi(\Sigma_{ij}) = \left(\frac{\rho_i M_i - \rho_j M_j}{\rho_i - \rho_j}, \frac{\rho_i M_i^2 - \rho_j M_j^2}{\rho_i - \rho_j} \right)$$

and its polar hyperplane H_{ij} (with respect to the paraboloid P) is:

$$(\rho_i - \rho_j) X_{d+1} - (2\rho_i M_i + 2\rho_j M_j) \cdot X + \rho_i M_i^2 - \rho_j M_j^2 = 0$$

Weighted Diagrams: Multiplicative Weight

We denote by Σ_i the spheres in E^{d+1} centered at $(\rho_i M_i, -\rho_i/2)$ and $\sigma_i = \rho_i M_i^2$, which have H_{ij} as their radical hyperplanes.

Theorem: The cell $\mathrm{Vor}_*(M_i)$ in $\mathrm{Vor}_*(M)$ is the projection of the intersection of the paraboloid P with the cell $P(\Sigma_i)$ in the power diagram of the Σ_i's .

Weighted Diagrams: Multiplicative Weight

Theorem: The cell $\mathrm{Vor}_*(M_i)$ in $\mathrm{Vor}_*(M)$ is the projection of the intersection of the paraboloid P with the cell $P(\Sigma_i)$ in the power diagram of the Σ_i's .

Proof:

$X \in \mathrm{Vor}_*(M_i) \Leftrightarrow \rho_i(X-M_i)^2 \leq \rho_j(X-M_j)^2$, for all $i \neq j$

$\Leftrightarrow H_{ij}(X, X^2) \leq 0$, for all $i \neq j$

$\Leftrightarrow \Sigma_i(\Phi(X)) \leq \Sigma_j(\Phi(X))$, for all $i \neq j$

$\Leftrightarrow \Phi(X)$ is in the cell $P(\Sigma_i)$.

Weighted Diagrams: Multiplicative Weight

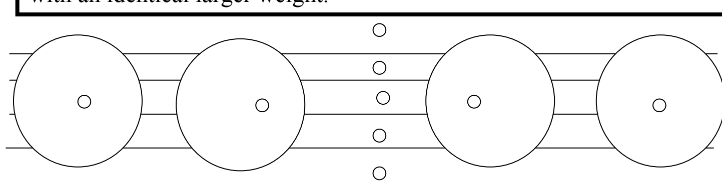
The multiplicative diagram can thus be computed as follows:

1. Compute Σ_i for i=1,…, n.
2. Compute the E^{d+1} power diagram of the Σ_i's.
3. For all i=1, …, n, project the intersection of the cell of the power diagram that corresponds to Σ_i, with the paraboloid P.

Weighted Diagrams: Multiplicative Weight - Conclusions

Theorem 18.3.2: The Voronoi diagram of a set of n points in E^d with multiplicative weights, has complexity $O(n^{\lfloor d/2 \rfloor +1})$ and can be computed in $O(n^{\lfloor d/2 \rfloor +1})$ time.

Example of an O(n^2) diagram in dimension 2: n/2 points are put on a vertical line and given the same weight, and n/2 points on a horizontal line with an identical larger weight.



L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$

The L_1 distance from a point X to a point A in E^d is defined as:

$$\delta_1(X, A) = \sum_{i=1}^{d} | X_i - A_i |$$

Let $M = \{M_1, ..M_n\}$ be a set of n points in E^d.

The Voronoi diagram of M for the L_1 distance is denoted $\mathrm{Vor}_{L1}(M)$.

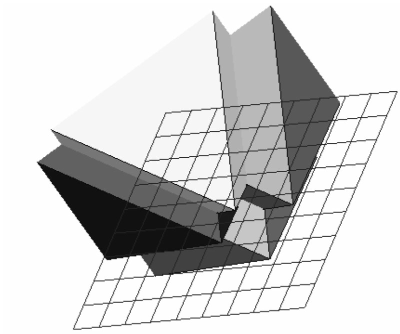
L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$

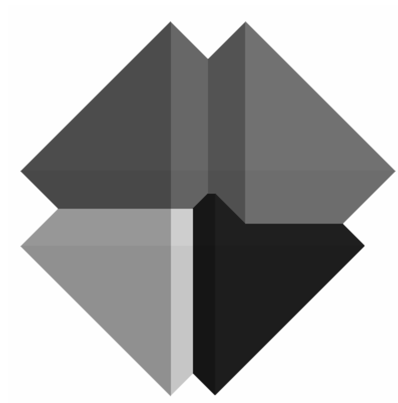
To each point M_i there corresponds a pyramid P_i of equation:

$$X_{d+1} = \delta_1(X, M_i)$$

The vertical projection of the lower envelope of the pyramids is the diagram $\mathrm{Vor}_{L1}(M)$.

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$





L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$

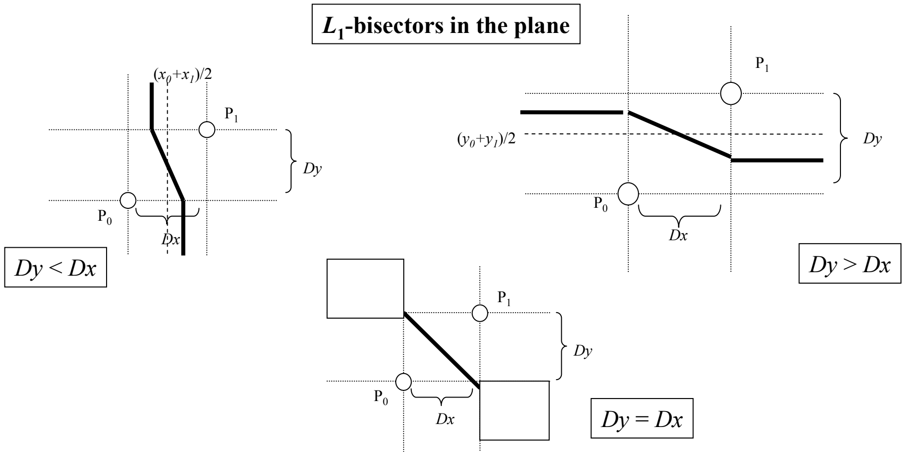
The complexity of the diagram $\mathrm{Vor}_{L1}(M)$ can be bounded by the complexity of the lower envelope of n d-simplices in E^{d+1}:

$$|\mathrm{Vor}_{L1}(M)| = O(n^d \alpha(n))$$

Where $\alpha(n)$ is an inverse of Ackerman's function.

Conjecture: For points in general position this bound is not attained (we prove this for d=2).

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$ in the Plane



L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$ in the Plane

Points are in ***L*₁-general position** if no two points are connected by a line parallel to one of the main bisectors, and no four points belong to a common co-cube.

If M is in L_1-general position in the plane then:

- The bisectors are polygonal lines consisting of three line segments (two of which are rays).
- $\mathrm{Vor}_{L1}(M)$ contains n connected cells.

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L1}(M)$ in the Plane

The diagram of n points in L_1-general position is a planar map with n cells whose vertices have degree two or three and whose edges consist of at most three segments.

From Euler's relation we get:

The complexity of $\mathrm{Vor}_{L1}(M)$ (for points in general position) in the plane is O(n).

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L\infty}(M)$

The L_∞ distance from a point X to a point A in E^d is defined as:

$$\delta_\infty(X, A) = \max_{i=1..d} | X_i - A_i |$$

Let $M = \{M_1, ..M_n\}$ be a set of n points in E^d.

The Voronoi diagram of M for the L_∞ distance is denoted $\mathrm{Vor}_{L\infty}(M)$.

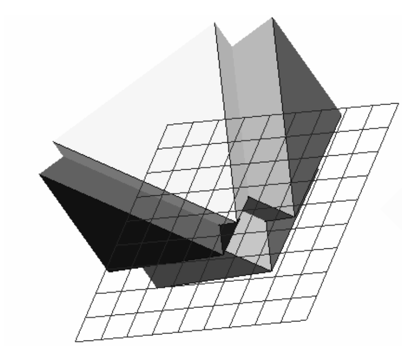
L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L_\infty}(M)$

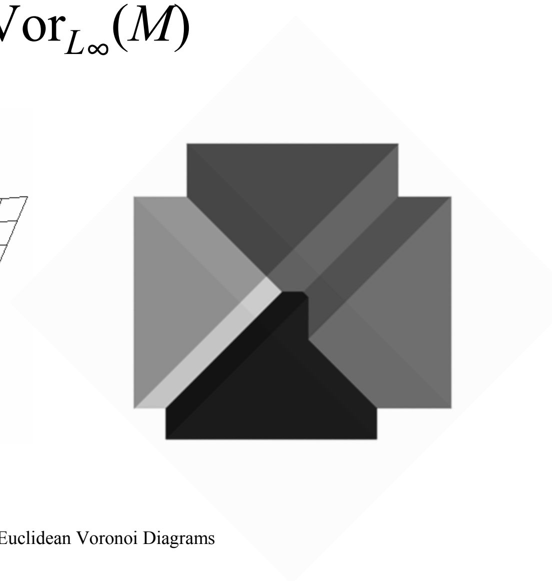
To each point M_i there corresponds a pyramid Q_i of equation:

$$X_{d+1} = \delta_\infty(X, M_i)$$

The vertical projection of the lower envelope of the pyramids is the diagram $\mathrm{Vor}_{L_\infty}(M)$.

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L_\infty}(M)$





L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L_\infty}(M)$

The complexity of the diagram $\mathrm{Vor}_{L_\infty}(M)$ can be bounded by the complexity of the lower envelope of n d-simplices in E^{d+1}:

$$|\ \mathrm{Vor}_{L_\infty}(M)| = \mathrm{O}(n^d \alpha(n))$$

Where $\alpha(n)$ is an inverse of Ackerman's function.

L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L_\infty}(M)$ in the Plane

Points are in ***L_∞*-general position** if no two points are connected by a line parallel to the axes, and no four points belong to a common co-cube whose facets are parallel to the coordinate axes.

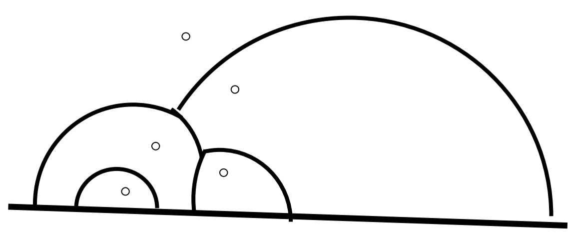
If M is in L_∞-general position the $\mathrm{O}(n^d \alpha(n))$ bound is not attained.

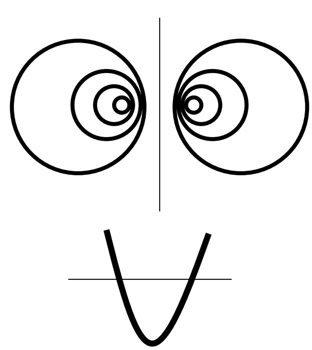
L_1 and L_∞ Metrics: Computing $\mathrm{Vor}_{L_\infty}(M)$

If the points are in L_∞-general position then the complexity of $\mathrm{Vor}_{L_\infty}(M)$ is the same as that of Vor(M), namely, $O(n^{\lceil d/2 \rceil})$.

For the case of d=2, this is easy to see – rotate the coordinate system by an angle of $\pi/4$ and the diagram is equivalent to an L_1-diagram.

Application - Voronoi Diagrams in Hyperbolic Space





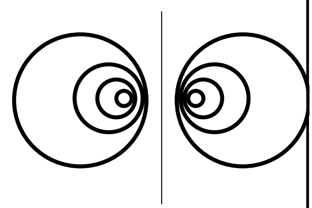
Voronoi Diagrams in Hyperbolic Space

Pencils of Spheres:

A **pencil of spheres** in E^d is a set of spheres that are affine combinations of two given spheres Σ_1 and Σ_2:

$$F = \left\{ \Sigma : \begin{vmatrix} \exists \lambda \in R, \forall X \in E^d, \\ \Sigma(X) = \lambda \Sigma_1(X) + (1-\lambda)\Sigma_2(X) \end{vmatrix} \right\}$$

Lifted to E^{d+1} with the mapping Φ, the pencil F is mapped to the line $\Phi(F)$ connecting $\Phi(\Sigma_1)$ and $\Phi(\Sigma_2)$.

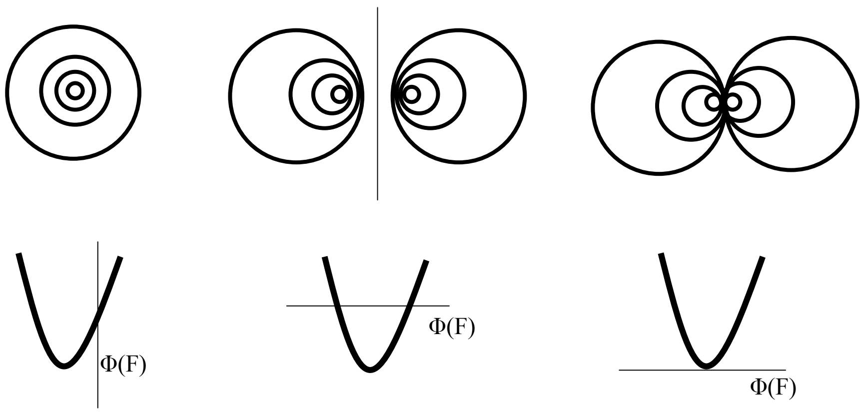


Voronoi Diagrams in Hyperbolic Space (Cont.)

Pencils Types:

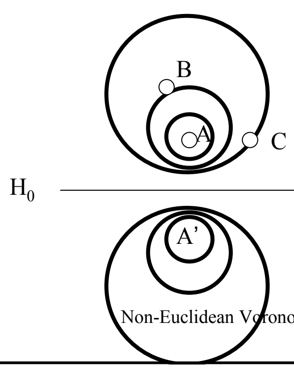
- If $\Phi(F)$ intersects the paraboloid P in only one point, then $\Phi(F)$ is a *pencil of concentric spheres*.
- If $\Phi(F)$ intersects P in two points, then $\Phi(F)$ is a *pencil with two limit points*.
- If $\Phi(F)$ is tangent to P, then $\Phi(F)$ is a *tangent pencil* (two limit points coincide).
- If the line $\Phi(F)$ does not intersect P, then $\Phi(F)$ is a *pencil with supporting sphere*.

Voronoi Diagrams in Hyperbolic Space (Cont.)



Voronoi Diagrams in Hyperbolic Space (Cont.)

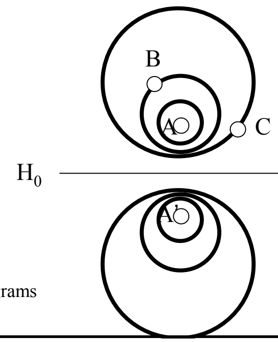
Let H_0 be the hyperplane X_d=0, and let A, B and C be points on the halfspace X_d>0. F_A is defined to be the pencil with two limit points A and A', where A' denoted the symmetric of A with respect to H_0.



Voronoi Diagrams in Hyperbolic Space (Cont.)

<u>Closer point comparison in hyperbolic space</u>:

Given two points B and C and an additional point A, the point B is closer than C to A for a hyperbolic distance, if the sphere F_A that passes through B has a smaller radius than that of the sphere that passes through C.



Voronoi Diagrams in Hyperbolic Space (Cont.)

$$X \in V_h(M_i)$$

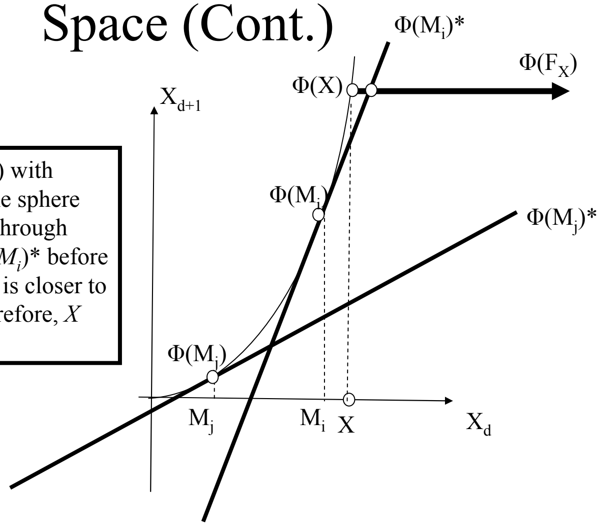
- If and only if the interior of the sphere in the pencil F_X that passes through M_i contains no point of M.

  X is closer to M_i than to any other point in M.

- If and only if the ray parallel to the X_d axis originating at $\Phi(X)$ (which corresponds to $\Phi(F_X)$) intersects the hyperplane $\Phi(M_i)^*$ before any other polar hyperplane $\Phi(M_j)^*$.

  The sphere of the pencil F_X that passes through M_i does not contain any other pencil sphere that passes through M_j.

Voronoi Diagrams in Hyperbolic Space (Cont.)



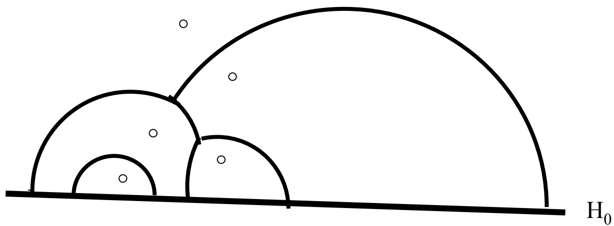
The intersection of $\Phi(F_X)$ with $\Phi(M_i)^*$ corresponds to the sphere of the pencil that passes through M_i. Since it intersects $\Phi(M_i)^*$ before any other hyperplane, M_i is closer to X than any other M_j. Therefore, X belongs to the cell of M_i.

Voronoi Diagrams in Hyperbolic Space (Cont.)

- The bisector surface of two points in hyperbolic distance is a half-sphere.
- A point X is equidistant from d+1 points $M_0,..,M_d$ iff $\Phi(X)$ is the parallel-to-the-X_d-axis projection of $\bigcap_{i=0}^{d} \Phi(M_i)^*$, onto the half-paraboloid.
- The hyperbolic Voronoi diagram can thus be obtained by X_d-parallel-projecting the polytope $V(M)=\cap\Phi(M_i)^{*+}$ onto the half-paraboloid, then projecting the result vertically onto the hyperplane $X_{d+1}=0$.

Voronoi Diagrams in Hyperbolic Space - Conclusion

- The complexity of the hyperbolic Voronoi diagram of n points in the hyperbolic half-space $\{X_d>0\}$ is $\Theta(n^{\lceil d/2 \rceil})$. We can compute such a diagram in $O(n\log n + n^{\lceil d/2 \rceil})$.



The End