Voronoi Diagram
Ordinary Voronoi Diagram - Recall

**Definition** – a subdivision of plane into cells

- **Sites**: \( S = \{s_1, s_2, \ldots, s_n\} \)
- **Euclidean distance in the plane**
  \[
  \text{dist}(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}.
  \]
- \( p \) lies in the cell of site \( s_i \) iff
  \[
  \text{dist}(p, s_i) < \text{dist}(p, s_j), \forall s_j \in S, j \neq i.
  \]

**Cells** - \( V(s_i) = \cap_{1 \leq j \leq n, j \neq i} h(s_i, s_j) \)

**Edges** - straight line segments
Multiplicatively Weighted Voronoi Diagram

Difference – Euclidean distance between points is divided by positive weights

Distance - \( \text{dist}(p, s_i) = \frac{||p - s_i||}{w_i} \).

Edges – circular arcs or straight line segments

For every point \( x \) on the edge separating \( V(s_i) \) and \( V(s_j) \),
\[
\text{dist}(x, s_i) = \text{dist}(x, s_j) \cdot \frac{w_i}{w_j}.
\]
Additively Weighted Voronoi Diagram

- **Difference** – positive weights are subtracted from the Euclidean distance

\[ \text{Distance} - \text{dist}(p, s_i) = \|p - s_i\| - w_i. \]

- **Edges** – hyperbolic arcs or straight line segments

  For every point \( x \) on the edge separating \( V(s_i) \) and \( V(s_j) \),
  \[ \text{dist}(x, s_i) = \text{dist}(x, s_j) + (w_i - w_j). \]
Voronoi Diagram in Different Metric

- **Difference** – Distance defined in $L_1$
- **Distance** - $\text{dist}(p, s_i) = |p_x - s_{i,x}| + |p_y - s_{i,y}|$.
- **Edges** – vertical, horizontal or diagonal at ±45 degree
Centroidal Voronoi Diagram (CVD)

- **Difference** – Each site is the mass centroid of each cell
  
  - Given a region $V \in \mathbb{R}^N$, and a density function $\rho$,
    
    mass centroid $z^*$ of $V$ is defined by $z^* = \frac{\int_V y \rho(y) \, dy}{\int_V \rho(y) \, dy}$
  
  - Centroid of polygon (CCW order of the vertices $(x_i, y_i)$)

\[
\text{Area} = A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i y_{i+1} - x_{i+1} y_i)
\]

\[
x_c = \frac{1}{6A} \sum_{i=0}^{N-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)
\]

\[
y_c = \frac{1}{6A} \sum_{i=0}^{N-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)
\]
CVD Computation — Lloyd’s Algorithm

1. Compute the Voronoi Diagram of the given set of sites \( \{s_i\}_{i=1}^n \); 
2. Compute the mass centroids of Voronoi cells \( \{V_i\}_{i=1}^n \) found in step 1, these centroids are the new set of sites; 
3. If this new set of sites meets the **convergence criterion**, terminate; Else, return to step 1.

**Note**
- Convergence criterion depends on specific application 
- Converges to a CVD slowly, so the algorithm stops at a tolerance value 
- Simple to apply and implement
Voronoi Diagram in Higher Dimensions

- **Cells** – convex polytopes
- **Bisectors** - $(d - 1)$-dimensional hyper-planes
- **Complexity** - $O \left( n \left\lceil \frac{d}{2} \right\rceil \right)$