Voronoi Diagram

Delaunay triangulation

Voronoi diagram

Delaunay and Voronoï

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How round is an object?
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Formal problem:
Given samples from the surface of a quasi-circular object, we would like to quantify how round it is.
Smallest width ring

We can come up with many measures.

We will consider the following measure:
What is the width of the minimal ring that contain all the samples?
Smallest width ring

Observations:
- It suffice to find the center of the ring
- The rings are determined by 4 points

Case 1:
3 outer 1 inner

Case 2:
1 outer 3 inner

Case 3:
2 outer 2 inner
**Ordinary Voronoi Diagram - Recall**

- **Definition** – a subdivision of plane into cells
  
  - Sites: $S = \{s_1, s_2, \ldots, s_n\}$
  
  - Euclidean distance in the plane
    
    $$
    \text{dist}(p, q) = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}.
    $$
  
  - $p$ lies in the cell of site $s_i$ iff
    
    $$
    \text{dist}(p, s_i) < \text{dist}(p, s_j), \forall s_j \in S, j \neq i.
    $$

- **Cells** - $V(s_i) = \bigcap_{1 \leq j \leq n, j \neq i} h(s_i, s_j)$

- **Edges** - straight line segments
Farthest point Voronoi diagram

Each cell is associated with the **farthest** point from the cell
Farthest point Voronoi diagram

Observations:
- The diagram is the intersection of the “Other side” of the bisector half-planes.
- A point $p$ has a cell iff $p$ is a vertex of the convex hull of the point.
- If the farthest point from $q$ is $p_i$, then, the ray from $q$ in the opposite direction to $p_i$ is also in the cell of $p_i$.
  - The cells are unbounded.
- The separator between the cells of $p_i$ and $p_j$ is the bisector of $p_i$ and $p_j$. 
Farthest point Voronoi diagram

Observations:

- The diagram is the intersection of the “Other side” of the bisector half-planes.
- A point $p$ has a cell iff $p$ is a vertex of the convex hull of the point.
- If the farthest point from $q$ is $p_i$, then, the ray from $q$ in the opposite direction to $p_i$ is also in the cell of $p_i$.
  $\Rightarrow$ The cells are unbounded.
- The separator between the cells of $p_i$ and $p_j$ is the bisector of $p_i$ and $p_j$. 
Farthest point Voronoi diagram

- Consider a random order of the CH vertices, $p_1, ..., p_h$
- Given a diagram for $p_1, ..., p_{i-1}$ we would like to add $p_i$
- We will denote the neighbors of $p_i$ (when $p_i$ is added) as $cw(p_i)$ and $ccw(p_i)$
- How do we find $cw(p_i)$ and $ccw(p_i)$?
  - Remove the points in the opposite order, the neighbors when $p_i$ is removed are $cw(p_i)$ and $ccw(p_i)$
Farthest point Voronoi diagram

- $ccw(p_i)$
- $cw(p_i)$
- Cell of $cw(p_i)$
- Cell of $ccw(p_i)$
- Cell of $cw(p_i)$
- Cell of $ccw(p_i)$
- Cell of $p_i$
Farthest point Voronoi diagram

- **Complexity:**
  - CH - $O(n \log n)$
  - Insertion of $p_i$: worst case $O(i)$
    - Expected: $O(1)$

- **Proof:**
  - The complexity of the $i$th insertion is as the complexity of the cell of $p_i$
  - There are at most $2i - 3$ edges after the $i$th insertion
  - $\Rightarrow$ The average cell complexity is $O(1)$
  - Each point from $p_1, \ldots, p_i$ have the same probability to be the last one added $\Rightarrow$ the expected complexity of insertion is $O(1)$
  - **Corollary:** the expected complexity is $O(n \log n)$ and the worst case complexity is $O(n^2)$. 
Back to the smallest width ring

◎ Case 1: the center is a vertex of the farthest point Voronoi diagram
◎ Case 2: the center is a vertex of the closest point Voronoi diagram
◎ Case 3: the center is an intersection of two edges from both diagrams.

Case 1: 3 outer 1 inner
Case 2: 1 outer 3 inner
Case 3: 2 outer 2 inner
Multiplicatively Weighted Voronoi Diagram

**Difference** – Euclidean distance between points is divided by positive weights

**Distance** - \( \text{dist}(p, s_i) = \frac{||p-s_i||}{w_i} \).

**Edges** – circular arcs or straight line segments

- For every point \( x \) on the edge separating \( V(s_i) \) and \( V(s_j) \),
  \[ \text{dist}(x, s_i) = \text{dist}(x, s_j) \cdot \frac{w_i}{w_j}. \]
Additively Weighted Voronoi Diagram

Difference – positive weights are subtracted from the Euclidean distance

**Distance** - \( \text{dist}(p, s_i) = \|p - s_i\| - w_i \).

Edges – hyperbolic arcs or straight line segments

For every point \( x \) on the edge separating \( V(s_i) \) and \( V(s_j) \),

\[ \text{dist}(x, s_i) = \text{dist}(x, s_j) + (w_i - w_j). \]
Voronoi Diagram in Different Metric

- **Difference** – Distance defined in $L_1$
  
  - **Distance** - $\text{dist}(p, s_i) = |p_x - s_{i,x}| + |p_y - s_{i,y}|$.

- Edges – vertical, horizontal or diagonal at $\pm 45$ degree
**Centroidal Voronoi Diagram (CVD)**

**Difference** – Each site is the mass centroid of each cell

- Given a region $V \in \mathbb{R}^N$, and a density function $\rho$,
  
  **mass centroid** $z^*$ of $V$ is defined by $z^* = \frac{\int_V y\rho(y) \, dy}{\int_V \rho(y) \, dy}$

- **Centroid of polygon** (CCW order of the vertices $(x_i, y_i)$)

  \[
  Area = A = \frac{1}{2} \sum_{i=0}^{N-1} (x_i y_{i+1} - x_{i+1} y_i)
  \]

  \[
  x_c = \frac{1}{6A} \sum_{i=0}^{N-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)
  \]

  \[
  y_c = \frac{1}{6A} \sum_{i=0}^{N-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)
  \]
1. Compute the Voronoi Diagram of the given set of sites \( \{s_i\}_{i=1}^n \);
2. Compute the mass centroids of Voronoi cells \( \{V_i\}_{i=1}^n \) found in step 1; these centroids are the new set of sites;
3. If this new set of sites meets the convergence criterion, terminate; Else, return to step 1.

**Note**
- Convergence criterion depends on specific application
- Converges to a CVD slowly, so the algorithm stops at a tolerance value
- Simple to apply and implement
Voronoi Diagram in Higher Dimensions

◎ **Cells** – convex polytopes

◎ **Bisectors** - \((d - 1)\)-dimensional hyper-planes

◎ **Complexity** - \(O\left(n\left\lfloor\frac{d}{2}\right\rfloor\right)\)