

Heilbronn's Triangle Problem

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ABSTRACT

In the famous Heilbronn's triangle problem, one aims to find a point set S (say, in the plane), in which the smallest area of a triangle defined by three points of S assumes its maximum. In this video segment we present some variants of the problem. We show a few optimal, or almost optimal, configurations of small numbers of points, and generalize the problem to higher dimensions. Then, we make the distinction between the off-line and on-line versions of the problem, and outline an efficient procedure for attacking the latter version of the problem.

Categories and Subject Descriptors

G.2.1 [Discrete Mathematics]: Combinatorics—*combinatorial algorithms*; G.1.6 [Numerical Analysis]: Optimization—*Unconstrained optimization*

General Terms

Algorithms, Theory

Keywords

Extremal configurations, packing arguments

1. INTRODUCTION

The *off-line* version of the now famous *triangle problem* was posed by Heilbronn [9] more than 50 years ago. It is formulated as follows:

Given n points in the unit square, what is $\mathcal{H}_2^{\text{off-line}}(n)$, the maximum possible area of the *smallest* triangle defined by some three of these points?

There is a large gap between the best currently-known lower and upper bounds on $\mathcal{H}_2^{\text{off-line}}(n)$, $\Omega(\log n/n^2)$ [7] and $O(1/n^{8/7-\varepsilon})$ (for any $\varepsilon > 0$) [6].

Barequet [1] generalized the off-line problem to d dimensions:

Given n points in the d -dimensional unit cube, what is $\mathcal{H}_d^{\text{off-line}}(n)$, the maximum possible volume of the *smallest* simplex defined by some $d+1$ of these points?

A “good” point set in d dimensions can be found on the d -dimensional moment curve (modulo n) [1]. The coordinates of the points are integer multiples of $1/n$: the i th point (for $0 \leq i \leq n-1$) is $(i/n, (i^2 \bmod n)/n, \dots, (i^d \bmod n)/n)$. This point set shows that $\mathcal{H}_d^{\text{off-line}}(n) = \Omega(1/n^d)$. The best currently-known lower bound on $\mathcal{H}_d^{\text{off-line}}(n)$ is $\Omega(\log n/n^d)$ [8]. Brass [4] showed that for an odd value of d we have $\mathcal{H}_d^{\text{off-line}}(n) = O(1/n^{1+1/(2d)})$.

The *on-line* version of the triangle problem is harder than the off-line version, because the value of n is not specified in advance. In other words, the points are positioned one after the other in a d -dimensional unit cube, while n is incremented by one after every point-positioning step. The procedure can be stopped at any time, and the already-positioned points must have the property that every subset of $d+1$ points defines a full-dimensional simplex whose volume is at least some quantity $\mathcal{H}_d^{\text{on-line}}(n)$, where the goal is to maximize this quantity.

Schmidt [10] showed that $\mathcal{H}_2^{\text{on-line}}(n) = \Omega(1/n^2)$. (He did not present his solution, however, in an on-line setting.) Barequet [2] used nested packing arguments to demonstrate that $\mathcal{H}_3^{\text{on-line}}(n) = \Omega(1/n^{10/3}) = \Omega(1/n^{3.333\dots})$ and that $\mathcal{H}_4^{\text{on-line}}(n) = \Omega(1/n^{127/24}) = \Omega(1/n^{5.292\dots})$. Barequet and Shaikhet [3] generalized this technique to higher dimensions, showing that $\mathcal{H}_d^{\text{on-line}}(n) = \Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$ in a fixed dimension d .

2. THE VIDEO

The video consists of two parts:

1. The first part deals with the off-line version of the problem. It demonstrates the best planar configurations of three, four, and five points. These configurations show that $\mathcal{H}_2^{\text{off-line}}(3) = 0.5$, $\mathcal{H}_2^{\text{off-line}}(4) = 0.5$, and $\mathcal{H}_2^{\text{off-line}}(5) = \sqrt{3}/9 = 0.192\dots$. It is also known that $\mathcal{H}_2^{\text{off-line}}(6) = 1/8 = 0.125$, $\mathcal{H}_2^{\text{off-line}}(7) \geq 0.0838\dots$, $\mathcal{H}_2^{\text{off-line}}(8) \geq (\sqrt{13}-1)/36 = 0.0723\dots$, $\mathcal{H}_2^{\text{off-line}}(9) \geq (9\sqrt{65}-55)/320 = 0.0548\dots$, and so on. See [5] for optimal (or close to optimal) configurations of 3 through 16 points. The problem is then generalized to higher dimensions.

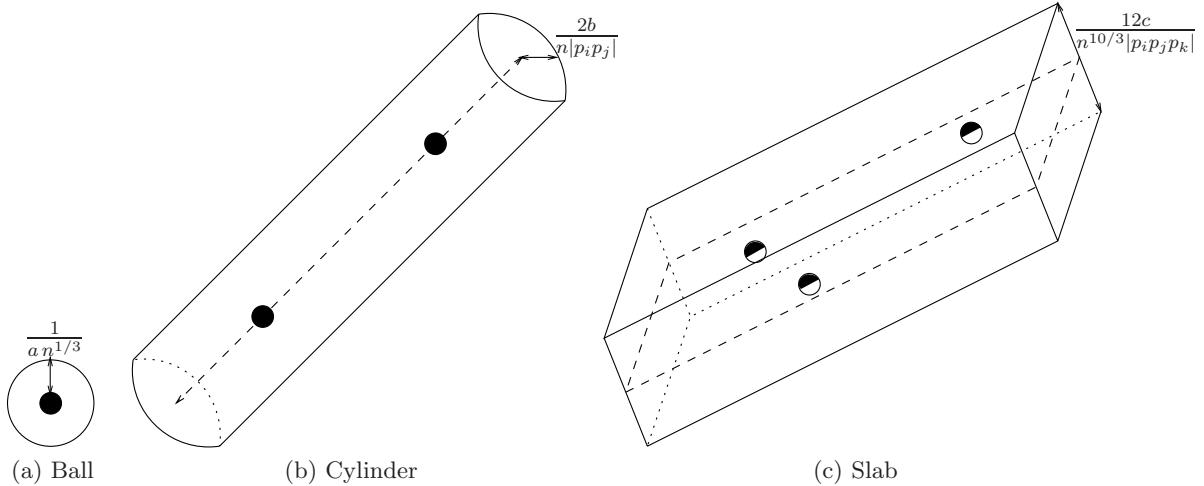


Figure 1: Forbidden zones defined by a point, a pair of points, and a triple of points

2. The second part of the video deals with the on-line version of the problem. The nested packing arguments technique [3] is demonstrated in three dimensions. In this method, the already-positioned points define forbidden zones in which the next positioned point cannot be located; see Figure 1. Specifically, each point defines a forbidden ball (centered at the point), whose radius depends on n , the number of already-positioned points; each pair of points defines a forbidden cylinder (centered at the line passing through the two points), whose radius depends on n and on the distance between the two points; and each triple of points defines a forbidden slab (centered at the plane passing through the three points), whose width depends on n and on the area of the triangle defined by the three points. When n grows, existing forbidden zones shrink, while new such zones appear, yet there is always room for the next point, so the procedure never fails. Similarly, in d dimensions, each subset of k of the already-positioned points (for $1 \leq k \leq d$) defines some forbidden zone, in which the next point cannot be located. A recursive formula, that relates between the volumes of the forbidden zones induced by different numbers of points, yields the lower bound on $\mathcal{H}_d^{\text{on-line}}(n)$.

The video was produced by using standard MS Windows tools, PhotoStudio (by ArcSoft), and 3DS-Max (by Autodesk Media and Entertainment).

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