Covering Points with a Polygon

Gill Barequet Yuval Scharf
Dept. of Computer Science
The Technion—IIT
Haifa 32000, Israel
[barequet,scharf]@cs.technion.ac.il

Matthew T. Dickerson
Dept. of Math. and Comp. Science
Middlebury College
Middlebury, VT 05753
dickerso@middlebury.edu

Categories and Subject Descriptors

I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling—*Geometric algorithms, languages, and systems*

General Terms

Algorithms, Theory

Keywords

Covering points

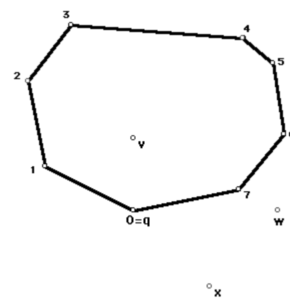
Abstract

We present a diagram that captures containment information for scalable rotated and translated versions of a convex polygon. For a given polygon P and a contact point q in a point set S, the diagram parameterizes possible translations, rotations, and scales of the polygon in order to represent containment regions for each additional point $v \in S$. We present geometric and combinatorial properties for this diagram, and describe how it can be computed and used for solving several geometric problems.

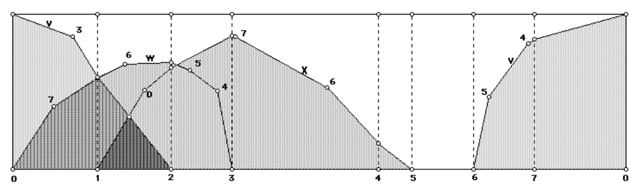
1. INTRODUCTION

Given a set of points in the plane and a convex polygon, we consider problems in which one tries to cover the points (or a maximum number of them) with the polygon. Depending on the specific problem, we allow the polygon to be translated, rotated, scaled, or any combination of the above.

The problem of finding a translation and rotation of a convex polygon P, that maximizes the number of contained points from a given input set S, was studied by Dickerson and Scharstein [2]. As part of their solution, they presented the so-called *rotation diagram*. The rotation diagram $R_{P,q}$ represents all possible placements of a convex polygon P in contact with a particular point $q \in S$. This two-dimensional diagram parameterizes translations along one axis and rotations along the other axis. For every other point $v \in S$, the diagram has a region R_v of all placements of P containing v. The cited work describes the combinatorial and geometric properties of rotation diagrams. The complete diagram with all such regions contains the necessary information to solve several placement problems for the polygon P and associated annulus regions.



(a) A point set and a polygon



(b) The Diagram

Figure 1: The translation-scale diagram of an 8-gon and a 4-point set

In [2], and also in the current work, the diagrams emphasize placements of a polygon that are *in contact* with some point of the point set. This is because any not-in-contact placement of the polygon can be modified to an in-contact placement without altering the set (or subset) of points contained in the polygon.

We first explore polygon placements that allow scaling of P and are limited to translation only. In particular, we present a two-dimensional containment diagram similar in nature to that of [2], but representing translation and scale instead of translation and rotation. We describe several combinatorial and geometric properties of the containing regions, and show that the complete translation-scale diagram has complexity $O(n(n+m))$, where m is the complexity of the polygon P and n is the cardinality of the point set S.

We then also explore translation-scale-rotation diagrams. Those diagrams capture the different ways of covering the point set using translation, scale, and rotation of P.

2. THE TRANSLATION-SCALE DIAGRAM

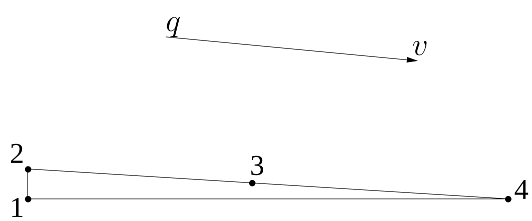
The goal is to create a diagram $D_{P,q}$ that represents translations and scales of a given polygon P. Specifically, we represent all translations that keep P in contact with a given point $q \in S$. For scalings of the polygon, the point q is also used as the center of scaling. That is, we have a two-dimensional diagram that represents along the x axis all translations of P in contact with q, and represents along the y axis the factors by which P can be scaled while maintaining the contact with q.

For each other point $v \in S$, $v \neq q$, we have a region R_v in $D_{P,q}$ that corresponds to those scales and translations of P that contain v. Actually, rather than parameterizing the scaling factor α of the polygon P, the diagram $D_{P,q}$ parameterizes the inverse value, $1/\alpha$, which is equivalent to scaling the entire plane by α while leaving P unscaled.

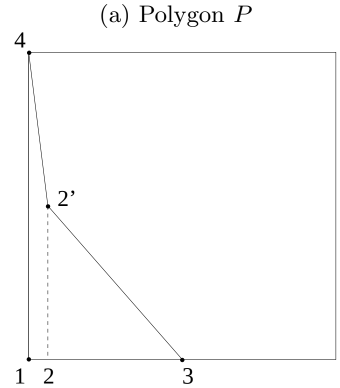
Here are two properties of the translation-scale diagram:

- Every region R_v in $D_{P,q}$ is an x-monotone polygon and exactly two vertices of R_v lie on the horizontal axis (at scale 0) at locations corresponding to vertices of P.

- The supremum of the internal angle between consecutive segments along the boundary of a region in the diagram is $\frac{5}{4}\pi$.



(a) Polygon P



(b) Translation-scale diagram of P and the point set $\{q, v\}$

Figure 2: Concave translation-scale diagram

We show the following about the diagram complexity:

- The boundaries of any two regions have at most one intersection (a point or a continuous line segment).

- The number of intersections between boundaries of regions is $\Theta(n^2)$ in the worst case.

- For any convex polygon P and point set S, the complexity of $D_{P,q}$ is $\Theta(n(n+m))$ in the worst case.

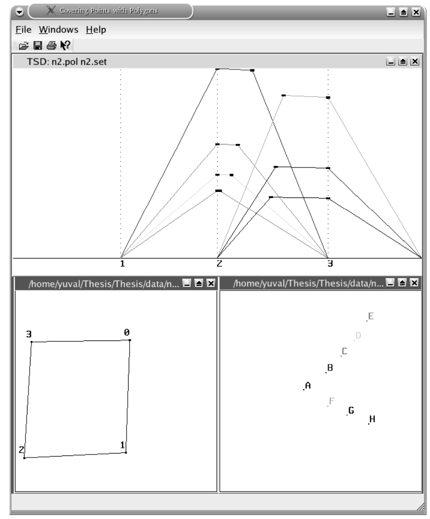


Figure 3: The number of intersections between boundaries of regions is $\Theta(n^2)$ in the worst case

3. APPLICATIONS OF THE DIAGRAM

We highlight some applications of the translation-scale diagram. For every problem we show how to solve it when either a precomputed diagram is given or not given. We solve the following problems:

- Compute the smallest scale of P, if it exists, such that there is a translation of P in contact with q and containing the entire point set S.

- Given a scale α and a point q, determine the maximum number of points that can be contained by a copy of P in contact with q and scaled by α.

- Compute the smallest scale of P in contact with q and containing at least k points.

4. TRANSLATION-SCALE-ROTATION DIAGRAM

We extend the translation-scale diagram to the third (z) dimension, where the additional dimension represents rotations of P. The translation-scale-rotation diagram of a polygon P and a point set is made of $n-1$ three-dimensional regions, one region for every other point $v \in S, v \neq q$. Each region corresponds to those scales, translations, and rotations of P that contain v. The complexity of the diagram is proven to be $O(n^3m^4)$ in the worst case. We describe a few applications of the translation-scale-rotation diagram, and show how to solve some of the problems presented in Section 3 when applied to the three-dimensional case.

5. REFERENCES

[1] G. Barequet and M. Dickerson. The translation-scale diagram for point-containing placements of a convex polygon. In *Proc. of 12th Canadian Conf. on Comp. Geom., (Fredericton, New Brunswick, Canada, 7–12, 2000)*, 2000.

[2] M. Dickerson and D. Scharstein. Optimal placement of a convex polygon to maximize point containment. *CGTA*, 11:1–16, 1998.

[3] Y. Scharf. *Covering points with a polygon.* M.Sc. Thesis, Dept. of Computer Science, The Technion–Israel Institute of Technology, Haifa, Israel, October 2004.