Davenport-Schintzel Sequences

And Geometric Applications

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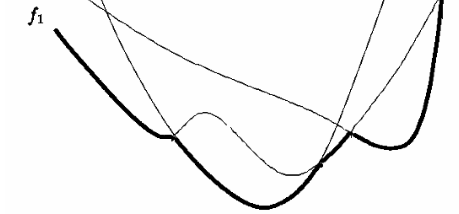
Motivation: Lower Envelopes

- Consider the lower envelope of a set of functions with up to ***s*** pairwise intersections
- What is the complexity of their lower envelope?



Davenport-Schintzel Sequences

- Let *n,s* be positive integers. A sequence $U = < u_1, u_2, ..., u_m >$ of symbols from a finite alphabet Σ of size *n* is a **Davenport-Schintzel Sequence - *DS(n,s)*** - if:

1. $\forall i < m \quad u_i \neq u_{i+1}$

2. There are no *s+2* indices $1 \le i_1 < i_2 < ... < i_{s+2} \le m$ for which there are $a, b \in \Sigma, a \neq b$ that hold:

$$u_{i_1} = u_{i_3} = .. = a, u_{i_2} = u_{i_4} = .. = b$$

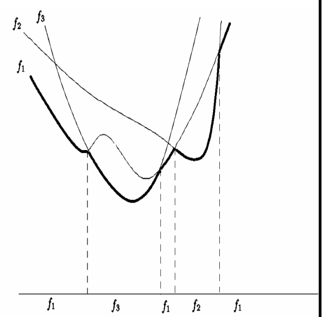
DS Sequences – cont'd

- A *DS(n,s)* is actually a sequence that doesn't allow alternation of two letters more than *s+1* times.
- "DAV**E**NPORT **S**CHINTZ**E**L **SE**QUENCE**S**" is a *DS(26,5) ("ESESES").*
- How does it apply to lower envelopes?

Lower Envelopes are DS sequences

- Let $F = \{f_1 .. f_n\}$ be a collection of *n* real-valued continuous functions on a common interval *L*, with at most *s* pairwise intersections. The lower envelope is: $E_F(x) = \min_{1 \le i \le n} f_i(x), \quad x \in L$
- The envelope is a (maximal) connected list of function portions, by function index:

$$U(f_1, f_2, .., f_n) = < u_1, u_2, ..., u_m >$$



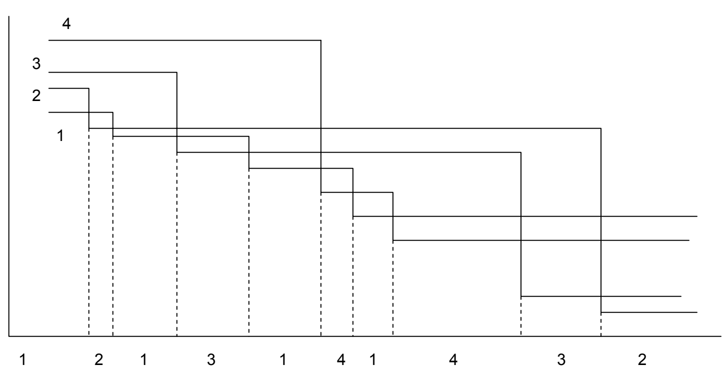
- **Lemma 1:** $< u_1, u_2, ..., u_m >$ is a *DS(n,s)*-sequence.
- **Proof:**
  - by definition there are no adjacent identical elements.
  - Suppose that there are *s*+*2* indices for which there is a substring of two alternately repeating functions f_1, f_2 (w.l.o.g. f_1 is first)*:*
    - *In Odd substring indices, we get* $f_1 < f_2$ *.*
    - *In even substring indices, we get* $f_1 > f_2$ *.*
  - Therefore, f_1, f_2 must intersect at least at *s+1* points! A contradiction.

- Conversely to Lemma 1, for any given *DS(n,s)*-sequence U, one can construct a collection of functions $f_1, f_2,.., f_n$ such that $U = U(f_1, f_2,.., f_n)$ (again, with at most *s* pairwise intersections).

Proof: Given a *DS(n,s)-sequence* $U = <u_1, u_2, ..., u_m>$, we define a set of functions $f_1, f_2,.., f_n$ so that:

  - w.l.o.g, a function's first appearance on the lower envelope is by order of indices.
  - *m-1* transition points are defined: $< x_1,.., x_{m-1} >$
  - *n+m-1* horizontal y-levels are defined.

- At $x < x_1$ all function assume $f_i(x) = i$.
- In each $x = x_j$, f_{u_j} falls down to the highest free y-level



Proof: $U = U(f_1, f_2, .., f_n)$

- By construction, the lower envelope is *U*.
- It is left to show that there are at most *s* pairwise intersections:
  - Intersection between f_i & f_j occurs when there is an appearance of *i* before a *j* in the sequence (or vice versa).
  - Therefore, should these functions intersect *s+1* times, there would be a subsequence <i..j..i..j..> of length *s*+2! A contradiction to *U=DS(n,s).*

Complexity of *DS(n,s)-sequences*

- $\lambda_s(n) = \max\{|U| : U \text{ is a } DS(n,s)-\text{sequence}\}$
- Some simple bounds:
  - $\lambda_1(n) = n$
    - $\lambda_1(n) \le n$ - no repeats allowed.
    - $\lambda_1(n) \ge n$ - *<1,2,3…,n>* is a *DS(n,1).*
  - $\lambda_2(n) = 2n-1$
    - $\lambda_2(n) \ge 2n-1$ - *<1,2,3,…..,n-1,n,n-1,……,1> is a DS(n,2).*
    - $\lambda_2(n) \le 2n-1$ - Proof using induction (in next slide).

- For *n=1* the upper bound is trivial.
- Assuming $\lambda_2(n-1) \le 2n-3$:
  - *U* is any *DS(n,2)-sequence.*
  - Removing from *U* the letter (denoted "*a*") which appear last for its first time. "*a"* only appears once (because of DS terms). The letter preceding "a" might be removed as well – we get a *DS(n-1,2)-sequence U*'.
  - $|U'| \le 2n-3$ by assumption, and so

    $|U| \le |U'| - 2 \le 2n-1$

  Q.E.D.

A Weak Bound for $\lambda_3(n)$

- $\lambda_3(n) \le 2n(\ln n + O(1)).$
- **Proof:** Let *U* be a *DS(n,3)-sequence*. Let *a* be the least frequent appearing symbol in *U.* Then, the number of occurrences of *a* must be at most $\lambda_3(n)/n$ times.
- Removing *a* and possible adjacent letters (can only appear in a first or final occurrence of *a)*, we get a *DS(n-1,3)-sequence*, and therefore: $\lambda_3(n) \le \lambda_3(n-1) + \lambda_3(n)/n + 2 \Rightarrow$

  $\Rightarrow \lambda_3(n) \le 2n(\ln n + O(1)).$

A Tight Bound for $\lambda_3(n)$

- It can be shown (through many-a-page..) that $\lambda_3(n) = \theta(n\alpha(n))$
- *α(n)* is the very-slow growing functional inverse of the ackermann's function.
- For any practical value of *n,* $\alpha(n) \leq 4$.

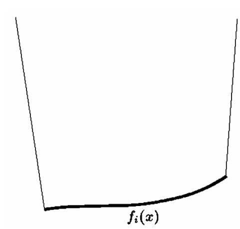
Bounds for Higher Orders

- It can also be shown, somehow, that:
  - $\lambda_4(n) = \theta(n \cdot 2^{\alpha(n)})$.
  - $$\lambda_s(n) \leq \begin{cases} n \cdot 2^{\alpha(n)^{(s-2)/2} + C_s(n)} & s \text{ is even} \\ n \cdot 2^{\alpha(n)^{(s-3)/2} \log \alpha(n) + C_s(n)} & s \text{ is odd} \end{cases}$$
  - $C_s(n)$ is a function of α(n) and *s*.

Some More Lower Envelopes

- **Lemma 2:** Let $F = \{f_1 .. f_n\}$ be a collection of partially defined functions on an interval *L*, with at most *s* pairwise intersections, then $U = U(f_1, f_2, .., f_n)$ is a *DS(n,s+2)-sequence*. Conversely, one can construct such a collection to fit a given DS-sequence.

- **Proof:** We extend each function's endpoints by an infinite, almost vertical, rays. It can be easily seen that we get fully continuous functions with at most *s+2* pairwise intersections.
- **Corollary to Lemma 2:** The lower envelope of *n* line segments in the plane is a *DS(n,3)-sequence* - Thus, $U = \theta(n\alpha(n))$.
  - There is a possible geometric realization of that bound – a simple example of the Ackermann's function in nature!

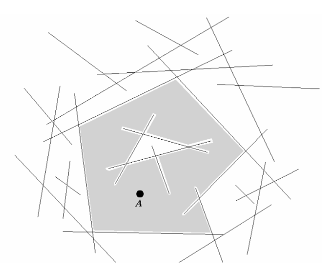


Geometric Applications of DS

- DS-sequences can be used to shed light on many problems. Demonstrated are:
  - Lower Envelopes
  - Cells in arrangement of segments
  - Nearest Neighbors (for dynamic points)
  - Geometric Graphs
  - …and many more.

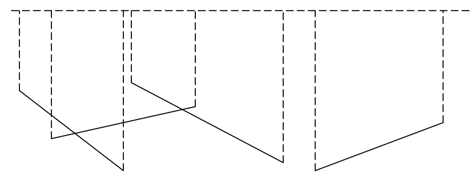
Arrangement of Segments

- Let *S* be a set of *n* segments in the plane.
- The arrangement *A(S)* is composed of:
  - **Vertices** – Endpoints & intersection points of segments.
  - **Edges** – Portions of segments between vertices.
  - **Cells –** Connected components of *E²/S*.
- Trivial cells do not contain endpoints, and are *O(n)*. We investigate non-trivial cells.



Lower Bound for a Single Cell

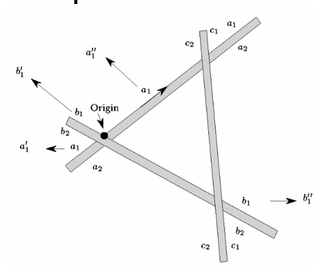
- According to Corollary 2, $U = \theta(n\alpha(n))$.
- This bound can be realized.
- With a choice of such *S, 2n* segments are added, almost vertical and long enough.
- A final horizontal segment is added.
- Now, the lower envelope is an unbounded cell of complexity $\Omega(n\alpha(n))$.



Upper Bound for A Single Cell

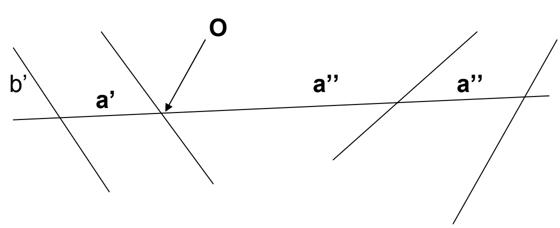
- General position assumed (as always).
- Segment sides (edges) are treated (*2n* sides).
- The boundary of a cell is a *single connected component,* denoted as Γ .
  - Should there be other components inside, the additive complexity will vouch for them.
- **Lemma 3:** Let $s \in S$ contain at least one edge of Γ , then the edges of Γ contained in *s* are traversed on the boundary of Γ in the same order they are traversed on *s*.
  - Proof: By Definition.

- Labeling the sequence Γ (considering different sides of segments) leads to a circular sequence (denoted Σ_Γ).
- It is linear by choosing a point of origin.
  - It is not always possible to choose an origin O such that the sequence would be *DS(2n,3).*



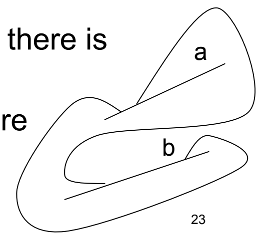
$$\Sigma_\Gamma = \{a_1, c_2, c_1, a_1, a_2, c_1, b_1, b_2, c_1, c_2, b_2, a_2, a_1, b_2, b_1\}$$

- To solve this, extra labeling is introduced:
  - Every portion of a segment side *s* intersecting with *a* before the origin point is now *s'*, and after that point it is *s''* (in the same orientation, say clockwise).

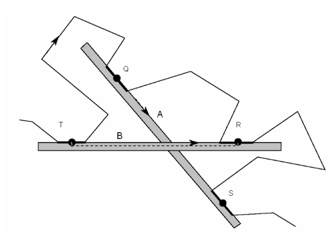


- **Lemma 4:** The new sequence (denoted Σ^*_Γ) is a *DS(4n,3)-sequence* (actually, a *DS(3n,3)-sequence..).*

- **Proof**: Σ^*_Γ has at most *4n* (*3n*) distinct labels, and does not contain identical consecutive elements. It is left to show that *ababa* is not a subsequence of Σ^*_Γ
  - **Aiding argument:** if *abab* is a subsequence of Σ^*_Γ, then *a* and *b* intersect.
  - **Proof:** Let *Q,R,S,T* be points on Γ in that order, and of the segment *A* and *B*, such that $Q,S \in A \quad R,T \in B$.
    - If any of these points are adjacent in Σ^*_Γ, there is an intersection.
    - Otherwise, disconnected components are created.



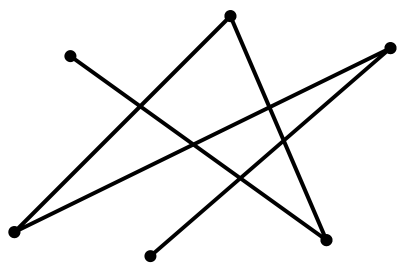
- If *ababa* is a subsequence of Σ^*_Γ, then *a* and *b* would have to intersect twice!
  - Because of the sequence *abab, a* and *b* intersect with *a* former in the sequence. having the sequence *baba* denotes another intersection with *b* first, which is a contradiction.



  - Therefore, Lemma 4 holds, and the complexity of a single cell is $\theta(n\alpha(n))$.

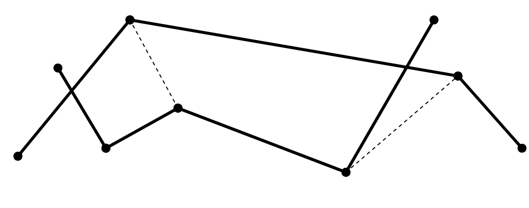
Geometric Graphs

- A geometric graph *G(V,E)* is a graph of straight line segments as edges between vertices.



Convex Positions

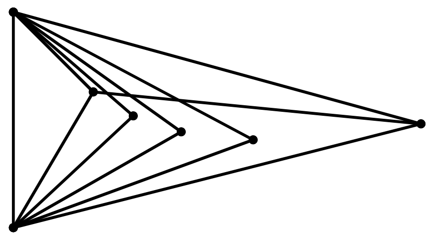
- Two Edges of a Geometric Graph are in *Convex Position*, if they can be a part of a convex quadrilateral.



- What are the bounds on the number of edges *e* of a graph with *n* vertices in which no two edges are in convex position? (also called an *improper graph)*.

Lower Bound of *e*

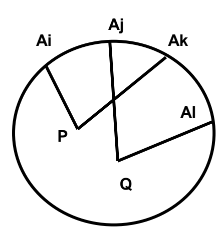
- The following construction shows that
  $e \geq 2n - 2$



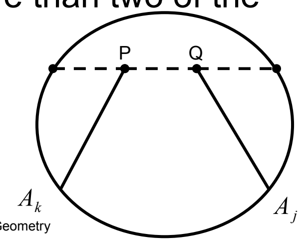
Upper bound of e

- The bound of $e \leq 2n - 1$ will be proven (seems familiar? But first some Lemmata):

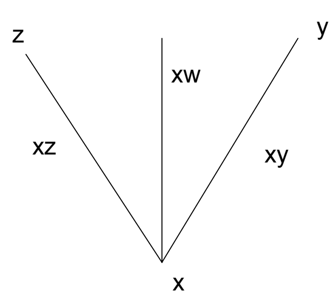
- **Lemma 5:** Let A_i, A_j, A_k, A_l be four points appearing in this order on a closed convex curve *γ*. Let *P,Q* be two points inside *γ*. Then, among the four segments
  $$PA_i, QA_j, PA_k, QA_l$$
  two will be in convex position if no segment contains one of *P,Q* and its supporting line ℓ contains both.



- **Proof:** Let $\ell = \ell(P,Q)$ be the line through these points, and ℓ^+ and ℓ^- be the two half-planes defined by it. Then, if ℓ^+ (ℓ^-) contains two disjoint segments, they are in convex position.
- Other cases:
  - One of the points is on ℓ *(*regarding the terms)
  - One half-plane contains more than two of the points.

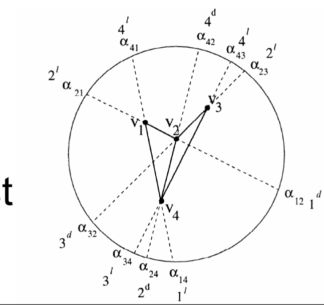


- An edge *xy* is *right (left)* of an edge *xz* if $\overrightarrow{xy}$ is obtained from $\overrightarrow{xz}$ by a (counter) clockwise rotation around *x* by positive angle, less than π. *xy* is the *rightmost (leftmost)* edge of *x*, if there is no edge to its right (left).



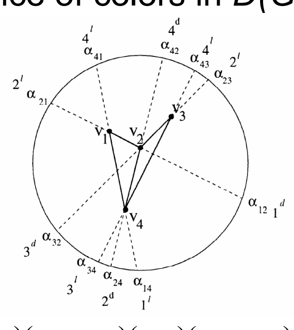
Proving The Upper Bound on *e*

- Let $V = \{v_1, v_2, .., v_n\}$ and a circle *C* that contains *G*. Each segment of *G* is extended from both sides to reach two points of *C*. Each point is labeled α_{ij} , being on $\overrightarrow{v_i v_j}$.
- The *Color* of α_{ij} is dark *i* if $v_i v_j$ is an interior edge of v_j , and light if it is leftmost or rightmost



Some More Definitions

- *D(G)* - The sequence of all points (cyclic).
- An *arc* – a maximal sequence of points with the same color (light or dark).
- *PS(G)* – The sequence of colors in *D(G)*



- In this example:
  - *Arcs:* $(\alpha_{41}, \alpha_{42}, \alpha_{43})(\alpha_{23})(\alpha_{12}, \alpha_{14})(\alpha_{24})(\alpha_{41}, \alpha_{41})(\alpha_{41})$
  - *PS(G)*=(4,2,1,2,3,2)

Key Lemmata

- **Lemma 6:** *PS(G)* is a *DS(n,2)-cycle* (the maximum length of a *DS(n,2)-cycle* is *2n-2*, proved in a similar fashion to the maximum length of a *DS(n,2)-sequence).*

- **Lemma 7:** An arc contains at most one dark point.

Upper bound approved

- Resulting from Lemmata 6 & 7:
  - *|D(G)|=2e=#light+#dark*
  - Every vertex has at most one rightmost (leftmost) arc, and so *#light≤2n.*
  - *#dark* ≤|*PS(G)*| (Lemma 7).
  - |*PS(G)*| ≤2n-2, and so:

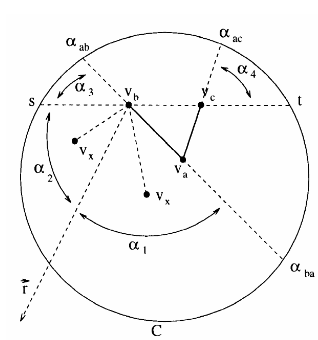
  - ***e ≤2n-1***
    - The proof will be complete after actually proving Lemmata 6 and 7.

Proof of Lemma 6

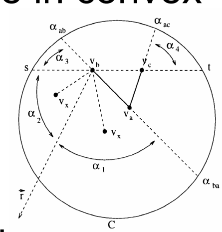
- It is obvious that there are no adjacent elements.
- Assume that there are four points $\alpha_{av_1}, \alpha_{bv_2}, \alpha_{av_3}, \alpha_{bv_4}$ along the circle. Then (assuming general position), according to Lemma 5, this is a contradiction, since they would be in convex position.

Proof of Lemma 7

- Suppose that points α_{ab}, α_{ac} are both dark. Assume (w.l.o.g.) that *ac* is to the right of *ab*. These edges are interiors of *b,c,* respectively.
- Let *bx (cy)* be an edge to the right (left) of *ba (ca)*. Let $\vec{r} \parallel ca$.



- If $x \in \alpha_1$, then α_{ab}, α_{ac} are not on the same arc.
- If $x \in \alpha_2$, then *bx* and *ca* are in convex position.
- Therefore, $x \in \alpha_3$ (and similarly, $y \in \alpha_4$), and then we get that *bx* and *cy* are in convex position! A contradiction.



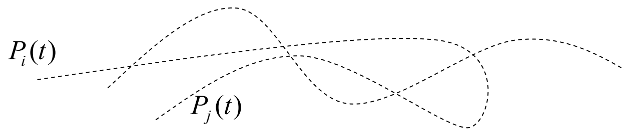
- Therefore, Lemma 6 & 7 hold, and as promised: **Q.E.D.**

Dynamic point – Nearest Neighbor

- Let $P = \{p_1, p_2, .., p_n\}$ be a set of points in the plane, whole locations are time-based functions: $p_i(t) = \{x_i(t), y_i(t)\}$
- The functions are assumed to be polynomials of maximum degree *s*.

Nearest Neighbors

- At time *t*, let *P(t)* denote the position of all points. $p_i(t) \in P(t)$ Is the *nearest neighbor* of $p_j(t) \in P(t)$ if $dist(p_i(t), p_j(t)) \leq \underset{k \neq i,j}{dist}(p_k(t), p_j(t))$.



- What is the maximum possible number of changes in the nearest neighbor for a given point?

- For every *i≠j,* define:

$$D_{ij}(t) = dist^2(p_i(t), p_j(t)) = (x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2$$

- We get a polynomial of degree at most *2s.*
- The lower envelope of all $D_{ij}(t)$ is the nearest neighbor, and its complexity is the number of changes.
- Therefore, the nearest neighbor changes at most $\lambda_{2s}(n)$ times.

Summary

- Davenport-Schinzel Sequences are sequences that doesn't allow long alternations.
- They can used to set bounds for known geometrical applications, such as lower envelopes, geometric graphs and nearest neighbors.

Bibliography

- Jean-Daniel Boissonnat and Mariette Yvinec. *Algorithmic geometry*. Cambridge University Press, 1998,
- Micha Sharir and Pankaj K. Agarwal. *Davenport-Schintzel sequences and their geometric applications.* Cambridge University Press, 1995
- M. Katchalski and H. Last. On geometric graphs with no two edges in convex position. *Discrete and computational geometry, 19, 1998.*

Thank You!

- Couldn't find a picture of A. Schintzel, so I brought a picture of A Schnitzel: