2-Point Site Voronoi Diagrams

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1. INTRODUCTION

The purpose of the video segment is to visualize three-dimensional surfaces, so-called the Voronoi surfaces of the 2-site distance functions described below. In order to build intuition, we begin by looking at analaguous surfaces for the regular planar Euclidean distance function, whose lower envelope determines the regular Voronoi diagram.

The standard Voronoi Diagram of a set of n given points (called sites) is a subdivision of the plane into n regions, one associated with each site. Each site's region consists of all points in the plane closer to it than to any of the other sites.

The Voronoi diagram has been rediscovered many times in dozens of fields of study including crystallography, geography, metrology, and biology, as well as mathematics and computer science. A comprehensive review of the various variations of Voronoi diagrams and of the hundreds of applications of them is given by Okabe, Boots, and Sugihara [3].

The regular (1-site) Voronoi diagram can be viewed as the result of growing circles around each point, where each point in the plane belongs to the region of the site whose circle sweeps it first. If this two-dimensional growth is modeled with time as a third dimension along the z-axis, the circle growing around a point p sweeps out a cone in space, with the p as its apex. That is, for every location (x, y), the z coordinate of the cone of the site p is the 2-dimensional distance from (x, y) to p. It is well known that the 1-site Voronoi diagram is the xy-projection of the lower envelope of these cones.

For 2-site distance functions, each pair of sites (p, q) is associated with a surface, where for every point (x, y, z) on the surface, the value z is the 2-site distance from (x, y) to the pair (p, q). The projection of the lower envelope of these surfaces gives the Voronoi diagram for these 2-point distance functions, where each region corresponds to a pair of points.

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2. THE VIDEO

This video first shows how the regular Voronoi diagram can be represented as the lower envelope of a set of cones as described above. The GEOMVIEW system is used to display a single cone, and then a set of intersecting colored cones corresponding to a set of points. These are rotated so that they are viewed from below, and an orthographic projection then shows that the visible part of each cone corresponds to a region in the Voronoi diagram.

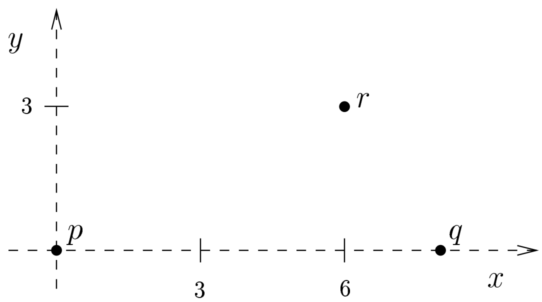
The process is then repeated for five 2-point distance functions. The surface generated for a pair of points under the given distance function is shown. Then for a set of points a cone is generated for each pair. This set of cones is rotated, and the view from below is shown to be the Vornoi diagram for that 2-point distance function.

We denote by $d(a, b)$ the Euclidean distance between the points a and b, and by $A(a, b, c)$ the area of the triangle defined by the points a, b, and c. Given two point sites p and q, we define the following distance functions from a point v to the pair (p, q):

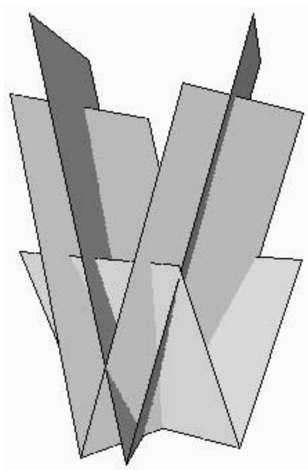
1. **Sum of distances**: $\mathcal{S}(v, (p, q)) = d(v, p) + d(v, q)$.
2. **Product of distances**: $\mathcal{M}(v, (p, q)) = d(v, p) \cdot d(v, q)$.
3. **Triangle perimeter**: $\mathcal{P}(v, (p, q)) = d(p, q) + d(v, p) + d(v, q)$.
4. **Triangle area**: $\mathcal{A}(v, (p, q)) = A(v, p, q)$.
5. **Difference between distances**: $\mathcal{D}(v, (p, q)) = |d(v, p) - d(v, q)|$.

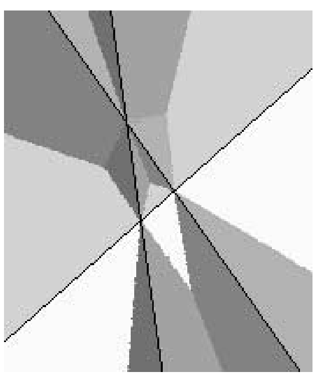
We show in the video that the first two functions generate very differently-looking surfaces, but almost identical Voronoi diagrams. (Both are equivalent to the second-order Voronoi diagram [2].) The first and third generate the same shaped surfaces, but because they are at different heights above the xy plane, the Voronoi diagrams are quite different.

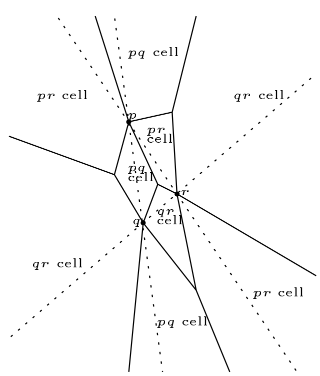
Consider, for example, the Voronoi diagram of three points with respect to the 2-site triangle-area distance function.



(a) Point set


(b) Surfaces


(c) Lower envelope



(d) Voronoi diagram

Figure 1: Computing VOR$_\mathcal{A}$

Figure 1(a) shows the three points, while Figure 1(b) shows the three respective Voronoi surfaces in a perspective view. Figure 1(c) shows the same construction from below. Figure 1(d) shows the Voronoi diagram of the three points, which is the xy-projection of the lower envelope of the surfaces.

A paper by three of the creators of this video analyzes the complexity of both the nearest-pair and the farthest-pair Voronoi diagram for a number of 2-point distance functions, and gives algorithms for computing them [1].

3. JAVA IMPLEMENTATION

We have implemented a Java applet that computes the Voronoi diagrams with respect to the distance functions discussed in this paper (and some more functions), and a Web page[1] which provides interface to this applet. The applet supports interactive selection of the point set and on-line computation and display of the Voronoi diagrams.

4. ACKNOWLEDGEMENTS

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5. REFERENCES

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[2] D.T. Lee, On k-nearest neighbor Voronoi diagrams in the plane, *IEEE Trans. on Computing*, 31 (1982), 478–487.

[3] A. Okabe, B. Boots, and K. Sugihara, Spatial Tessellations: Concepts and Applications of Voronoi Diagrams, John Wiley & Sons, Chichester, England, 1992.

[1] `http://www.middlebury.edu/~dickerso/TrivialApplet.html`; see also `http://www.cs.technion.ac.il/~barequet/ndf/plate*`