Quotient-difference type generalizations of the power method and their analysis

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Received 24 September 1989
Revised 15 March 1990

Abstract: The recursion relations that were proposed by Ford and Sidi (1988) for implementing vector extrapolation methods are used for devising generalizations of the power method for linear operators. These generalizations are shown to produce approximations to largest eigenvalues of a linear operator under certain conditions. They are similar in form to the quotient-difference algorithm and share similar convergence properties with the latter. These convergence properties resemble also those obtained for the basic LR and QR algorithms. Finally, it is shown that the convergence rate produced by one of these generalizations is twice as fast for normal operators as it is for nonnormal operators.

Keywords: Vector extrapolation methods, power method, quotient-difference algorithm.

1. Introduction

Let $B$ be a normed linear space over the field of complex numbers, and denote the norm associated with $B$ by $\| \cdot \|$. In case $B$ is also an inner product space, we adopt the following convention for the homogeneity property of the inner product: for $y, z \in B$ and $\alpha, \beta$ complex numbers, the inner product $(\cdot, \cdot)$ satisfies $(\alpha y, \beta z) = \beta \alpha (y, z)$. The norm in this case is the one induced by the inner product, i.e., if $x \in B$, $\| x \| = \sqrt{(x, x)}$.

Let $x_i, i = 0, 1, \ldots$, be a sequence in $B$. We shall assume that

$$x_m \sim \sum_{i=1}^{\infty} v_i \lambda_i^n$$

as $m \to \infty$. (1.1)

Here $v_1, v_2, \ldots$ are linearly independent vectors in $B$. $\lambda_i$ are distinct scalars ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots,$$

and satisfy

$$\lambda_i \neq 0, \quad i = 1, 2, \ldots, \quad \lambda_i \neq \lambda_j \text{ if } i \neq j.$$ (1.3)

Also there can be only a finite number of $\lambda_i$'s having the same modulus. The meaning of (1.1) is
that for any positive integer $N$ there exist a positive constant $K$ and a positive integer $m_0$ that depend only on $N$, such that for every $m \geq m_0$,

$$
\left\| x_m - \sum_{i=1}^{N-1} \nu_i \lambda_i^m \right\| \leq K |\lambda_N|^m. \quad (1.4)
$$

Sequences of the form described above arise naturally, for example, in the matrix iterative process

$$
x_{j+1} = Ax_j, \quad j = 0, 1, \ldots,
$$

in which $x_0$ is an arbitrary column vector, and $A$ is a square matrix whose nonzero eigenvalues have only associated eigenvectors and no principal vectors. In this case, $\lambda_1, \lambda_2, \ldots$ are some or all of the nonzero eigenvalues of $A$, depending on the spectral decomposition of $x_0$, and $\nu_1, \nu_2, \ldots$ are proportional to its corresponding eigenvectors. Also the infinite sum $\sum_{i=1}^{\infty} \nu_i \lambda_i^m$ in (1.1) is now finite, and the asymptotic relation there becomes an equality.

In the next section we present three recursive techniques, which we designate the qd-MMPE, qd-TEA and qd-MPE algorithms, that are based solely on the vector sequence $x_0, x_1, x_2, \ldots$, and that can be used for obtaining approximations to $\lambda_1, \lambda_2, \lambda_3, \ldots$, in this order. These techniques generalize the well-known power method and resemble the quotient-difference (qd) algorithm. In Section 3 we analyze the convergence properties of these techniques and show that they indeed behave like the qd algorithm. We recall that the qd algorithm is used in approximating the poles of meromorphic functions in general, and the zeros of polynomials in particular. For detailed descriptions of the qd algorithm, see, for example, [3, Chapter 7] and [4, Chapter 3]. The convergence rates derived for the methods of the present work are also very similar to those derived for the basic LR and QR algorithms, see, for example, [5].

In Section 4 we will analyze one of the recursive techniques of Section 2, namely the qd-MPE algorithm, in conjunction with vector sequences $x_m, m = 0, 1, \ldots$, that are generated by iterating with a normal operator, and we will show that the rate of convergence for this case is twice that obtained for an arbitrary nonnormal operator that has the same spectrum.

Finally, we mention that the recursive techniques developed in this work are based on the recursive algorithms that were developed in [2] for implementing some vector extrapolation methods, such as the modified minimal polynomial extrapolation (MMPE), the topological epsilon algorithm (TEA), and the minimal polynomial extrapolation (MPE), from which their names are derived. The methods of proof relating to the convergence of these techniques are similar to those that were used in [6,8] in the analysis of the above-mentioned vector extrapolation methods, and in [7] in the analysis of some other recent extensions of the power method as they are applied to normal operators. We also mention that the scalar and vector epsilon algorithms of [9,10] can also be used as generalizations of the power method and produce results similar to the ones given in Section 3 of the present work. This has been shown in [1].

2. Development of the algorithms

In this section we follow very closely the developments and notation of [2].

In all three methods that we develop in this section we assume that the vector sequence $x_0, x_1, x_2, \ldots$, is given. We also assume that a sequence of scalar quantities $\mu_n^m, m, n \geq 0$, is given.
These quantities are determined in different forms from the vectors $x_i$. Their exact definition will be given later in this section. In addition, we denote by $b$ an arbitrary sequence $b_0, b_1, b_2, \ldots$. Thus $\mu^j$ stands for the sequence $\mu^0_0, \mu^1_1, \mu^2_2, \ldots$.

We define $G^{n,m}_k$ to be the determinant

$$G^{n,m}_k = \begin{vmatrix} \mu^m_n & \mu^m_{n+1} & \cdots & \mu^m_{n+k-1} \\ \mu^{m+1}_n & \mu^{m+1}_{n+1} & \cdots & \mu^{m+1}_{n+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^{m+k-1}_n & \mu^{m+k-1}_{n+1} & \cdots & \mu^{m+k-1}_{n+k-1} \end{vmatrix},$$

and $f^{n,m}_k(b)$ to be the determinant

$$f^{n,m}_k(b) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+k} \\ \mu^m_n & \mu^m_{n+1} & \cdots & \mu^m_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ \mu^{m+k-1}_n & \mu^{m+k-1}_{n+1} & \cdots & \mu^{m+k-1}_{n+k} \end{vmatrix}.$$

We also define

$$G^{n,m}_0 = 1 \quad \text{and} \quad f^{n,m}_0(b) = b_n.$$  

Finally, we let

$$T^{n,m}_k(b) = \frac{f^{n,m}_k(b)}{G^{n+1,m}_k}.$$  

Note that by (2.3) and (2.4),

$$T^{n,m}_0(b) = b_n.$$  

2.1. The qd-MMPE algorithm

Let $Q_1, Q_2, \ldots$ be a sequence of linearly independent bounded linear functionals on $B$, and let

$$\mu^m_n = Q_{m+1}(x_n), \quad m, n \geq 0.$$  

We recall that in case $B$ is a complete inner product space, each $Q_j$ has a unique representer $q_j$ in $B$ in the sense that $Q_j(z) = (q_j, z)$ for all $z$ in $B$.

As is shown in [2, Theorem 2.1], the quantities $T^{n}_k(b) = T^{n,0}_k(b)$ satisfy the three-term recursion relation

$$T^{n}_k(b) = T^{n}_k(b) - d^{n}_k T^{n+1}_k(b),$$

where

$$d^{n}_k = \frac{T^{n-1}_k(\mu^{k-1}_n)}{T^{n+1}_k(\mu^{k-1}_n)}.$$  

Note that $d^{n}_1 = \mu^0_n/\mu^0_{n+1}$ from (2.5) and (2.8). Note also that $T^{n}_k(\mu^j) = 0$ for $0 \leq j \leq k - 1$. 

As will be shown in Section 3, for the vector sequences \( x_0, x_1, x_2, \ldots \) described in the beginning of Section 1, \( 1/d_k^n \to \lambda_k \) as \( n \to \infty \) under certain conditions. Thus the developments above provide us with a technique that can produce approximations to \( \lambda_1, \lambda_2, \ldots \), in this order. This technique is summarized below.

**qd-MMPE algorithm**

Given the sequence \( x_0, x_1, x_2, \ldots \), compute \( \mu_n^m, m, n \geq 0 \), by (2.6), and set \( T_0^n(\mu^j) = \mu_n^j \). For \( k = 1, 2, \ldots \), use (2.7) to compute \( d_k^n \), and then use (2.7) to compute \( T_k^n(\mu^j), j \geq k \).

We note that \( d_k^n \) is constructed from the \( k + 1 \) vectors \( x_n, x_{n+1}, \ldots, x_{n+k} \).

Finally, we observe that \( 1/d_1^n = \mu_{n+1}^0/\mu_n^0 = Q_1(x_{n+1})/Q_1(x_n) \), which is exactly the approximation provided by the power method for \( \lambda_1 \), the largest eigenvalue in modulus of a linear operator, provided this eigenvalue is simple and satisfies \( |\lambda_1| > |\lambda_2| \). In this sense the qd-MMPE algorithm generalizes the power method.

**2.2. The qd-TEA algorithm**

Let \( Q \) be a bounded linear functional on \( B \), and let

\[
\mu_n^m = Q(x_{m+n}), \quad m, n \geq 0.
\]  

(2.9)

Again, in case \( B \) is a complete inner product space, \( Q \) has a unique representer \( q \) in \( B \) in the sense that \( Q(z) = (q, z) \) for all \( z \) in \( B \). The rest is exactly as in the qd-MMPE algorithm. We only have to observe that \( d_k^n \) now is determined by the \( 2k \) vectors \( x_n, x_{n+1}, \ldots, x_{n+2k-1} \). We also observe that for \( k = 1 \) the qd-TEA and qd-MMPE algorithms are the same when \( Q = Q_1 \). Consequently, as with the qd-MMPE algorithm, also with the qd-TEA algorithm the power method is obtained for \( k = 1 \). Hence the qd-TEA algorithm also generalizes the power method.

**2.3. The qd-MPE algorithm**

The scalar quantities \( \mu_n^m \) are now defined by

\[
\mu_n^m = (x_m, x_n), \quad m, n \geq 0.
\]  

(2.10)

As is shown in [2, Theorem 3.1], the quantities \( T_k^n(b) = T_k^{n,n}(b) \) and \( \tilde{T}_k^n(b) = T_{k-1}^{n,n-1}(b) \) satisfy the coupled recursion relations

\[
T_k^n(b) = T_{k-1}^{n,n}(b) - d_k^n \tilde{T}_{k-1}^{n,n+1}(b), \quad n \geq 0,
\]

\[
\tilde{T}_k^n(b) = T_{k-1}^{n,n}(b) - \tilde{d}_k^n \tilde{T}_{k-1}^{n,n+1}(b), \quad n \geq 1,
\]  

(2.11)

where

\[
d_k^n = \frac{T_{k-1}^{n,n+1}(\mu_n^{n+k-1})}{\tilde{T}_{k-1}^{n,n+1}(\mu_n^{n+k-1})}, \quad n \geq 0,
\]

\[
\tilde{d}_k^n = \frac{T_{k-1}^{n,n-1}(\mu_n^{n-1})}{\tilde{T}_{k-1}^{n,n-1}(\mu_n^{n-1})}, \quad n \geq 1.
\]  

(2.12)

In fact, there exists a very interesting four-term (lozenge) recursion relation among \( T_k^n(b), T_k^{n+1}(b), T_{k-1}^{n+1}(b) \) and \( T_{k-1}^{n+1}(b) \), see [2]. Note that \( d_1^n = \mu_n/\mu_{n+1}, n \geq 0 \), and \( \tilde{d}_1^n = \mu_n^{-1}/\mu_{n+1}^{-1}, n \geq 1 \), from (2.5) and (2.12). Note also that \( T_k^n(\mu^j) = 0 \) for \( n \leq j \leq n + k - 1 \), and \( \tilde{T}_k^n(\mu^j) = 0 \) for \( n - 1 \leq j \leq n + k - 2 \).
As will be shown in Section 3, for the vector sequences \( x_0, x_1, x_2, \ldots \), described in the beginning of Section 1, \( 1/d^n_k \to \lambda_k \) and \( 1/d^n_k \to \lambda_k \) as \( n \to \infty \) under certain conditions. Thus, the developments above provide us with yet another technique that can produce approximations to \( \lambda_1, \lambda_2, \ldots \), in this order. This technique is summarized below.

**qd-MPE algorithm**

Given the sequence \( x_0, x_1, x_2, \ldots \), compute \( \mu^n_m, m, n \geq 0 \), by (2.10), and set \( T^n_j(\mu^j) = \mu^n_j = T^n_j(\mu^j) \). For \( k = 1, 2, \ldots \), use (2.12) to compute \( d^n_k \) and \( d^n_k \), and then use (2.11) to compute \( T^n_j(\mu^j), j \leq n - 1, j \geq n + k, \) and \( T^n_j(\mu^j), j \leq n - 2, j \geq n + k - 1 \).

Observe that \( d^n_k \) and \( d^n_k \) are constructed from the \( k + 2 \) vectors \( x_{n-1}, x_n, x_{n+1}, \ldots, x_{n+k} \) only.

Finally, we observe that \( 1/d^n_k = \mu^n_{n+1}/\mu^n_n = (x_n, x_{n+1})/(x_n, x_n) \), which is exactly the Rayleigh quotient that provides an approximation to \( \lambda_1 \), the largest eigenvalue in modulus of a linear operator, provided this eigenvalue is simple and satisfies \( |\lambda_1| > |\lambda_2| \). In this sense the qd-MPE algorithm generalizes the power method (the Rayleigh quotient). It is known that in case the vector sequence \( x_0, x_1, x_2, \ldots \) is generated by a normal operator, the Rayleigh quotient converges to \( \lambda_1 \) twice as quickly as it does for a nonnormal operator.

So far we have shown that the three algorithms above generalize the power method for a linear operator. We would now like to explain why they are of the quotient-difference type. First, the algorithms are somewhat similar in form to the quotient-difference method. Second, as is shown in the next section (see (3.20) and (3.24)), the \( d^n_k \) are all expressed in terms of four determinants the way the quantities in the quotient-difference method are. Furthermore, again as will be shown in the next section, these determinants have asymptotic expansions very similar in form to those obtained for the determinants involved in the quotient-different method. We note in passing that the determinants that appear in the quotient-difference algorithm are Hankel determinants. The same holds true for the qd-TEA algorithms as will be clear in the next section, although the indices of these determinants are different in the two methods.

### 3. Convergence analysis of the algorithms

We now state the main convergence results for the qd-type algorithms that were devised in the previous section.

The following assumption is common to all three methods:

\[
|\lambda_{k-1}| > |\lambda_k| > |\lambda_{k+1}| \quad \text{for some integer } k \geq 1, \quad \lambda_0 \equiv \infty.
\]

Let us also define

\[
\epsilon_k = \max \left( \left| \frac{\lambda_k}{\lambda_{k-1}} \right|, \left| \frac{\lambda_{k+1}}{\lambda_k} \right| \right), \quad k \geq 1.
\]

Thus, \( \epsilon_k < 1 \).
Theorem 3.1. In the qd-MMPE algorithm denote
\[ S_j = \begin{vmatrix} Q_1(v_1) & \cdots & Q_1(v_j) \\ \vdots & \ddots & \vdots \\ Q_j(v_1) & \cdots & Q_j(v_j) \end{vmatrix}, \quad j = 1, 2, \ldots \tag{3.3} \]
Provided (3.1) holds, and
\[ S_{k-1} \neq 0 \quad \text{and} \quad S_k \neq 0, \tag{3.4} \]
we have
\[ \frac{1}{d_k^n} = \lambda_k + O(\epsilon_k^n) \quad \text{as} \quad n \to \infty. \tag{3.5} \]

Theorem 3.2. In the qd-TEA algorithm, provided (3.1) holds and
\[ \prod_{i=1}^{k} Q(v_i) \neq 0, \tag{3.6} \]
(3.5) is satisfied.

Theorem 3.3. In the qd-MPE algorithm, provided (3.1) holds, (3.5) is satisfied. Equation (3.5) is satisfied also when \( d_k^n \) is replaced by \( \tilde{d}_k^n \).

Note. In relation to Theorem 3.3 we realize that if the sequence \( x_0, x_1, x_2, \ldots \) is generated by the matrix \( A \) in the way \( x_{j+1} = Ax_j, \ j = 0, 1, \ldots, x_0 \) being arbitrary, then the qd-MPE algorithm performs asymptotically like the qd-MMPE and qd-TEA algorithms, although it does not require additional conditions like (3.4) and (3.6).

In the proofs of Theorems 3.1–3.3 we shall make use of the following results, which are of interest in themselves.

Lemma 3.4. Let \( \sigma_1, \sigma_2, \ldots \) and \( \xi_1, \xi_2, \ldots \) be two sequences of nonzero scalars, such that the \( \sigma_i \) are distinct, and
\[ |\xi_1| \geq |\xi_2| \geq \cdots. \tag{3.7} \]
Assume, furthermore, that there can be only a finite number of \( \xi_j \)'s having the same modulus. Consider the determinant
\[ \psi_r^n = \begin{vmatrix} u_{1,0}^{(n)} & u_{1,1}^{(n)} & \cdots & u_{1,r-1}^{(n)} \\ u_{2,0}^{(n)} & u_{2,1}^{(n)} & \cdots & u_{2,r-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{r,0}^{(n)} & u_{r,1}^{(n)} & \cdots & u_{r,r-1}^{(n)} \end{vmatrix}, \tag{3.8} \]
in which
\[ u_{p,q}^{(n)} \sim \sum_{j=1}^{\infty} z_{j,p} \sigma_j^n \xi_j \quad \text{as } n \to \infty, \quad (3.9) \]
z_{j,p} being some scalars. Define
\[ Z_{j_1,j_2,\ldots,j_r} = \begin{vmatrix} z_{j_1,1} & z_{j_1,2} & \cdots & z_{j_1,r} \\ z_{j_2,1} & z_{j_2,2} & \cdots & z_{j_2,r} \\ \vdots & \vdots & \ddots & \vdots \\ z_{j_r,1} & z_{j_r,2} & \cdots & z_{j_r,r} \end{vmatrix} \quad (3.10) \]

Then \( \psi_r \), for \( n \to \infty \), has the asymptotic expansion
\[ \psi_r^n \sim \sum_{1 \leq j_1 < j_2 < \cdots < j_r} Z_{j_1,j_2,\ldots,j_r} V(\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_r}) \left( \prod_{p=1}^{r} \xi_{j_p}^n \right), \quad (3.11) \]
where \( V(a_1, \ldots, a_r) \) is the Vandermonde determinant defined by
\[ V(a_1, \ldots, a_r) = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{r-1} \\ 1 & a_2 & \cdots & a_2^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_r & \cdots & a_r^{r-1} \end{vmatrix}. \quad (3.12) \]

(When \( \sim \) is replaced by \( = \) in (3.9), \( \sim \) is replaced by \( = \) in (3.11) too.) If, in addition,
\[ |\xi_r| > |\xi_{r+1}| \quad \text{and} \quad Z_{1,2,\ldots,r} \neq 0, \quad (3.13) \]
then
\[ \psi_r^n = Z_{1,2,\ldots,r} V(\sigma_1, \sigma_2, \ldots, \sigma_r) \left( \prod_{j=1}^{r} \xi_j^n \right) \left[ 1 + O\left( \left| \frac{\xi_{r+1}}{\xi_r} \right|^n \right) \right] \quad \text{as } n \to \infty. \quad (3.14) \]

Lemma 3.4 can be obtained from [7, Lemmas 2.2 and 2.3], whose technique is a generalization of that employed in [8, Theorem 3.2].

**Lemma 3.5.** Let \( \xi_1, \xi_2, \ldots \) be a sequence of nonzero scalars exactly as described in Lemma 3.4. Consider the determinant \( \psi_r^n \) as given in (3.8), but with
\[ u_{p,q}^{(n)} \sim \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} z_{i,j} \xi_i^{n+p-1} \xi_j^{n+q} \quad \text{as } n \to \infty, \quad (3.15) \]
\( z_{ij} \) being some scalars. Define
\[
Z_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} = \begin{vmatrix}
  z_{i_1 j_1} & z_{i_1 j_2} & \cdots & z_{i_1 j_r} \\
  z_{i_2 j_1} & z_{i_2 j_2} & \cdots & z_{i_2 j_r} \\
  \vdots & \vdots & & \vdots \\
  z_{i_r j_1} & z_{i_r j_2} & \cdots & z_{i_r j_r}
\end{vmatrix}.
\] (3.16)

Then \( \psi_n^r, \) for \( n \to \infty, \) has the asymptotic expansion
\[
\psi_n^r \sim \sum_{1 \leq i_1 < i_2 < \cdots < i_r} \sum_{1 \leq j_1 < j_2 < \cdots < j_r} Z_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} \
 \times V(\xi_{i_1}, \ldots, \xi_{i_r}) V(\xi_{j_1}, \ldots, \xi_{j_r}) \left( \prod_{p=1}^r \xi_{p,i}^n \right) \left( \prod_{p=1}^r \xi_{j,p}^n \right).
\] (3.17)

(When \( \sim \) is replaced by \( = \) in (3.15), \( \sim \) is replaced by \( = \) in (3.17) too.) If, in addition,
\[
|\xi_r| > |\xi_{r+1}| \quad \text{and} \quad Z_{i_1, \ldots, i_r} \neq 0,
\] (3.18)
then
\[
\psi_n^r = Z_{i_1, \ldots, i_r}^{j_1, \ldots, j_r} \left| V(\xi_1, \ldots, \xi_r) \right|^2 \left( \prod_{j=1}^r \xi_{j}^n \right) \left[ 1 + O\left( \frac{\xi_{r+1}}{\xi_r} \right) \right] \quad \text{as} \quad n \to \infty.
\] (3.19)

Lemma 3.5 can be proved exactly as in [6, Lemma 3.2 and Theorem 3.1].

**Proof of Theorem 3.1.** Combining (2.8) with (2.4), (2.2) and (2.1), and recalling that \( T^{n}_k(b) = T^{n,0}_k(b) \), we obtain
\[
d_n^m = \frac{G^{n,0}_k G^{n+2,0}_{k-1}}{G^{n+1,0}_k G^{n+1,0}_{k-1}}.
\] (3.20)

Now by (2.6) and (1.1) we have
\[
\mu_n^m \sim \sum_{j=1}^\infty Q_{m+1}(v_j) \lambda_j^n \quad \text{as} \quad n \to \infty.
\] (3.21)

Substituting (3.21) in the determinant expression for \( G^{n,0}_k \), we see that \( G^{n,0}_k \) is simply \( \psi_k^r \) of Lemma 3.4 with \( u^{(n)}_{p,q} = \mu^{-1}_{p,q} \), thus \( z_{j,p} = Q_p(v_j), \xi_j = \lambda_j \) and \( \sigma_j = \lambda_j \). Furthermore, \( S_j \) in (3.3) is \( Z_{1,2, \ldots, i} \) in Lemma 3.4. Therefore, Lemma 3.4 applies, and we have
\[
G^{n,0}_k = S_k V(\lambda_1, \ldots, \lambda_k) \left( \prod_{j=1}^k \lambda_j^n \right) \left[ 1 + O\left( \left| \frac{\lambda_{k+1}}{\lambda_k} \right|^n \right) \right] \quad \text{as} \quad n \to \infty.
\] (3.22)

Treating \( G^{n+1,0}_k, G^{n+1,0}_{k-1} \) and \( G^{n+2,0}_{k-1} \) similarly, and combining the results in (3.20), we obtain (3.5). \( \square \)
Proof of Theorem 3.2. We observe that by (2.9) and (2.1)
\[ \mu_n^m = \sum_{j=1}^{\infty} \left[ Q(v_j) \lambda_j^m \right] \lambda_j^n \quad \text{as } n \to \infty. \] (3.23)

The rest of the proof is exactly the same as that of Theorem 3.1, with \( Q_{m+1}(v_j) \) in the latter being replaced by \( Q(v_j) \lambda_j^n \) now. On account of this, \( Z_{1,\ldots,j} \) now becomes \([\prod_{\ell=1}^{j} Q(v_{\ell})] V(\lambda_1, \ldots, \lambda_j)\). We leave out the rest of the details. \( \square \)

Here we note that \( G_k^n \) for the qd-TEA algorithm is a Hankel determinant. Thus the qd-TEA algorithm resembles very strongly the qd algorithm. Furthermore, when the sequence \( x_0, x_1, x_2, \ldots \) is one of scalars, then the qd-TEA algorithm can still be employed by picking the functional \( Q \) to be the identity operator, in which case \( \mu_n^m = x_{m+n} \). Thus for scalar sequences satisfying all the conditions mentioned in Section 1 of this work the qd-TEA algorithm provides another form of the qd algorithm.

Proof of Theorem 3.3. Combining (2.12) with (2.4), (2.2) and (2.1), and recalling that \( T_n^m(b) = T_{k,n}^m(b) \) and \( T_n^m(b) \equiv T_{n+1,n}^m(b) \), we obtain
\[ d_k^n = \frac{G_k^n G_{k+1,n}^{n+1} G_{k-1}^{n+2}}{G_{k+1}^{n+1,n} G_{k-1}^{n+2,n}}, \quad \tilde{d}_k^n = \frac{G_k^n G_{k-1}^{n+1} G_{k+1}^{n+2,n}}{G_{k+1}^{n+1,n} G_{k-1}^{n+2,n}}. \] (3.24)

Substituting (2.10) in (2.1), and comparing with (3.8), we see that \( G_k^n = G_{n+1}^n \) with \( u_n^{p,q} = \mu_n^{p+q-1} \) and \( \xi_i = \lambda_i \). Also \( Z_{1,\ldots,k} \) in Lemma 3.5 now becomes
\[ Z_{1,\ldots,k} = \left( \prod_{i=1}^{k} \tilde{\lambda}_i \right) \begin{pmatrix} (v_1, v_1) & \cdots & (v_1, v_k) \\ \vdots & \ddots & \vdots \\ (v_k, v_1) & \cdots & (v_k, v_k) \end{pmatrix} \equiv \left( \prod_{i=1}^{k} \tilde{\lambda}_i \right) U_{1,\ldots,k}. \] (3.25)

Since the vectors \( v_1, v_2, \ldots \) in (1.1) are assumed to be linearly independent, the Gram determinant \( U_{1,\ldots,k} \) of the vectors \( v_1, \ldots, v_k \) is positive. Thus \( Z_{1,\ldots,k} \neq 0 \). Consequently, Lemma 3.5 applies, and we obtain
\[ G_k^{n+s} = \left( \prod_{i=1}^{k} \tilde{\lambda}_i \right) \left| V(\lambda_1, \ldots, \lambda_k) \right|^{2n} \left( \prod_{j=1}^{k} \lambda_j \right)^{2n} \left( 1 + O\left( \left\| \lambda_{k+1} \right\| \right) \right)^n \] (3.26)
as \( n \to \infty \).

Invoking (3.26) in (3.24), we obtain the required result. \( \square \)

4. The qd-MPE algorithm for normal operators

As we have shown in Theorem 3.3, the convergence rate of the qd-MPE algorithm is the same as those of the qd-MMPE and qd-TEA algorithms, in general. When the vectors \( v_1, v_2, \ldots \) in the expansion (1.1) are orthogonal with respect to the inner product \( (\cdot, \cdot) \), however, the convergence
rate of the qd-MPE method becomes twice that given in Theorem 3.3, as will be shown shortly in Theorem 4.1. When the vector sequence $x_0, x_1, \ldots$ is generated by the matrix iterative process $x_{j+1} = Ax_j$, $j = 0, 1, \ldots$, with $x_0$ arbitrary, then, as we mentioned in the Introduction, $x_m$ satisfies (1.1), with $\lambda_i$ there being nonzero eigenvalues of $A$ that possess only eigenvectors and no principal vectors, and $v_i$ there being proportional to the corresponding eigenvectors. Now, when $A$ is a normal operator, it has only eigenvectors, and these form an orthogonal set with respect to the inner product. We recall that Hermitian and anti-Hermitian operators are normal. Thus Theorem 4.1 applies to vector sequences generated by matrix iterative processes with normal matrices and with the Euclidean inner product.

**Theorem 4.1.** In the qd-MPE algorithm, provided (3.1) holds and the vectors $v_1, v_2, \ldots$ in (1.1) satisfy

\[
(v_i, v_j) = z_j \delta_{ij}, \quad \text{(thus } z_j > 0),
\]

we have

\[
\frac{1}{d_k^n} = \lambda_k + O(\epsilon_k^{2n}) \quad \text{as } n \to \infty.
\]

When $\lambda_j > 0$, $j = 1, 2, \ldots$, (4.2) can be refined considerably. (In this case, the condition in (3.1) is automatically satisfied.) If $\lambda_{k+1}/\lambda_k > \lambda_k/\lambda_{k-1}$, then

\[
\frac{1}{d_k^n} \sim \lambda_k - \beta_k \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n} \quad \text{as } n \to \infty, \text{ some } \beta_k > 0,
\]

i.e., the sequence $\{1/d_k^n\}_{n=0}^\infty$ converges to $\lambda_k$ monotonically from below, and if $\lambda_{k+1}/\lambda_k < \lambda_k/\lambda_{k-1}$, then

\[
\frac{1}{d_k^n} \sim \lambda_k + \gamma_k \left( \frac{\lambda_k}{\lambda_{k-1}} \right)^{2n} \quad \text{as } n \to \infty, \text{ some } \gamma_k > 0,
\]

i.e., the sequence $\{1/d_k^n\}_{n=0}^\infty$ converges to $\lambda_k$ monotonically from above. $\beta_k$ and $\gamma_k$ are given in the proof below.

All the results above hold when $d_k^n$ is replaced by $\tilde{d}_k^n$, though with different $\beta_k$ and $\gamma_k$.

**Proof.** We saw in the proof of Theorem 3.3 that $G_k^{n,n+1} = \psi_k^n$ with $u_p^{(n)} = (x_{n+p-1}, x_n)$. Invoking now (1.1) and (4.1), we see that $u_p^{(n)}$ is of the form (3.9), $z_{j,p} = z_j \lambda_{j+p-1}$, $a_j = \lambda_j$ and $z_j = |\lambda_j|^2$. Also $Z_{j_1, \ldots, j_k}$ in Lemma 3.4 becomes

\[
Z_{j_1, \ldots, j_k} = \left( \prod_{p=1}^k z_{j,p} \right) \left( \prod_{p=1}^k \lambda_{j,p} \right) V(\lambda_{j_1}, \ldots, \lambda_{j_k}).
\]

Substituting all this in (3.11), we obtain for $n \to \infty$

\[
G_k^{n,n+1} \sim \sum_{1 \leq j_1 < \cdots < j_k} \left( \prod_{p=1}^k \lambda_{j,p} \right) \left( \prod_{p=1}^k z_{j,p} \right) |V(\lambda_{j_1}, \ldots, \lambda_{j_k})|^2 \left( \prod_{p=1}^k \lambda_{j,p} \right)^{2n}.
\]
Consequently, all the conditions of Lemma 3.4 are satisfied, and
\[ G_k^{n+s} = \left( \prod_{j=1}^{k} \lambda_j^s \right) \left( \prod_{j=1}^{k} z_j \right) |V(\lambda_1, \ldots, \lambda_k)|^2 \left( \prod_{j=1}^{k} \lambda_j \right)^{2n} \left[ 1 + O\left( \frac{\lambda_{k+1}}{\lambda_k}^{2n} \right) \right] \]

as \( n \to \infty \). \( \quad (4.7) \)

Invoking (4.7) in (3.24), (4.2) follows.

Now when the \( \lambda_j \)'s are all real, from (4.6) and (4.1), we see that \( G_k^{n+s}, s = 0, \pm 2, \pm 4, \ldots \), as \( n \to \infty \), are all real and positive, and when the \( \lambda_j \)'s are all positive, \( G_k^{n+s} \) are positive for all \( s \). If the vectors \( x_j \) are generated by the matrix iterative process given in (1.5), then this holds for all \( n \), and not only as \( n \to \infty \). In case all \( \lambda_j \)'s are positive, (1.2) and (3.1) become
\[ \lambda_1 > \lambda_2 > \cdots > \lambda_{k-1} > \lambda_k > \lambda_{k+1} > \cdots, \]

so that (3.1) is automatically satisfied now. The two most dominant terms in the summation in (4.6) are those with the indices \( j_1, \ldots, j_k = 1, 2, \ldots, k \) and \( j_1, \ldots, j_k = 1, 2, \ldots, k-1, k+1 \). Thus
\[ G_k^{n+s} = \left( \prod_{j=1}^{k} z_j \right) |V(\lambda_1, \ldots, \lambda_k)|^2 \left( \prod_{j=1}^{k} \lambda_j^{2n+s} \right) \times \left[ 1 + \alpha_k \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n+s} + O\left( \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n} \right) \right] \text{ as } n \to \infty, \]

where
\[ \alpha_k = \frac{z_{k+1}}{z_k} \left[ \frac{V(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)}{V(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)} \right]^2 = \frac{z_{k+1}}{z_k} \frac{1}{\lambda_{k-1}} \left( \frac{\lambda_{k+1}}{\lambda_k} - \frac{\lambda_j}{\lambda_k} \right)^2 > 0. \] \( \quad (4.10) \)

Consequently, for \( n \to \infty \),
\[ d_k^n \sim \frac{1}{\lambda_k} \left[ 1 + \alpha_k \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n} \right] \left[ 1 + \alpha_{k-1} \left( \frac{\lambda_{k-1}}{\lambda_{k-1}} \right)^{2n+2} \right] \]
\[ \sim \frac{1}{\lambda_k} \left[ 1 + \alpha_k \left( 1 - \frac{\lambda_{k+1}}{\lambda_k} \right) \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n} \right] \left[ 1 - \alpha_{k-1} \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \left( \frac{\lambda_k}{\lambda_{k-1}} \right)^{2n+1} \right] \]
\[ \sim \frac{1}{\lambda_k} \left[ 1 + \alpha_k \left( 1 - \frac{\lambda_{k+1}}{\lambda_k} \right) \left( \frac{\lambda_{k+1}}{\lambda_k} \right)^{2n} - \alpha_{k-1} \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \left( \frac{\lambda_k}{\lambda_{k-1}} \right)^{2n+1} \right]. \] \( \quad (4.11) \)

The rest of the proof can now be completed easily. We only note that
\[ \beta_k = \alpha_k \left( 1 - \frac{\lambda_{k+1}}{\lambda_k} \right) \]
in (4.3), and
\[ \gamma_k - \alpha_{k-1} \left( 1 - \frac{\lambda_k}{\lambda_{k-1}} \right) \frac{\lambda_k}{\lambda_{k-1}} \]
in (4.4).  □

References