Delaunay Triangulation of Four Cocircular Points

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Chapeau to the student who pointed the following fact. Assume that we have four cocircular points. Naturally, these points have two triangulations, each of which can be obtained from the other by flipping the single diagonal of the triangulation. Indeed, it is **not** true that, in general, the two triangulations have the same six angles!

In fact, it is easy to see that if the angle vector associated with one triangulation is (not in ascending order) $(\alpha, \beta, \gamma, \delta, \alpha+\beta, \gamma+\delta)$ (see Figure 1(left)), where $\alpha + \beta + \gamma + \delta = 180°$, then the angle vector associated with the other triangulation is $(\alpha, \beta, \gamma, \delta, \alpha+\delta, \beta+\gamma)$ (see Figure 1(right)). Thus, the two angle vectors are identical if and only if $\alpha = \beta$ or $\alpha = \gamma$, that is, the four points form an isosceles trapezoid. (And, then, obviously, the two diagonals have the same length; therefore, the respective arcs have the same length and the corresponding angles are equal.)

For a point set in general position (that is, a point set with no four cocircular points, where the formed circle contains other points of the set), everything taught in class (about the Delaunay Triangulation) is accurate. However, we need to be more careful with degenerate cases.

Denote by $t = t(S)$, as in class, the number of triangles in any triangulation of

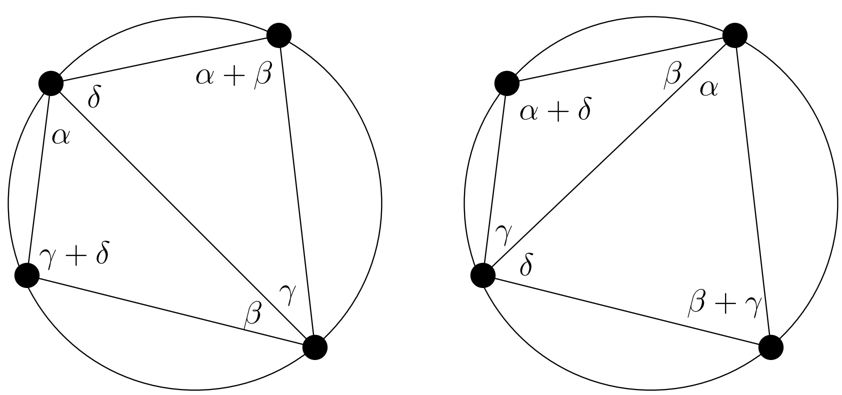

Figure 1: Two triangulations of four cocircular points

a point set S. Instead of taking all $3t$ angles of the triangulation, an alternative definition of the Delaunay triangulation of S refers to picking from each triangle only the *minimum* angle, forming the associated angle vector (of length t) in which entries are in ascending order, and defining the Delaunay triangulation as the triangulation whose angle vector is maximal lexicographically.

In nondegenerate situations, this modified definition is identical to the definition given in class (vectors of length $3t$). This is because an improving flip will always affect the minimum angle of the six angles of the two triangles involved in the flip.

The more delicate situation occurs when the point set contains four circular points which define a circle empty of other points of the set. As shown in Figure 1, these points define four angles $\alpha, \beta, \gamma, \delta$ which are common to the two triangulations. None of the other involved angles ($\alpha + \beta$, $\gamma + \delta$, $\alpha + \delta$, and $\beta + \gamma$) can be the minimal of the six angles. (For example, $\alpha + \beta$ cannot be minimal since $\alpha + \beta > \alpha$.) However, it **is** possible to have the angle $\alpha + \beta$ minimal in its respective triangle. (Refer again to Figure 1(left), and imagine that both γ and δ are almost $90°$, that is, the lower left triangle is very small.) Assume, without loss of generality, that $\alpha < \beta$. In such a case the two local minimum-angle vectors are $(\alpha, \alpha + \beta)$ (left) and (α, β) (right), which makes the left triangulation bigger (lexicographically) than the right one, thus, it is the Delaunay triangulation of the four points.

Please have a look at a discussion of this issue in Section 3 (p. 47) in the book "Triangulations and Applications" by Øyvind Hjelle and Morten Dæhlen, specifically, in Subsection 3.2 (p. 50) and the figure in p. 51. The book is available in `http://books.google.co.il/books?id=cRXAe8CVbBYC` .

In the upcoming week I will investigate further how the issue is referred to in the textbook of the course [BKOS].