where \( C(x) = 1 - 3x + x^2/2 + x^3/30 \). Note that \( C'(x) \) is positive for \( x > 3 \) and \( C(9/2) = 53/80 \). Hence \( C(x) \) is positive for \( x > 9/2 \) and the bounds are established. We can now write

\[
F(z) > z \int_0^2 e^{-zt} dt + z \int_2^\infty 2te^{-(z+1)t} dt,
\]

and thus

\[
F(z) > 1 + e^{-2z-2}(w(z) - e^2),
\]

where

\[
w(z) = 8 - \frac{4}{(z+1)^2} - \frac{4}{(z+1)^3}.
\]

Note that \( w(z) \) increases from \( 200/27 \) at \( z = 2 \) to 8 at \( z = \infty \). Since \( 200/27 > e^2 \), we are done.

**REFERENCES**


**A Family of Matrix Problems**

**Problem 97-11**, by Dan Givoli (Technion, Haifa, Israel).

The following family of matrix problems arises in the design of some high-order local non-reflecting boundary conditions [1,2,3]:

\[
\begin{pmatrix}
1 & 1^2 & 1^4 & \ldots & 1^{2(N-1)} \\
2^0 & 2^2 & 2^4 & \ldots & 2^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N^0 & N^2 & N^4 & \ldots & N^{2(N-1)}
\end{pmatrix}
\begin{bmatrix}
\alpha_{1}^{(N)} \\
\alpha_{2}^{(N)} \\
\vdots \\
\alpha_{N}^{(N)}
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
2 \\
\vdots \\
N
\end{bmatrix}.
\]

For example, the solutions of this system for \( N = 1, 2, 3 \) are:

\[
\begin{align*}
N = 1 : & \quad \alpha_{1}^{(1)} = 1 \\
N = 2 : & \quad \alpha_{1}^{(2)} = 2/3, \quad \alpha_{2}^{(2)} = 1/3, \\
N = 3 : & \quad \alpha_{1}^{(3)} = 3/5, \quad \alpha_{2}^{(3)} = 5/12, \quad \alpha_{3}^{(3)} = -1/60.
\end{align*}
\]

1. Numerical solution for various values of \( N \) shows that \( \alpha_{m}^{(N)} \) is positive for even \( m \) and for \( m = 1 \), and is negative for odd \( m \neq 1 \). Thus, the pattern of the signs of the solutions \( \alpha_{m}^{(N)} \) is \(+, +, -, +, +, -\), and so on. (This has a bearing on the stability of the non-reflecting boundary condition.) Prove that this is indeed the case for all \( N \).

2. Estimate the condition number of the matrix as a function of \( N \).
3. Give an asymptotic approximation for the solution of the system for large $N$.

REFERENCES


Solution by A. Sidi (Technion, Haifa, Israel).

Part 1. We prove part 1 for the more general problem

\[
\begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_N & x_N^2 & \cdots & x_N^{N-1}
\end{pmatrix}
\begin{pmatrix}
\alpha_1^{(N)} \\
\alpha_2^{(N)} \\
\vdots \\
\alpha_N^{(N)}
\end{pmatrix} = \begin{pmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_N)
\end{pmatrix},
\]

where $0 < x_1 < x_2 < \cdots < x_N < X$ for some $X > 0$ and

(i) $f \in C[0, X]$ and $f(0) \geq 0$, and

(ii) $f \in C^N(0, X)$ and $(-1)^{j-1}f^{(j)}(0) > 0$ for $x \in (0, X), j = 1, 2, \ldots, N$.

Thus, the original problem is a special case of this one, with $f(x) = \sqrt{x}$ and $x_k = k^2$, $k = 1, \ldots, N$. Obviously, $Q_N(x) = \sum_{i=1}^{N} \alpha_i^{(N)}x^{i-1}$ is the polynomial of interpolation to $f(x)$ at the points $x_1, x_2, \ldots, x_N$.

Substituting $x = 0$ in the well-known error formula

\[
f(x) - Q_N(x) = \frac{f^{(N)}(\xi(x))}{N!} \prod_{i=1}^{N} (x - x_i), \quad \text{for some } \xi(x) \in (\min\{x, x_1\}, \max\{x, x_N\}),
\]

that is valid also for $x = 0$ even though $f(x)$ is not necessarily differentiable there, we obtain

\[
\alpha_1^{(N)} = Q_N(0) = f(0) + (-1)^{N+1} \frac{f^{(N)}(\xi(0))}{N!} \prod_{i=1}^{N} x_i, \quad \text{with } \xi(0) \in (0, x_N).
\]

By the assumptions that $f(0) \geq 0$ and $(-1)^{N-1}f^{(N)}(x) > 0$ for $x \in (0, X)$, (3) gives $\alpha_1^{(N)} > 0$.

Next, let us look at the Newton form of $Q_N(x)$, namely,

\[
Q_N(x) = f(x_1) + \sum_{k=2}^{N} f[x_1, \ldots, x_k](x - x_1) \cdots (x - x_{k-1}).
\]

Here $f[x_1, \ldots, x_k]$ are the divided differences of $f(x)$, and we know that they satisfy

\[
f[x_1, \ldots, x_k] = \frac{f^{(k-1)}(\xi_k)}{(k-1)!}, \quad \text{for some } \xi_k \in (x_1, x_k), \ k = 2, \ldots, N.
\]
We also observe that \( x_i > 0 \) for all \( i \) implies that

\[
(x - x_1) \cdots (x - x_{k-1}) = \sum_{i=1}^{k} (-1)^{k-i} C_{ki} x^{i-1}, \quad k = 2, 3, \ldots ,
\]

for some positive constants \( C_{ki} \). From

\[
\alpha_i^{(N)} = \sum_{k=1}^{N} f[x_1, \ldots , x_k] (-1)^{k-i} C_{ki}, \quad i = 2, \ldots , N,
\]

that follows from (4) and (6), from (5), and from the assumption that \((-1)^{j-1} f^{(j)}(x) > 0 \) for \( x \in (0, X) \), \( j = 1, 2, \ldots , N \), we obtain \((-1)^{i} \alpha_i^{(N)} > 0 \), \( i = 2, \ldots , N \). (Note that even though (7) holds also for \( i = 1 \), it can not be used to show that \( \alpha_1^{(N)} > 0 \). This is the reason we have treated \( \alpha_1^{(N)} \) separately.)

Now apply the above to \( f(x) = \sqrt{x} \) with \( x_k = k^2 \), \( k = 1, 2, \ldots , \) and with arbitrary \( N \).

(It is interesting to note that if \((-1)^{j} f^{(j)}(x) > 0 \) for \( x \in (0, X) \), \( j = 0, 1, \ldots , N \), then with the help of (4)–(7) we can show that \((-1)^{i-1} \alpha_i^{(N)} > 0 \), \( i = 1, 2, \ldots , N \). This is the case for \( f(x) = x^a \) with \( a < 0 \), for example.)

Part 2. Let us denote the matrix of the linear system in (1) by \( A_N \). Then the \( l_1 \) condition number \( \kappa_1(A_N) \) of \( A_N \) is \( \kappa_1(A_N) = \|A_N\|_1 \|A_N^{-1}\|_1 \), where

\[
\|A_N\|_1 = \max_{1 \leq j \leq N} \left( \sum_{i=1}^{N} x_i^{j-1} \right) \quad \text{and} \quad \|A_N^{-1}\|_1 = \max_{1 \leq k \leq N} \left( \prod_{1 \leq i \leq k} \frac{1 + x_i}{|x_k - x_i|} \right).
\]

(for \( \|A_N^{-1}\|_1 \), see Gautschi [1]).

For the problem at hand, we have \( x_k = k^2 \), \( k = 1, 2, \ldots \). Thus

\[
\|A_N\|_1 = \sum_{i=1}^{N} i^{2(N-1)} \in (N^{2N-2} , N^{2N-1})
\]

and

\[
\|A_N^{-1}\|_1 = \frac{\Pi_N}{(N^N)^{2N-2}} \max_{1 \leq k \leq N} \left[ \frac{2k^2}{1 + k^2} \left( \frac{2N}{N + k} \right) \right] \in \left( \frac{\Pi_N}{(N + 1)(N + 1)(N^2 + 1)} , \frac{2N^3 \Pi_N}{(N + 1)(N + 1)(N^2 + 1)} \right)
\]

where \( \Pi_N \equiv \prod_{i=1}^{N} (1 + i^{-2}) \). Consequently,

\[
\frac{N}{N + 1} \Pi_N N^{2N-2} < \kappa_1(A_N) < \frac{2N^3 \Pi_N}{(N + 1)(N^2 + 1)} \Pi_N N^{2N-1}.
\]

In other words, \( \kappa_1(A_N) \) is at best \( O(N^{2N-2}) \) and at worst \( O(N^{2N-1}) \), as \( N \to \infty \). (Here we have used the fact that \( \lim_{N \to \infty} \Pi_N \) exists and is finite.)

Part 3. Using the well-known recursion relation that is used in defining the divided differences, we can show by induction that when \( f(x) = \sqrt{x} \) and \( x_k = k^2 \), \( k = 0, 1, 2, \ldots \),

\[
f[x_i, x_{i+1}, \ldots , x_{i+k}] = (-4)^{k-1} \frac{(\frac{1}{2})^{k-1} (2i)!}{k! (2i + 2k - 1)!}, \quad i = 0, 1, \ldots , \ k = 1, 2, \ldots \]

We thus have
\[ \alpha_N(x) = f[x, x_1, x_2, \ldots, x_N] = \frac{(-1)^N}{\sqrt{\pi}N(2N)!} \sim \frac{1}{2} \text{ as } N \to \infty. \]

Also, substituting \( x = x_0 = 0 \) in the divided difference form of the error
\[ f(x) - Q_N(x) = f[x, x_1, x_2, \ldots, x_N] \prod_{i=1}^{N} (x - x_i), \]
we obtain
\[ \alpha_1 = Q_N(0) = f(0) - f[x_0, x_1, x_2, \ldots, x_N](-1)^N \prod_{i=1}^{N} x_i, \]
which, by (12), becomes
\[ \alpha_1 = \frac{4^{N-1}N^{-1}(N)!}{(2N)!} \sim \frac{1}{2} \text{ as } N \to \infty. \]

In obtaining the asymptotic behaviors of \( \alpha_N \) and \( \alpha_1 \) for \( N \to \infty \) in (13) and (16) we have made use of the Stirling formula for the gamma function.

We next discuss the asymptotic behaviors of the \( C_{Ni} \) as these are important in determining the asymptotic behaviors of the \( \alpha_i(N) \). We first note that
\[ C_{ki} = C_{k-1,i-1} + x_{k-1}C_{k-1,i}, \quad i = 1, 2, \ldots, k, \]
with \( C_{kk} = 1 \) and \( C_{k0} = C_{k,k+1} = 0 \) for all \( k \). (From this we obtain \( C_{k1} = \prod_{i=1}^{k-1} x_i \) and \( C_{k,k-1} = \sum_{i=1}^{k-1} x_i \) for all \( k \), which are, of course, true.) We look at two different cases.

(i) Letting \( C_{ki} = C_{k1}D_{ki}, \quad i = 1, 2, \ldots, \) with \( D_{k1} = 1 \), we rewrite (17) in the form of a difference equation as in
\[ D_{ki} - D_{k-1,i} = \frac{1}{x_{k-1}}D_{k-1,i-1}, \quad i = 2, 3, \ldots. \]

We can now show that \( D_{k2} = \sum_{i=1}^{k-1} 1/x_i = \sum_{i=1}^{k-1} i^{-2} \sim \zeta(2) \) as \( k \to \infty \). With this knowledge, we can next show that
\[ D_{k3} = \sum_{1 \leq i < j \leq k-1} \frac{1}{xixj} = \frac{1}{2} \left( \sum_{i=1}^{k-1} \frac{1}{x_i} \right)^2 - \sum_{i=1}^{k-1} \frac{1}{x_i^2} \sim \frac{1}{2} [\zeta(2)^2 - \zeta(4)] \text{ as } k \to \infty. \]

(Here \( \zeta(z) \) is the Riemann zeta function.) In general, by induction, we obtain from (18) that
\[ D_{ki} \sim \hat{D}_i \quad \text{as } k \to \infty, \quad i = 1, 2, \ldots, \quad i \text{ fixed}, \]
for some constants \( \hat{D}_i \) that are independent of \( k \). As a result,
\[ C_{ki} \sim \hat{D}_i [(k-1)!]^2 \text{ as } k \to \infty, \quad i = 1, 2, \ldots, \quad i \text{ fixed}. \]
(ii) Replacing $i$ in (17) by $k - s$, we obtain the difference equation

\[ C_{k,k-s} - C_{k-1,(k-1)-s} = x_{k-1}C_{k-1,(k-1)-(s-1)}, \quad s = 0, 1, \ldots, k - 1. \]

For $s = 1$ we obtain from (21) that $C_{k,k-1} = \sum_{i=1}^{k-1} x_i = \sum_{i=1}^{k-1} i^2 = (k - 1)k(2k - 1)/6 \sim k^3/3$ as $k \to \infty$. Continuing with $s = 2, 3, \ldots$, and realizing that $C_{k,k-s}$ are polynomials in $k$ whose degrees depend on $s$, we obtain

\[ C_{k,k-s} \sim \frac{k^{3s}}{3!(s!)} \quad \text{as} \quad k \to \infty, \quad s = 0, 1, 2, \ldots, \text{fixed}. \]

We now proceed to the asymptotic behavior of the $\alpha_{i}^{(N)}$. Again, we look at two different cases.

(i) From (7) and from the fact that for fixed $i$

\[ (-1)^{k-i} f[x_1, \ldots, x_k]C_{ki} \sim (-1)^i \frac{D_i}{4k^2} \quad \text{as} \quad k \to \infty, \]

that follows from (13) and (19), we see that, for $i = 1, 2, \ldots$, and $i$ fixed,

\[ \lim_{N \to \infty} \alpha_{i}^{(N)} = \hat{\alpha}_i \quad \text{for some constant} \quad \hat{\alpha}_i. \]

(ii) Let us replace $i$ in (7) by $N - s$. Then the summation there has only $s + 1$ terms independently of $N$. The asymptotic behavior for $N \to \infty$ of this summation is determined solely by its last term, namely, by $(-1)^{s} f[x_1, \ldots, x_N]C_{N,N-s}$, since

\[ \frac{f[x_1, \ldots, x_N]C_{N,N-s}}{f[x_1, \ldots, x_{N-1}]C_{N-1,N-s}} \sim -\frac{1}{3s} N \quad \text{as} \quad N \to \infty. \]

Thus, for $s = 0, 1, \ldots$, and $s$ fixed,

\[ \alpha_{N-s}^{(N)} \sim (-1)^{s} \frac{f[x_1, \ldots, x_{N-s}]C_{N,N-s}}{\sqrt{\pi N(2N)!}} \frac{N^{3s}}{3!(s!)} \quad \text{as} \quad N \to \infty. \]

That is to say, $\alpha_{N-s}^{(N)}$ tends to 0 as $N \to \infty$ like $N^{N^{3s-\frac{1}{2}}}/(2N)!$.

Note that the result in (24) is not contained in that given in (26) and vice versa. Note also that neither (24) nor (26) covers $\alpha_{i}^{(N)}$, $i = 1, \ldots, N$, uniformly in $i$. It is, however, possible to show that $\alpha_{i}^{(N)}$ are uniformly bounded both in $N$ and in $i$. To do this we first show that $C_{ki} \leq \Pi_{k-1}[(k-1)!]^2$ for all $k$ and $i$, with $\Pi_m = \prod_{i=1}^{m} (1+i^{-2})$ as before. This is achieved by using induction in (17). Consequently, $C_{ki} < \Pi_{\infty} [(k-1)!]^2$ for all $k$ and $i$, since $\Pi_{\infty} = \lim_{m \to \infty} \Pi_m$ exists. Next, we substitute this upper bound on $C_{ki}$ in (7) to obtain

\[ \left| \alpha_{i}^{(N)} \right| \leq \Pi_{\infty} \sum_{k=1}^{N} \left| f[x_1, \ldots, x_k] \right| [(k-1)!]^2, \quad i = 1, 2, \ldots, N. \]

Invoking (12) in (27) and using Stirling’s formula, we can show that $|f[x_1, \ldots, x_k]|[(k-1)!]^2 = O(k^{-2})$ as $k \to \infty$. The result now follows.
A Laplace Transform from a Diffusion Problem

Problem 97-12*, by M. L. Glasser (Clarkson University).

In an investigation [1] of the scaling properties of diffusion in a space of dimensionality $d$, the authors require knowledge of the Laplace transform

$$
\phi(s) = \int_0^\infty e^{-st} \sin^{-1} \left( \text{sech}^{d/2}(t/2) \right) dt.
$$

1. Show that for $d = 1, 2$ it is possible to express $\phi(s)$ in closed form in terms of the digamma function.
2. Can this be done for $d = 3, 4$?

REFERENCE


Solution by Carl C. Grosjean (University of Ghent, Ghent, Belgium).

The given integral is convergent for $\text{Re}(s) > -d/4$ since the integrand approximately behaves like $2^{d/2}e^{-(s+d/4)t}$ as $t \to +\infty$. For $s \neq 0$, integration by parts can be carried out as follows:

$$
\phi(s) = -\frac{1}{s} \int_0^\infty \sin^{-1} \left( \text{sech}^{d/2}(t/2) \right) de^{-st}
$$

$$
= \frac{\pi}{2s} - \frac{d}{4s} \int_0^\infty e^{-st} \frac{\sinh(t/2)}{\cosh(t/2)|\cosh^{d}(t/2) - 1|^{1/2}} dt.
$$

For $s \to 0$, this right-hand side tends to

$$
\frac{d}{4} \int_0^\infty \frac{t \sinh(t/2)}{\cosh(t/2)|\cosh^{d}(t/2) - 1|^{1/2}} dt
$$

representing $\phi(0)$. Note that, with the substitution of a new integration variable, $x^2 = \cosh^{d}(t/2) - 1$,

$$
\frac{d}{4} \int_0^\infty \frac{\sinh(t/2)}{\cosh(t/2)|\cosh^{d}(t/2) - 1|^{1/2}} dt = \frac{\pi}{2},
$$

as required.

The simplest case is that of $d = 2$. For $s \neq 0$, we have

(1) \hspace{1cm}

$$
\phi_2(s) = \frac{\pi}{2s} - \frac{1}{2s} \int_0^\infty \frac{e^{-st}}{\cosh(t/2)} dt
$$

$$
= \frac{\pi}{2s} - \frac{1}{s} \int_0^\infty \frac{e^{-(s+1/2)t}}{1 + e^{-t}} dt
$$

$$
= \frac{\pi}{2s} - \frac{1}{s} \left[ \frac{1}{s + 1/2} - \frac{1}{s + 3/2} + \frac{1}{s + 5/2} - \frac{1}{s + 7/2} + \cdots \right].
$$