Voronoi Diagrams
in
Euclidean Space

Alina Shaikhet

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Roadmap of the Talk

1. Introduction. Main results
2. Theoretical part
3. Possible extensions
4. Conclusions

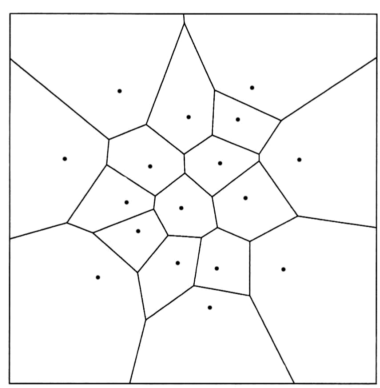
Introduction. Main Results

a) Definition of Voronoi diagrams.
b) Construction in 2-dim space. Higher dimensions.
c) Parabolic representation. Voronoi diagrams and polytopes.
d) Delaunay triangulation. Connection with Voronoi diagrams.

Voronoi Diagram: Definition

For a given set of sites (points) *on space* E^d,

<u>*Voronoi Diagram*</u> is the subdivision of the space into <u>cells</u>, such that *each point of the space is assigned to the nearest site.*



Voronoi Diagram: Definition (cont.)

Given $M = \{M_1, \ldots, M_n\}$: set of ***sites*** (points) in E^d, to each M_i attach the ***cell*** $V(M_i)$ as follows:

$$V(M_i) = \{X \in E^d : \delta(X, M_i) \le \delta(X, M_j) \;\; for \;\; any \;\; j \ne i\}.$$

Here $\delta(\cdot,\cdot)$ - Euclidean distance in E^d.

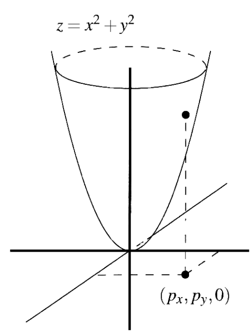
Parabolic Construction: Adding Dimension

We consider the construction in *d* dimensions with the time complexity $O(n \log n + n^{\lceil d/2 \rceil})$.

1. For each point $p = (p_1, .., p_d) \in E^d$ construct $p^* = (p_1, .., p_d, p_{d+1}) \in E^{d+1}$, where $p_{d+1} = p_1^2 + .. + p_d^2$.

2. Now space E^d is represented as a parabolic surface $Q \in E^{d+1}$

$$Q = \{q \in E^{d+1} : q_{d+1} = q_1^2 + .. + q_d^2\}$$

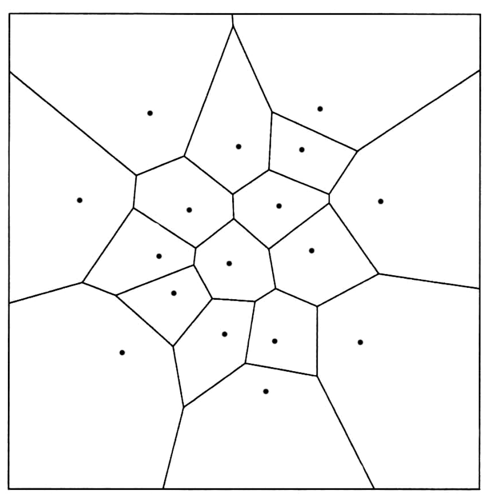


Parabolic Construction: Adding Dimension (cont.)

3. For each site $M_i \in M$ consider the corresponding $M_i^* \in Q$
4. For each projected site $M_i^* \in Q$ construct the hyperplane H_i tangent to Q at point M_i^*.
5. The intersection of the *n* half spaces lying above the hyperplanes defines a polytope in E^{d+1}.
6. The facets of the *obtained polytope* are projected down to E^d to get *exactly* the cells of the Voronoi Diagram.

Delaunay Triangulation: Definition

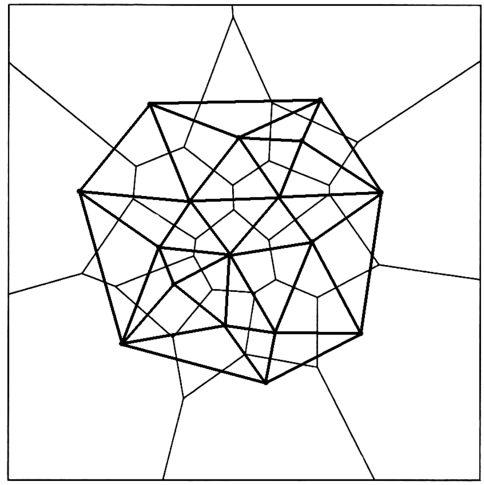
1. Connect all the pairs of sites whose Voronoi cells are adjacent.



Delaunay Triangulation: Definition

1. Connect all the pairs of sites whose Voronoi cells are adjacent.

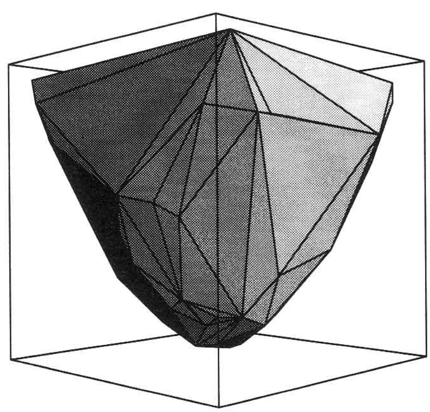
2. The resulting set of segments forms the Delaunay triangulation.



Delaunay triangulation: Parabolic Construction

1. – 3. the same
4. Construct L, the lower envelope of the convex hull of points $M_i^* \in Q$.

5. The facets of L are projected down to get *exactly* the cells of the *Delaunay triangulation*.



Theoretical Part

a) Power of a point w.r.t. a sphere.
b) Point representation of spheres.
c) Polarity. Polar hyperplanes.
d) Orthogonal spheres.
e) The connection with Voronoi Diagram.

Power of a Point w.r.t. a Sphere

Let Σ be a sphere in E^d with center C and radius r.

For any point $X \in E^d$ define its power w.r.t. Σ as

$$\Sigma(X) = XC^2 - r^2.$$

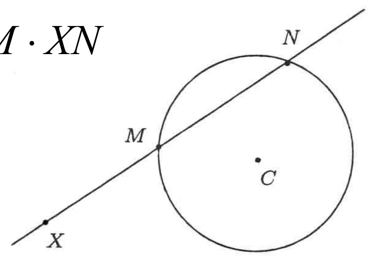
Here XC is a vector ($XC = \overrightarrow{XC}$).

Properties of $\Sigma(\cdot)$

1. $\Sigma(X) = 0, for\ \ X \in \Sigma$
2. $\Sigma(X) > 0, for\ \ X\ \ outside\ \Sigma$
3. $\Sigma(X) < 0, for\ \ X\ \ inside\ \Sigma$

4. If D is any line that contains X; if M and N are the intersection points of D with the sphere $\sum$, then

$$\Sigma(X) = XM \cdot XN$$



Properties of $\Sigma(\cdot)$ (cont.)

$$\Sigma(X) = XM \cdot XN = XC^2 - r^2$$

- Proof:

1. If D is a line that connecting X and C:

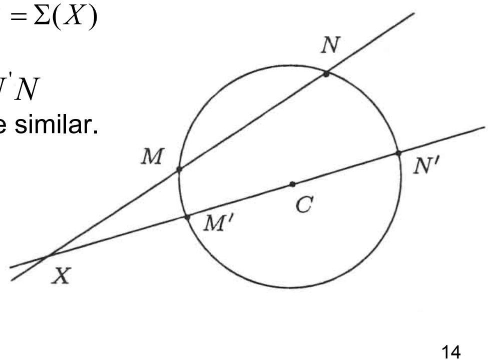
$$XM \cdot XN = |XM| \cdot |XN| \cdot \cos(\angle MXN) = |XM| \cdot |XN| =$$
$$= (|XC| - r)(|XC| + r) = XC^2 - r^2 = \Sigma(X)$$

2. Otherwise: $\angle XMM^{'} = \angle XN^{'}N$

The triangles *XMM'* and *XNN'* are similar.

So, $\frac{|XM|}{|XN'|} = \frac{|XM'|}{|XN|}$

$$|XM| \cdot |XN| = (|XC| + r)(|XC| - r)$$
$$|XM| \cdot |XN| \cdot \cos(0) = XC^2 - r^2$$
$$XM \cdot XN = XC^2 - r^2$$



Point Representation of Spheres

For each sphere $\sum$ with center *C* and radius *r*, define

$$\sigma = \Sigma(O) = \Sigma(origin) = OC^2 - r^2 = C^2 - r^2$$

Pay attention, that pair (C,σ) describes sphere $\sum$ completely.

Introduce the mapping $\phi : \phi(\Sigma) = (C,\sigma) \in E^{d+1}$.

The mapping ϕ takes a sphere $\sum$ in E^d to the <u>point</u> $(C,\sigma) \in E^{d+1}$.

We embed E^d as the hyperplane in E^{d+1} whose equation is $x_{d+1} = 0$.

Point Representation of Spheres

<u>Connection with paraboloid:</u>

Consider a point $X \in E^d$ as a sphere with center X and $r = 0$.
The correspondent σ equals $\sigma = X^2 = x_1^2 + \ldots + x_d^2$
and the mapping is $\phi(X) = (X,\sigma)$.

Thus, for any point in $X \in E^d$ the correspondent $\phi(X)$
lies on the paraboloid $Q = \{q \in E^{d+1} : q_{d+1} = q_1^2 + .. + q_d^2\}$.

Polarity. Polar Hyperplanes

Useful notation: $X \in E^d$, but $\underline{X} \in E^{d+1}$.

For each point $\underline{P} = (p_1, \dots, p_{d+1}) \in E^{d+1}$ define a unique (*polar*) hyperplane $\underline{H}$ as follows:

$$\underline{H} = \{\underline{X} \in E^{d+1} : \; x_{d+1} = 2\sum_{i=1}^{d} p_i x_i - p_{d+1}\}$$

Actually, there is a one-to-one correspondence between the points of E^{d+1} and the non-vertical hyperplanes in E^{d+1}.

Polarity. Polar Hyperplanes (cont.)

Our case: for a sphere Σ take the corresponding $\phi(\Sigma) = (C, \sigma) \in E^{d+1}$ and construct the polar hyperplane $\phi(\Sigma)^*$.

$$\phi(\Sigma)^* = \{\underline{X} \in E^{d+1} : \; x_{d+1} = 2\sum_{i=1}^{d} c_i x_i - \sigma\}$$

or

$$\phi(\Sigma)^* = \{\underline{X} = (X, x_{d+1}) \in E^{d+1} : \; x_{d+1} = 2C \cdot X - \sigma\}$$

Connection with Paraboloid:

Recall...

1. For a point $X \in E^d$ as a sphere with center X and $r = 0$ we have $\sigma = X^2$ and the correspondent $\phi(X) = (X, \sigma)$ lies on the paraboloid
$$Q = \{q \in E^{d+1} : q_{d+1} = q_1^2 + .. + q_d^2\}.$$
2. Then the polar hyperplane $\phi(X)^*$ is tangent to Q at point $\phi(X)$.

   Proof outline:

   a) By definition we get that $\phi(X) \in \phi(X)^*$.

   b) The hyperplane $\phi(X)^*$ intersects a paraboloid in only one point:

   for any $\underline{Y} \in \phi(X)^* \cap Q$ we get that necessarily $\underline{Y} = \phi(X)$.

Orthogonal Spheres-1

Two spheres Σ_1 and Σ_2 are orthogonal ($\Sigma_1 \perp \Sigma_2$)

if their centers C_i and radii r_i satisfy
$\Sigma_1(C_2) = r_2^2$ or, equivalently, $\Sigma_2(C_1) = r_1^2$.

The power of C_2 w.r.t. a sphere Σ_1 equals r_2^2.

Actually: $\Sigma_1 \perp \Sigma_2$ iff the angle (IC_1, IC_2) at any intersection point $I \in \Sigma_1 \cap \Sigma_2$ is a right angle (90^0).

Orthogonal Spheres-2

Recalling that $\sigma_i = C_i^2 - r_i^2,\ i = 1,2$ we have

$$\Sigma_1 \perp \Sigma_2 \Leftrightarrow C_1 \cdot C_2 - \frac{1}{2}(\sigma_1 + \sigma_2) = 0 \Leftrightarrow \sigma_2 = 2C_1 \cdot C_2 - \sigma_1$$

So, two spheres Σ_1 and Σ_2 are *orthogonal* if $\phi(\Sigma_1) \in \phi(\Sigma_2)^*$
or, equivalently, $\phi(\Sigma_2) \in \phi(\Sigma_1)^*$.

It is said that $\phi(\Sigma_1)$ *and* $\phi(\Sigma_2)$ are *conjugate.*

Lemma 17.2.1 : *The set of spheres in* E^d *that are orthogonal to a given sphere* Σ *is mapped by* ϕ *to the polar hyper-plane* $\phi(\Sigma)^*$ *of* $\phi(\Sigma)$.

Orthogonal Spheres-3

The sphere that passes through a given point $X \in E^d$ is *orthogonal* to X as *to a zero-radius sphere centered at* X.

Corollary*: The set of spheres in* E^d *that pass through a given point* X *is mapped by* ϕ *to the polar hyper-plane* $\phi(X)^*$ *of* $\phi(X)$*, tangent to the paraboloid Q at* $\phi(X)$.

Orthogonal Spheres-4

Lemma *(location of a polar hyperplane)* :

1. *The intersection of* $\phi(\Sigma)^*$ *with Q is the image under* ϕ *of the set of spheres with radius* 0*, that are orthogonal to* Σ*, namely,* Σ *itself.*

2. *Let* Σ *be a sphere in* E^d*. Then, the points of* Σ *, lifted on the paraboloid Q in* E^{d+1}*, belong to a unique hyperplane that intersects Q exactly at these points. This hyperplane is the polar hyperplane* $\phi(\Sigma)^*$ *of* $\phi(\Sigma)$.

Orthogonal Spheres-5

Lemma 17.2.3 *The power of X with respect to a sphere* Σ *equals the signed vertical distance from the point* $\phi(X)$ *to the hyperplane* $\phi(\Sigma)^*$.

Proof: Construct a sphere Σ_X, centered at X and $\Sigma_X \perp \Sigma$.
By definition, its radius satisfies $r_{\Sigma_X}^2 = \Sigma(X)$.
Now, $\phi(\Sigma_X)$ and $\phi(X)$ are placed on the same vertical line, that passes through X and intersects $\phi(\Sigma)^*$. This is since $\phi(\Sigma_X) \in \phi(\Sigma)^*$.
The x_{d+1} - coordinates of $\phi(\Sigma_X)$ and $\phi(X)$ are

X^2 *and* $\Sigma_X(0) = X^2 - r_{\Sigma_X}^2 = X^2 - \Sigma(X)$, respectively.

The difference between these coordinates is the *signed vertical distance*.

Orthogonal Spheres-6

Lemma 17.2.4 Let X *and* Σ *be a point and a sphere in* E^d. *Then,*

1. $X \in \Sigma \quad \Leftrightarrow \quad \phi(X) \in \phi(\Sigma)^* \quad \Leftrightarrow \quad \phi(\Sigma) \in \phi(X)^*$

2. $X \in \mathrm{int}(\Sigma) \quad \Leftrightarrow \quad \phi(X) \in \phi(\Sigma)^{*-} \quad \Leftrightarrow \quad \phi(\Sigma) \in \phi(X)^{*-}$

3. $X \in ext(\Sigma) \quad \Leftrightarrow \quad \phi(X) \in \phi(\Sigma)^{*+} \quad \Leftrightarrow \quad \phi(\Sigma) \in \phi(X)^{*+}$

Here H^+ *and* H^- *define the halfspaces lying above and below the hyperplane* H, *respectively.*

Radical Hyperplane

Let Σ_1 and Σ_2 be two spheres in E^d.
The radical hyperplane H_{12} satisfies

$$H_{12} = \{X \in E^d : \ \Sigma_1(X) - \Sigma_2(X) = 0\}.$$

I.e., H_{12} *is the set of points of* E^d *that have the same power with respect to both spheres.*

Observation 1: *The spheres that are orthogonal to* Σ_1 *and* Σ_2 *are mapped by* ϕ *to the intersection of* $\phi(\Sigma_1)^* \cap \phi(\Sigma_2)^*$ *, which can be projected onto* E^d *to* H_{12} .

Think about spheres, that pass through two given points....

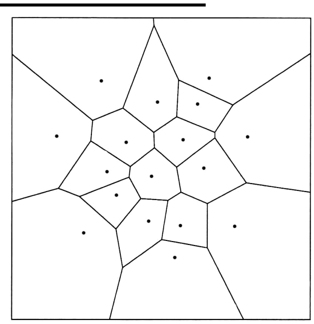
Voronoi Diagrams

Let $M = \{M_1, \ldots, M_n\}$ be a set of points in E^d.
Embed the space E^d into E^{d+1} as the hyperplane $x_{d+1} = 0$.

As before, construct the paraboloid Q, specify the points $\phi(M_i) \in Q$ and the corresponding polar hyperplanes $\phi(M_i)^*$, which are tangent to Q at points $\phi(M_i)$.

Let $V(M)$ denote the polytope that is the intersection of the n halfspaces, lying *above* the hyperplanes $\phi(M_i)^*$.

Voronoi Diagrams-Intuition



$M = \{M_1, \ldots, M_n\}$ - sites in E^d.

1. *Each cell* $V(M_i)$ *is the set of centers of spheres, such that the boundary of such a sphere contains* M_i *and its interior does not contain another site* M_j, $i \neq j$.

2. *For each such sphere* Σ *holds* $\phi(\Sigma) \in \phi(M_i)^*$, *but*

$$\phi(\Sigma) \in \phi(M_j)^{*+}, \; i \neq j$$

lies above the hyperplanes $\phi(M_j)^*$, $i \neq j$ *(see Lemma 17.2.4(1-2)).*

3. *Therefore, the Voronoi diagram can be represented as a polytope in* E^{d+1}.

Voronoi Diagrams-2

Theorem 17.2.5. *The Voronoi diagram of M, $Vor(M)$, is the cell complex of dimension d in E^d, whose faces are obtained by projecting onto E^d the proper faces of the Voronoi polytope $V(M)$.*

Proof: The boundary of $V(M)$ is a pure cell complex of dimension d, hence so is $Vor(M)$. Let $\overline{A}$ be a point on a facet of $V(M)$, that is contained in $\phi(M_i)^*$. This point $\overline{A}$ is the image under ϕ of some sphere Σ_A that passes through M_i and whose interior contains no other point of M. There cannot be a site in M closer to the center of Σ_A than M_i. That is, *A,* as the center of Σ_A, belongs to the cell $V(M_i)$ of the Voronoi diagram. *A* is the projection of $\overline{A}$ onto E^d.

Properties of Voronoi Diagrams

- L_2 - general position assumption: *no* $d+2$ *points in* M *lie on the boundary of a sphere*. If it is satisfied, then $V(M)$ is a simple (*d*+1) – dimensional polytope, with each vertex incident to *d*+1 hyperplanes. Also, $Vor(M)$ is a complex whose vertices are all equidistant from some *d*+1 points in M and closer to these points than to any other point in M.

- The problem of computing the Voronoi diagram of *n* points in E^d is reduced to the computation of the intersection of *n* half-spaces of E^{d+1}.

  Corollary 17.2.6: *The complexity (namely, the number of faces) of the Voronoi diagram of n points in* E^d *is* $\Theta(n^{\lceil d/2 \rceil})$. *The diagram may be computed in time* $O(n \log n + n^{\lceil d/2 \rceil})$ *, which is optimal in the worst case.*

Delaunay Complexes

Let $M = \{M_1, \ldots, M_n\}$ be a set of points in E^d .
Embed the space E^d into E^{d+1} as the hyperplane $x_{d+1} = 0$.

As before, construct the paraboloid Q and specify the points $\phi(M_i) \in Q$.

Let $D(M)$ be the convex hull of the points $\phi(M_1), \ldots, \phi(M_n)$ and let $L(M)$ be the *lower envelope of* $D(M)$.

The projection of $L(M)$ onto E^d form a complex, whose vertices are exactly the points $M_1, \ldots, M_n$. The domain of this complex is the projection of the convex hull of $\phi(M_1), \ldots, \phi(M_n)$, hence, it is a convex hull of $M_1, \ldots, M_n$.

This complex is called the *Delaunay complex* - $Del(M)$.

Delaunay. Connection with Voronoi

1. For $k = 0, \ldots, d$ the k-faces (*k*-dimensional faces) of $Del(M)$ are in a one-to-one correspondence with the *k*- faces of $L(M)$.

2. There exists a one-to-one correspondence between the vertices of $L(M)$ and the faces of $V(M)$: it maps the facet of $V(M)$, containing $\phi(M_i)^*$, to the point $\phi(M_i)$. More generally, the *k*-faces of $V(M)$ are in one-to-one correspondence with the (*d-k*) - faces of $L(M)$. Also, the bijection reverses inclusion relationships.

3. In addition, the *k*-faces of $V(M)$ are in a one-to-one correspondence with the *k* - faces of $Vor(M)$.

Delaunay. Connection with Voronoi

Therefore, we have the bijection between the k-faces of $Del(M)$ *and the (d-k) - faces of* $Vor(M)$.

The Delaunay complex is therefore dual *to the Voronoi diagram. The above duality maps a face of Vor(M), formed by the points, equidistant from m sites in M, to the face of Del(M), that is the convex hull of these sites.*

Delaunay. Connection with Voronoi

Theorem 17.3.1 *The Delaunay complex of points* $M_1,\ldots,M_n \in E^d$ *is a complex dual to the Voronoi diagram. Its faces are obtained by projecting the faces of the lower envelope of the convex hull of the points* $\phi(M_1),\ldots,\phi(M_n)$*, obtained by lifting the* $M_1,\ldots,M_n \in E^d$ *onto the paraboloid Q.*

Therefore, the computation of the Delaunay complex in E^d is reduced to the computation of the convex hull of *n* points in E^{d+1}.

Corollary: *The Delaunay complex of n points in* E^d *can be computed in time* $O(n\log n + n^{\lceil d/2 \rceil})$*, which is optimal in the worst case.*

Delaunay Triangulations

Under L_2 general assumption, $L(M)$ is a simplicial polytope and $Del(M)$ is a simplicial complex which is called then the *Delaunay Triangulation.*

If the assumption is not satisfied, then some d-face of $Del(M)$ will be formed by more than d+1 points and, hence, will not be a simplex.

Delaunay . Properties-1

Theorem 17.3.3 : *Let M be a set of points $M_1,\ldots,M_n \in E^d$. Then, any d- face in the Delaunay complex can be circumscribed by a sphere that passes through all its vertices, and whose interior contains no point in M.*

Proof: Assume L_2 general condition. Pick a d-face T of the Delaunay complex. Then, T is the convex hull $conv(M_{i_0},\ldots,M_{i_d})$ of $d+1$ cospherical points $M_{i_0},\ldots,M_{i_d}$. By the bijection between $L(M)$ and $Del(M)$, the convex hull $conv(\phi(M_{i_0}),\ldots,\phi(M_{i_d}))$ is a d-face of $L(M)$.

The points $\phi(M_{i_0}),\ldots,\phi(M_{i_d})$ lie on paraboloid Q and also belong to $\phi(\Sigma)^*$, where Σ circumscribes $M_{i_0},\ldots,M_{i_d}$. In turn, by the orthogonality argument, $\phi(\Sigma)$ belongs to the intersection of $\phi(M_{i_0})^*,\ldots,\phi(M_{i_d})^*$ and, hence, is a vertex of $V(M)$.

Again, by the bijection between $V(M)$ and $Vor(M)$, the point C (center of Σ) is the vertex of $Vor(M)$, incident to the cells that correspond to the sites $M_{i_0},\ldots,M_{i_d}$, therefore, the interior of Σ cannot contain any other points in M.

Delaunay . Properties-2

Theorem 17.3.4 : *Let* M *be a set of points* $M_1,\ldots,M_n \in E^d$ *and let* $\widetilde{M}_k = \{M_{i_0},\ldots,M_{i_k}\}$ *be a subset of k points in* M.

Then, the convex hull of $\widetilde{M}_k$ *is a face of the Delaunay complex if and only if there exists a (d-1)- sphere passing through* $M_{i_0},\ldots,M_{i_k}$ *and such that no point in* M *belongs to its interior.*

Corollary: *Any Delaunay triangulation of a set* $M = \{M_1,\ldots,M_n\} \in E^d$ *is such that the sphere circumscribed to any d- simplex in the triangulation contains no points of* M *in its interior. Conversely, any triangulation satisfying this property is a Delaunay triangulation.*

Higher-order Voronoi diagrams

Given $M = \{M_1,\ldots,M_n\}$: set of <u>*sites*</u> (points) in E^d.
To each subset $\widetilde{M}_k \subset M$ of size *k* attach the *cell* $V_k(\widetilde{M}_k)$

$$V_k(M_k) = \{X \in E^d : \ \forall M_i \in \widetilde{M}_k,\ \forall M_j \in M \setminus \widetilde{M}_k,\ \| XM_i \| \le \| XM_j \|\}.$$

In other words, it is the set of points in E^d, that are closer to all the sites in $\widetilde{M}_k$ than to any other site in $M \setminus \widetilde{M}_k$.

The total complexity of the Voronoi diagrams of all orders *k*, $1 \le k \le n-1$, is $O(n^{d+1})$.

Note: some of cells now can be empty…

Example: Voronoi Diagram of Order 2