Uniqueness of Padé Approximants
From Series of Orthogonal Polynomials

By Avram Sidi

Abstract. It is proved that whenever a nonlinear Padé approximant, derived from a series of orthogonal polynomials, exists, it is unique.

Let \( \phi_r(x), r = 0, 1, 2, \ldots, \) be a set of polynomials which are orthogonal on an interval \([a, b]\), finite, semi-infinite, or infinite, with weight function \( w(x) \), whose integral over any subinterval of \([a, b]\) is positive; i.e.,

\[
\int_a^b w(x)\phi_r(x)\phi_s(x) \, dx = 0 \quad \text{if } r \neq s.
\]

Then it is known that \( \phi_r(x) \) is a polynomial of degree exactly \( r \).

Suppose now \( f(x) \) is a function which has a formal expansion of the form

\[
f(x) = \sum_{r=0}^{\infty} a_r \phi_r(x)
\]

on \([a, b]\). The \((m, n)\) Padé approximant to \( f(x) \) is defined to be the rational function

\[
S_{m,n}(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{r=0}^{m} p_r \phi_r(x)}{\sum_{s=0}^{n} q_s \phi_s(x)}
\]

having an expansion in \( \phi_r(x), r = 0, 1, 2, \ldots, \), which agrees with that of \( f(x) \) given in (2) up to and including the term \( a_{m+n} \phi_{m+n}(x) \). It is assumed that the polynomials \( P(x) \) and \( Q(x) \) have no common factor, apart from a constant, and that \( Q(x) \) does not vanish on \([a, b]\). It is worth mentioning that the approximations defined above are the ones called "nonlinear Padé approximants" in [2].

**Theorem 1.** If \( g(x) \) is any continuous function on \([a, b]\) such that

\[
\int_a^b w(x)g(x)\phi_r(x) \, dx = 0, \quad r = 0, 1, \ldots, k - 1,
\]

then \( g(x) \) either changes sign at least \( k \) times in the interval \((a, b)\) or is identically zero.

The proof of this theorem can be found in [1, p. 110].

As a consequence of Theorem 1, it follows that if \( Q(x) \) is nonzero on \([a, b]\), then \( q_0 \neq 0 \); hence one can normalize \( Q(x) \) by taking \( q_0 = 1 \).

**Theorem 2.** If the \((m, n)\)th nonlinear Padé approximant \( P(x)/Q(x) \) to \( f \) exists, in the sense of (3), and, after dividing out common factors, if \( Q \) is of one sign on \([a, b]\), then it is unique.

**Proof.** By the definition of \( S_{m,n}(x) = P(x)/Q(x) \) one has

\[
f(x) - S_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r \phi_r(x).
\]
If \( \bar{S}_{m,n}(x) = \frac{\bar{P}(x)}{\bar{Q}(x)} \) is another \((m, n)\) Padé approximant to \((1)\), then
\[
(5) \quad f(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} A_r \phi_r(x).
\]
Subtracting (4) from (5) one obtains
\[
(6) \quad S_{m,n}(x) - \bar{S}_{m,n}(x) = \sum_{r=m+n+1}^{\infty} (\bar{A}_r - A_r) \phi_r(x).
\]
Now since \( S_{m,n}(x) \) and \( \bar{S}_{m,n}(x) \) are continuous on \([a, b]\) so is \( D(x) \equiv S_{m,n}(x) - \bar{S}_{m,n}(x) \). Then from (6) it follows that \( D(x) \) satisfies \( \int_a^b w(x) D(x) \phi_r(x) \, dx = 0, \ r = 0, 1, \ldots, m + n \). Hence by Theorem 1, \( D(x) \) either changes sign at least \( m + n + 1 \) times on \((a, b)\), or is identically zero there. But
\[
(7) \quad D(x) = \frac{P(x) - \bar{P}(x)}{Q(x) - \bar{Q}(x)} = \frac{P(x)Q(x) - \bar{P}(x)Q(x)}{Q(x)\bar{Q}(x)},
\]
\( \text{i.e., the numerator of } D(x) \text{ is a polynomial of degree at most } m + n, \text{ therefore, can have at most } m + n \text{ zeros on } (a, b). \) Since \( Q(x) \) and \( \bar{Q}(x) \) are nonzero on \([a, b]\), \( D(x) \) changes sign at most \( m + n \) times on \((a, b)\). Therefore, \( D(x) \equiv 0; \) hence \( S_{m,n}(x) \equiv \bar{S}_{m,n}(x) \). Q.E.D.

So far Padé approximants from Legendre series [2] and Chebyshev series have been considered [3], [4]. As is explained in [2], the determination of the \( q_s, \ s = 1, 2, \ldots, n \), in general, involves the solution of \( n \) nonlinear equations, the determination of the \( p_r \) being trivial then. However, these \( n \) equations may have several solutions. But, as is mentioned in [2], only one solution with \( Q(x) \neq 0 \) on \([a, b]\) has been found for the examples in [2]. By Theorem 2 there is no other solution, and it is at this point that the result of Theorem 2 becomes important.

The author wishes to thank Professor I. M. Longman for encouragement and help.

Department of Geophysics and Planetary Sciences
Tel-Aviv University
Ramat-Aviv, Israel